The Distribution of Sum, Product and Ratio for the Absolutely Continuous Bivariate Generalized Exponential Random Variables

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Abstract

Block and Basu bivariate exponential distribution is a standout amongst the most absolutely continuous bivariate distributions. This idea can be extended to the generalized exponential distribution also. In this case this distribution is called as the Block and Basu bivariate generalized exponential (BBBGE) distribution. Some properties of BBBGE distribution can be obtained as the moment generating function, median and mode . The exact forms for the distribution of sum; ratio and product of dependent variables follow the Block and Basu bivariate generalized exponential distribution are derived. The maximum likelihood estimation (MLE) procedure is performed for the parameters of the BBBGE distribution. A numerical illustration performed to see the performances of the MLEs.

Keywords: *Block and Basu bivariate exponential distribution, moment generating function, Marshal-Olkin bivariate exponential distribution.*

1 Introduction

Block and Basu bivariate generalized exponential (BBBGE) distribution has been obtained from the Marshal- Olkin bivariate generalized exponential (MOBGE) distribution by removing the singular part and that makes BBBGE distribution as an absolutely continuous bivariate distribution.

An important operation in probability theory is to obtain the distribution of the sum of two correlated random variables X_1 and X_2. Applications of the sums appear in many areas of mathematics, probability theory, physics and engineering. In many applications a random variable Z is a functionally related to two or more different random variables X_1 and X_2. A good example is the random signal S at the input of an amplifier consists of a random signal X_1 to which is add independent random noise X_2. Hence the random signals S is the sum of X_1 and X_2. Now an important question assess, what is the probability density function of the random variable S, which represents the amplifiers input. Also, many signal processing systems use electronic multipliers to multiply two signals together. If X_1 is the signal of one input and X_2 is another signal input, what is the probability density function of $P = X_1 X_2$.

The ratios of two random variables X_1 and X_2 is the stress-strength model in the reliability theory. An important example is that model which describes the lifetimes of a component which has a random strength X_1 and subject to random stress X_2. These components fail at the time instant that the stress exceeds the strength and this component will function whenever $X_1 > X_2$. Hence the probability $P(X_2 > X_1) = P(2X_2/(X_1 + X_2) < 1)$ is a measure of the reliability of the component.

The paper is organized as follows: In Section 2, the BBBGE distribution is introduced and the representations for the probability density function (pdf), cumulative distribution function (cdf), marginal distributions and moment generating function (mfg) are obtained . The exact forms for the distribution of sum; ratio and product of dependent variables follow the BBBGE distribution are derived in Section 3. The maximum likelihood estimation, estimated variance-covariance matrix and asymptotic confidence intervals for BBBGE distribution are provided in Section 4. Simulation results are presented in Section 5. Finally conclude the paper in Section 6.

2 The BBBGE Distribution

If X has univariate generalized exponential (GE) distribution with the shape and scale parameters as $\alpha > 0$ and $\lambda > 0$ respectively, then the cdf and pdf of the GE distribution is as follows respectively,

$$F_{GE}(x;\alpha,\lambda) = (1-e^{-\lambda y})^{\alpha} \quad , \qquad y > 0$$

$$f_{GE}(x;\alpha,\lambda) = \alpha\,\lambda\,(1-e^{-\lambda y})^{\alpha-1} e^{-\lambda y} \quad , \qquad y > 0, \alpha, \lambda > 0$$

Kundu and Gupta (2009) introduced that (Y_1, Y_2) have MOBGE distribution if

The joint cdf of (Y_1, Y_2) can be written as

$$F_{MOBGE}(y_1, y_2) = \begin{cases} F_1(y_1, y_2) & if \quad 0 < y_1 < y_2 < \infty \\ F_2(y_1, y_2) & if \quad 0 < y_2 < y_1 < \infty \\ F_3(y) & if \quad 0 < y_1 = y_2 = y < \infty \end{cases} \tag{2.1}$$

Where

$$F_1(y_1, y_2) = F_{GE}(y_1; \alpha_{13}) \cdot F_{GE}(y_2; \alpha_2)$$
$$F_2(y_1, y_2) = F_{GE}(y_1; \alpha_1) \cdot F_{GE}(y_2; \alpha_{23})$$
$$F_3(y) = F_{GE}(y; \alpha_{123})$$

Where $\alpha_{13} = \alpha_1 + \alpha_3$, $\alpha_{23} = \alpha_2 + \alpha_3$ and $\alpha_{123} = \alpha_1 + \alpha_2 + \alpha_3$.

They observed that the joint distribution function of (Y_1, Y_2) can be written as a mixture of an absolutely continuous part and a singular part as follows;

$$F_{MOBGE}(y_1, y_2) = \frac{\alpha_{12}}{\alpha_{123}} F_a(y_1, y_2) + \frac{\alpha_3}{\alpha_{123}} F_s(y)$$

where $y = \min(y_1, y_2)$.

$F_s(y) = (1 - e^{-y})^{\alpha_{123}}$,

and $F_a(y_1, y_2) = \frac{\alpha_{123}}{\alpha_{12}} (1 - e^{-y_1})^{\alpha_1} (1 - e^{-y_2})^{\alpha_2} (1 - e^{-y})^{\alpha_3} - \frac{\alpha_3}{\alpha_{12}} (1 - e^{-y})^{\alpha_{123}}$.

Here $F_s(.,.)$ and $F_a(.,.)$ are the singular and the absolutely continuous part respectively.

The BBBGE distribution can be obtained from MOBGE distribution by removing the singular part and keeping only the continuous part. The joint pdf of BBBGE distribution can be written as

$$f_{BBBGE}(x_1, x_2) = \begin{cases} cf_1(x_1, x_2) & if \quad 0 < x_1 < x_2 < \infty \\ cf_2(x_1, x_2) & if \quad 0 < x_2 < x_1 < \infty. \end{cases} \tag{2.2}$$

Where

$$f_1(x_1,x_2) = f_{GE}(x_1;\alpha_{13},\lambda)\ f_{GE}(x_2;\alpha_2,\lambda)$$
$$=\alpha_{13}\alpha_2\lambda^2 e^{-\lambda(x_1+x_2)}[1-e^{-\lambda x_1}]^{\alpha_{13}-1}\,[1-e^{-\lambda x_2}]^{\alpha_2-1},$$

and

$$f_2(x_1,x_2) = f_{GE}(x_1;\alpha_1,\lambda)\ f_{GE}(x_2;\alpha_{23},\lambda)$$
$$=\alpha_1\alpha_{23}\lambda^2 e^{-\lambda(x_1+x_2)}[1-e^{-\lambda x_1}]^{\alpha_1-1}\,[1-e^{-\lambda x_2}]^{\alpha_{23}-1}.$$

Here c is the normalizing constant and $c=\dfrac{\alpha_{123}}{\alpha_{12}}$. Therefore, the joint pdf of (X_1,X_2) can be written as (2.2) and will be denoted by $BBBGE(\alpha_1,\alpha_2,\alpha_3,\lambda)$.

In what follows the joint cdf corresponding to Equation (2.2), the marginal distributions of the BBBGE are presented.

Proposition 2.1. Let $(X_1,X_2)\sim BBBGE(\alpha_1,\alpha_2,\alpha_3,\lambda)$. The joint cdf is given as

$$F_{X_1,X_2}(x_1,x_2) = \frac{\alpha_{123}}{\alpha_{12}}F_{GE}(x_1;\lambda,\alpha_1)\ F_{GE}(x_2;\lambda,\alpha_2)F_{GE}(x;\lambda,\alpha_3)$$
$$-\frac{\alpha_3}{\alpha_{12}}F_{GE}(x;\lambda,\alpha_{123});$$

Where $x=\min(x_1,x)$. Moreover, the marginal cdfs are given by

$$F_{X_1}(x_1) = \frac{\alpha_{123}}{\alpha_{12}}F_{GE}(x_1;\lambda,\alpha_{13}) - \frac{\alpha_3}{\alpha_{12}}F_{GE}(x_1;\lambda,\alpha_{123})$$
$$F_{X_2}(x_2) = \frac{\alpha_{123}}{\alpha_{12}}F_{GE}(x_2;\lambda,\alpha_{23}) - \frac{\alpha_3}{\alpha_{12}}F_{GE}(x_2;\lambda,\alpha_{123})$$

Proof : The joint cdf given in (2.1) can be written as follows

$$F_{MOBGE}(y_1,y_2) = \frac{\alpha_{12}}{\alpha_{123}}F_a(y_1,y_2)+\frac{\alpha_3}{\alpha_{123}}F_s(y)$$

$$F_s(y)=(1-e^{-y})^{\alpha_{123}}$$

where $F_s(.,.)$ and $F_a(.,.)$ are the singular and the absolutely continuous part respectively. For $y=\min(y_1,y_2)$,

$$F_s(y)=F_{GE}(y;\lambda,\alpha_{123}),$$

and $F_a(y_1,y_2) = \frac{\alpha_{123}}{\alpha_{12}}F_{GE}(y_1;\lambda,\alpha_1)\ F_{GE}(y_2;\lambda,\alpha_2)F_{GE}(y;\lambda,\alpha_3)$

$$-\frac{\alpha_3}{\alpha_{12}}F_{GE}(y;\lambda,\alpha_{123}).$$

Once $F_{Y_1,Y_2}(y_1, y_2) = F_a(y_1, y_2)$, the result holds. The marginal cdfs are obtained simply.

Proposition 2.2. The marginal pdfs correspond to the cdf given in Proposition 2.1 are as follows

$$f_{X_1}(x_1) = cF_{GE}(x_1; \lambda, \alpha_{13}) - c\frac{\alpha_3}{\alpha_{123}} f_{GE}(x_1; \lambda, \alpha_{123}), \qquad x_1 > 0$$

And

$$f_{X_2}(x_2) = cF_{GE}(x_1; \lambda, \alpha_{23}) - c\frac{\alpha_3}{\alpha_{123}} f_{GE}(x_2; \lambda, \alpha_{123}), \qquad x_2 > 0$$

Proof: By apply $f_{X_1}(x_1) = \frac{dF_{X_1}(x_1)}{dx_1}$ and $f_{X_2}(x_2) = \frac{dF_{X_2}(x_2)}{dx_2}$, the results obtained.

Unlike those of the MOBGE distribution, the marginals of the BBBGE distribution are not GE distributions. If $\alpha_3 \to 0^+$, then X_1 and X_2 follow GE distributions and in this case, X_1 and X_2 become independent .

Proposition 2.3. Let $(X_1, X_2) \sim BBBGE(\alpha_1, \alpha_2, \alpha_3, \lambda)$. Then

i. the Stress- Strength parameter has the following form;

$$R = P(X_1 < X_2) = \frac{\alpha_1}{\alpha_{12}},$$

ii. $max(X_1, X_2) \sim GE(\alpha_{123})$.

The BBBGE density may be unimodal depending on the values of $\alpha_1, \alpha_2, \alpha_3$ and λ that is $f_{BBBGE}(x_1, x_2)$ is unimodal and the respective modes are

$$\{\frac{1}{\lambda}\ln(\alpha_{13}), \frac{1}{\lambda}\ln(\alpha_2)\} \text{ and } \left\{\frac{1}{\lambda}\ln(\alpha_1), \frac{1}{\lambda}\ln(\alpha_{23})\right\}.$$

The median for the BBBGE distribution is obtained as

$$-\frac{1}{\lambda}\left\{1 - \left(\frac{1}{2}\right)^{1/\alpha_{123}}\right\}.$$

Proposition 2.4. the joint moment generating function (mgf) for the BBBGE distribution is given by

$$\begin{aligned} M(t_1,t_2) &= \frac{\alpha_2\alpha_{123}}{\alpha_{12}} B(1-t_2,\alpha_{123})\ {}_3F_2[1-t_2,\alpha_{13},t_1;\alpha_{13}+1,\alpha_{123}+1-t_2;1] \\ &+ \frac{\alpha_1\alpha_{123}}{\alpha_{12}} B(1-t_1,\alpha_{123})\ {}_3F_2[1-t_1,\alpha_{23},t_2;\alpha_{23}+1,\alpha_{123}+1-t_1;1] \end{aligned} \tag{2.3}$$

Where

$$\mathrm{B}(\alpha,\beta)=\int_0^1 \mathrm{u}^{\alpha-1}(1-\mathrm{u})^{\beta-1}\,\mathrm{du}\,,$$

$${}_p\mathrm{F}_q(\mathrm{b}_1,\ldots,\mathrm{b}_\mathrm{P};\mathrm{c}_1,\ldots,\mathrm{c}_\mathrm{q};\mathrm{u})=\sum_{\mathrm{i}=0}^{\infty}\frac{(\mathrm{b}_1)_\mathrm{i}\ldots(\mathrm{b}_\mathrm{P})_\mathrm{i}}{(\mathrm{c}_1)_\mathrm{i}\ldots(\mathrm{c}_\mathrm{q})_\mathrm{i}}\frac{\mathrm{u}^\mathrm{i}}{\mathrm{i}!}\,,$$

$$(\mathrm{b})_\mathrm{i}=\mathrm{b}(\mathrm{b}+1)\ldots(\mathrm{b}+\mathrm{i}-1)=\frac{\Gamma(\mathrm{b}+\mathrm{i})}{\Gamma(\mathrm{b})}(\mathrm{b}\neq 0,\mathrm{i}=1,2,\ldots),$$

and p,q are nonnegative integers.

Proof: The mgf of BBGE is defined as

$$M(t_1,t_2)=\int_0^\infty\int_0^\infty e^{t_1x_1+t_2x_2}\, f_{BBBGE}(x_1,x_2)\, dx_1dx_2$$

By Substitute $f_{BBBGE}(x_1,x_2)$ from (2.2) gets

$$\begin{aligned} M(t_1,t_2) &= c\lambda^2\alpha_2\alpha_{13}\int_0^\infty (1-e^{-\lambda x_2})^{\alpha_2-1}e^{-x_2(\lambda-t_2)}\int_0^{x_2}(1-e^{-\lambda x_1})^{\alpha_1+\alpha_3-1}e^{-x_1(\lambda-t_1)}dx_1dx_2 \\ &+ c\lambda^2\alpha_1\alpha_{23}\int_0^\infty (1-e^{-\lambda x_1})^{\alpha_1-1}e^{-x_1(\lambda-t_1)}\int_0^{x_1}(1-e^{-\lambda x_2})^{\alpha_2+\alpha_3-1}e^{-x_2(\lambda-t_2)}dx_2dx_1 \end{aligned} \tag{2.4}$$

then by using the following relation

$$\mathrm{B}_x(\alpha,\beta)=\int_0^x u^{\alpha-1}\left(1-u\right)^{\beta-1}du=\frac{x^\alpha}{\alpha}\ {}_2F_1(\alpha,1-\beta;\alpha+1;x),\qquad (0\le x\le 1)$$

where $\mathrm{B}_\mathrm{x}(\alpha,\beta)$ is an incomplete beta function

and the identity

$$\int_0^1 u^{\alpha-1}\left(1-u\right)^{\beta-1}\ {}_2F_1(c,d;\rho;u)du=\mathrm{B}(\alpha,\beta)\ {}_3F_2(\alpha,c,d;\rho,\alpha+\beta;1)$$

$$for\quad \alpha,\beta>0\quad and\quad d+\beta-\alpha-c>0,$$

For $\lambda = 1$ the expression for $M(t_1, t_2)$ that given in (2.3) is obtained.

To check whether $M(t_1, t_2)$ **mgf or not:** set $t_1 = t_2 = 0$ in (2.3)

hence

$$ {}_3F_2[1, \alpha_i + \alpha_3, 0; \alpha_i + \alpha_3 + 1, \alpha_1 + \alpha_2 + \alpha_3 + 1; 1] = 1 \quad , \quad i = 1,2 $$

And then

$$ B(1, \alpha_{123}) = \alpha_{123}, \quad M(0,0) = 1.. $$

3 The Distribution of Sum, Product and Ratio for BBBGE Random Variables

This Section is devoted to derive the exact forms of the distribution of sum; ratio and also product of dependent variables follow the BBBGE distribution.

Proposition 3.1. If X_1 and X_2 are jointly distributed according to (2.2), then the probability density function of $S = X_1 + X_2$ is given by

$$ \begin{aligned} f_S(s) = \lambda \frac{\alpha_{13}\alpha_2\alpha_{123}}{\alpha_{12}} \sum_{i,j=0}^{\infty} \frac{K_{ij}}{i-j} [e^{-\lambda s(i+1)} - e^{-\lambda s(\frac{i+j}{2}+1)}] \\ + \lambda \frac{\alpha_{13}\alpha_2\alpha_{123}}{\alpha_{12}} \sum_{i,j=0}^{\infty} \frac{\widehat{K}_{ij}}{i-j} [e^{-\lambda s(\frac{i+j}{2}+1)} - e^{-\lambda s(i+1)}]; \quad i < j. \end{aligned} \tag{3.1} $$

Where

$$ K_{ij} = (-1)^{i+j} \binom{\alpha_{13} - 1}{i} \binom{\alpha_2 - 1}{j} \text{and} \qquad \widehat{K}_{ij} = (-1)^{i+j} \binom{\alpha_1 - 1}{i} \binom{\alpha_{23} - 1}{j}. $$

Proof: based on the following transformation

$$ S = X_1 + X_2 \text{ and } R = \frac{X_1}{X_1 + X_2}. $$

So, $\quad X_1 = RS$ and $X_2 = S(1 - R). \qquad (3.2)$

Now, there exist the following two possibilities $X_1 > X_2$ and $X_1 < X_2$.

i. if $X_1 > X_2$ then $r > \frac{1}{2}, s > 0$ and the Jacobian is $-s$.

ii. if $X_1 < X_2$ then $r < \frac{1}{2}, s > 0$ and the Jacobian is $-s$

Using (2.2), (3.2) and the Jacobian values, the joint density function of S and R is given by

$$g_{S,R}(s,r) = \begin{cases} g_1(s,r), & 0 < r < \frac{1}{2} \\ g_2(s,r), & \frac{1}{2} < r < 1, \end{cases} \tag{3.3}$$

where

$$g_1(s,r) = \lambda^2 \frac{\alpha_{13}\alpha_2\alpha_{123}}{\alpha_{12}} s\, e^{-\lambda s}[1-e^{-\lambda sr}]^{\alpha_{13}-1}[1-e^{-\lambda s(1-r)}]^{\alpha_2-1},$$

$$g_2(s,r) = \lambda^2 \frac{\alpha_{23}\alpha_1\alpha_{123}}{\alpha_{12}} s\, e^{-\lambda s}[1-e^{-\lambda sr}]^{\alpha_1-1}[1-e^{-\lambda s(1-r)}]^{\alpha_{23}-1}$$

Now to derive the pdf of S

$$f_S(s) = \int_0^1 g_{S,R}(s,r)dr. \tag{3.4}$$

Substituting from (3.3) into (3.4), gets

$$f_S(s) = \int_0^{\frac{1}{2}} g_1(s,r)dr + \int_{\frac{1}{2}}^1 g_2(s,r)dr.$$

After solving the above integral the exact pdf of $S = X_1 + X_2$ as given in (3.1).

Proposition 3.2. If X_1 and X_2 are jointly distributed according to (2.2), then the probability density function of $R = \frac{X_1}{X_1 + X_2}$ is given by

$$f_R(r)=\begin{cases} f_1(r), & 0<r<\frac{1}{2} \\ f_2(r), & \frac{1}{2}<r<1, \end{cases} \tag{3.5}$$

where

$$f_1(r)=\frac{\alpha_{13}\alpha_2\alpha_{123}}{\alpha_{12}}\sum_{i,j=0}^{\infty}K_{ij}[r(i-j)+j+1]^{-2},$$

$$f_2(r)=\frac{\alpha_1\alpha_{23}\alpha_{123}}{\alpha_{12}}\sum_{i,j=0}^{\infty}\widehat{K}_{ij}[r(i-j)+j+1]^{-2}.$$

Proof: To derive the pdf there are two possibilities

First, when $r<\frac{1}{2}$ in (3.3), then

$$f_1(r)=\int_0^{\infty}g_1(s,r)ds$$

by using the binomial series expansion

$$(1-e^{-x})^{\alpha-1}=\sum_{i=1}^{\infty}(-1)^i\binom{\alpha-1}{i}e^{-ix}$$

and solving the above integral, gets

$$f_1(r)=\frac{\alpha_{13}\alpha_2\alpha_{123}}{\alpha_{12}}\sum_{i,j=0}^{\infty}K_{ij}[r(i-j)+j+1]^{-2},$$

Next, when $r>\frac{1}{2}$ in (3.3), then

$$f_2(r)=\int_0^{\infty}g_2(s,r)ds$$

$$=\frac{\alpha_1\alpha_{23}\alpha_{123}}{\alpha_{12}}\sum_{i,j=0}^{\infty}\widehat{K}_{ij}[r(i-j)+j+1]^{-2}$$

By summarizing the two parts of $f_R(r)$, the exact form of probability density function of the ratio R as given in (3.5) is obtained.

Proposition 3.3. If X_1 and X_2 are jointly distributed according to (2.2), then the probability density function of $P=X_1X_2$ is given by

$$f_P(p)=\lambda^2\frac{\alpha_{13}\alpha_2\alpha_{123}}{\alpha_{12}}\sum_{i,j,k=0}^{\infty}A_{ijk}\,p^k\,\Gamma^*(-k,\lambda(j+1)\sqrt{p})$$
$$+\lambda^2\frac{\alpha_1\alpha_{23}\alpha_{123}}{\alpha_{12}}\sum_{i,j,k=0}^{\infty}\widehat{A}_{ijk}\,p^k\,\Gamma(-k,\lambda(j+1)\sqrt{p}). \qquad (3.6)$$

where

$$A_{ijk}=\frac{(-1)^{i+j+k}}{k!}\binom{\alpha_{13}-1}{i}\binom{\alpha_2-1}{j}[\lambda^2(i+1)(j+1)]^k,$$

$$\widehat{A}_{ijk}=\frac{(-1)^{i+j+k}}{k!}\binom{\alpha_1-1}{i}\binom{\alpha_{23}-1}{j}[\lambda^2(i+1)(j+1)]^k,$$

$$\Gamma^*(c,z)=\int_0^z u^{c-1}\,e^{-u}du \quad ;c>0$$

and

$$\Gamma\ (c,z)=\int_z^\infty u^{c-1}\,e^{-u}du \quad ;c>0\,.$$

Proof: based on the the following transformation

$$P=X_1X_2 \ \text{ and } \ X=X_1. \qquad (3.7)$$

So, $X_1=X$ and $X_2=\dfrac{P}{X}$.

Now there exist the following two possibilities $X_1>X_2$ and $X_1<X_2$.

i. if $X_1>X_2$ then $x>\sqrt{p}$ and the Jacobian is $\dfrac{1}{x}$

ii. $X_1<X_2$ then $x<\sqrt{p}$ and the Jacobian is $\dfrac{1}{x}$

Using (2.2), (3.7) and the Jacobian values, the joint density function of X and P is given by

$$h_{X,P}(x,p)=\begin{cases}h_1(x,p), & 0<x<\sqrt{p}\\ h_2(x,p), & x>\sqrt{p},\end{cases} \qquad (3.8)$$

Where

$$h_1(x,p)=\lambda^2\frac{\alpha_{13}\alpha_2\alpha_{123}}{\alpha_{12}}\frac{1}{x}e^{-\lambda(x+\frac{p}{x})}[1-e^{-\lambda x}]^{\alpha_{13}-1}[1-e^{-\lambda\frac{p}{x}}]^{\alpha_2-1},$$

$$h_2(x,p)=\lambda^2\frac{\alpha_{23}\alpha_1\alpha_{123}}{\alpha_{12}}\frac{1}{x}e^{-\lambda(x+\frac{p}{x})}[1-e^{-\lambda x}]^{\alpha_1-1}[1-e^{-\lambda\frac{p}{x}}]^{\alpha_{23}-1}.$$

Now to derive the pdf of P

$$f_P(p)=\int_0^\infty h_{X,P}(x,p)dx \tag{3.9}$$

Substituting from (3.8) into (3.9), becomes

$$\begin{aligned} f_P(p) &= \int_0^{\sqrt{p}} h_1(x,p)dx+\int_{\sqrt{p}}^\infty h_2(x,p)dx. \\ &= \lambda^2\frac{\alpha_{13}\alpha_2\alpha_{123}}{\alpha_{12}}\sum_{i,j=0}^\infty(-1)^{i+j}\binom{\alpha_{13}-1}{i}\binom{\alpha_2-1}{j}\int_0^{\sqrt{p}}\frac{1}{x}e^{-\lambda[x(i+1)+\frac{p}{x}(j+1)]}dx \\ &+\lambda^2\frac{\alpha_{23}\alpha_1\alpha_{123}}{\alpha_{12}}\sum_{i,j=0}^\infty(-1)^{i+j}\binom{\alpha_1-1}{i}\binom{\alpha_{23}-1}{j}\int_{\sqrt{p}}^\infty\frac{1}{x}e^{-\lambda[x(i+1)+\frac{p}{x}(j+1)]}dx \end{aligned} \tag{3.10}$$

Set $y=\frac{1}{x}$ and use the series expansion

$$e^{-x}=\sum_{k=0}^\infty\frac{(-x)^k}{k!}$$

Equation (3.10) becomes

$$\begin{aligned} f_P(p) &= \lambda^2\frac{\alpha_{13}\alpha_2\alpha_{123}}{\alpha_{12}}\sum_{i,j,k=0}^\infty A_{ijk}\int_{\frac{1}{\sqrt{p}}}^\infty y^{-(k+1)}\ e^{-\lambda p(j+1)y}dy \\ &+\lambda^2\frac{\alpha_{23}\alpha_1\alpha_{123}}{\alpha_{12}}\sum_{i,j,k=0}^\infty \widehat{A}_{ijk}\int_0^{\frac{1}{\sqrt{p}}} y^{-(k+1)}\ e^{-\lambda p(j+1)y}dy \end{aligned} \tag{3.11}$$

By the definition of the complementary incomplete gamma function,

$$\int_{\frac{1}{\sqrt{p}}}^\infty y^{-(k+1)}\ e^{-\lambda p(j+1)y}dy=\Gamma^*(-k,\lambda(j+1)\sqrt{p}), \tag{3.12}$$

$$\int_{0}^{\frac{1}{\sqrt{p}}} y^{-(k+1)}\ e^{-\lambda p(j+1)y} dy = \Gamma(-k, \lambda(j+1)\sqrt{p}). \tag{3.13}$$

The result of the theorem follows by substituting (3.12) and (3.13) into (3.11).

4 Maximum likelihood Estimation

In this Section, the maximum likelihood estimators (MLEs) of the unknown parameters of the BBBGE distribution are obtained. Suppose $\{(x_{11}, x_{21}), \ldots, (x_{1n}, x_{2n})\}$ is a random sample from $BBBGE(\alpha_1, \alpha_2, \alpha_3, \lambda)$ distribution. Consider the following notation

$$I_1 = \{i; x_{1i} < x_{2i}\}, \quad I_2 = \{i; x_{1i} > x_{2i}\}, \qquad I = I_1 \cup I_2,$$

$$|I_1| = n_1, \quad |I_2| = n_2 \quad \text{and} \quad n_1 + n_2 = n.$$

The log-likelihood function of the sample of size n is given by

$$\ln L(\Theta) = \sum_{i \in I_1} \ln f_1(x_{1i}, x_{2i}) + \sum_{i \in I_2} \ln f_2(x_{1i}, x_{2i}) \tag{4.1}$$

$$\begin{aligned} \ln L(\Theta) = {} & 2(n_1 + n_2)\ln\lambda + n_1 \ln(\alpha_2) + n_2 \ln(\alpha_1) + n_1 \ln(\alpha_1 + \alpha_3) \\ & + n_2 \ln(\alpha_2 + \alpha_3) - \lambda[\sum_{i=1}^{n_1}(x_{1i} + x_{2i}) + \sum_{i=1}^{n_2}(x_{1i} + x_{2i})] \\ & + (\alpha_1 + \alpha_3 - 1)\sum_{i=1}^{n_1} \ln(1 - e^{-\lambda x_{1i}}) + (\alpha_2 - 1)\sum_{i=1}^{n_1} \ln(1 - e^{-\lambda x_{2i}}) \\ & + (\alpha_1 - 1)\sum_{i=1}^{n_2} \ln(1 - e^{-\lambda x_{1i}}) + (\alpha_2 + \alpha_3 - 1)\sum_{i=1}^{n_2} \ln(1 - e^{-\lambda x_{2i}}). \end{aligned} \tag{4.2}$$

where $\Theta = (\alpha_1, \alpha_2, \alpha_3, \lambda)$.

On differentiating (4.2) with respect to $\alpha_1, \alpha_2, \alpha_3$ and λ and equating to zero, obtain the following likelihood equations are obtained.

$$\frac{n_2}{\hat{\alpha}_1} + \frac{n_1}{\hat{\alpha}_1 + \hat{\alpha}_3} + \sum_{i=1}^{n_1} \ln\left(1 - e^{-\hat{\lambda} x_{1i}}\right) + \sum_{i=1}^{n_2} \ln(1 - e^{-\hat{\lambda} x_{1i}}) = 0,$$

$$\frac{n_1}{\hat{\alpha}_2} + \frac{n_2}{\hat{\alpha}_2 + \hat{\alpha}_3} + \sum_{i=1}^{n_1} \ln\left(1 - e^{-\hat{\lambda} x_{2i}}\right) + \sum_{i=1}^{n_2} \ln\left(1 - e^{-\hat{\lambda} x_{2i}}\right) = 0,$$

$$\frac{n_1}{\hat{\alpha}_1+\hat{\alpha}_3}+\frac{n_2}{\hat{\alpha}_2+\hat{\alpha}_3}+\sum_{i=1}^{n_1}\ln\left(1-e^{-\hat{\lambda}x_{1i}}\right)+\sum_{i=1}^{n_2}\ln\left(1-e^{-\hat{\lambda}x_{2i}}\right)=0,$$

and

$$(\hat{\alpha}_1-1)\sum_{i=1}^{n_2}\frac{x_{1i}\,e^{-\hat{\lambda}x_{1i}}}{1-e^{-\hat{\lambda}x_{1i}}}+(\hat{\alpha}_1+\hat{\alpha}_3-1)\sum_{i=1}^{n_1}\frac{x_{1i}\,e^{-\hat{\lambda}x_{1i}}}{1-e^{-\hat{\lambda}x_{1i}}}+\frac{2(n_1+n_2)}{\hat{\lambda}}-\sum_{i=1}^{n_2}(x_{1i}+x_{2i})$$
$$-\sum_{i=1}^{n_1}(x_{1i}+x_{2i})+(\hat{\alpha}_2-1)\sum_{i=1}^{n_1}\frac{x_{2i}\,e^{-\hat{\lambda}x_{2i}}}{1-e^{-\hat{\lambda}x_{2i}}}+(\hat{\alpha}_2+\hat{\alpha}_3-1)\sum_{i=1}^{n_2}\frac{x_{2i}\,e^{-\hat{\lambda}x_{2i}}}{1-e^{-\hat{\lambda}x_{2i}}}=0. \tag{4.3}$$

The system (4.3) of nonlinear equation will be solved numerically to obtain $\hat{\alpha}_1$, $\hat{\alpha}_2$, $\hat{\alpha}$ and $\hat{\lambda}$.

Based on Cohen(1965), the approximate variance-covariance matrix is given as

$$I^{-1}=\begin{bmatrix} a_{11} & a_{12} & a_{13} & a_{14} \\ a_{21} & a_{22} & a_{23} & a_{24} \\ a_{31} & a_{32} & a_{33} & a_{34} \\ a_{41} & a_{42} & a_{43} & a_{44} \end{bmatrix}^{-1}$$

Where

$$a_{11}=-\frac{\partial^2 \ln L}{\partial \alpha_1^2}\bigg|_{\hat{\alpha}_1,\hat{\alpha}_2,\hat{\alpha}_3,\hat{\lambda}}=\frac{n_1}{(\hat{\alpha}_1+\hat{\alpha}_3)^2}+\frac{n_2}{\hat{\alpha}_1^2},$$

$$a_{22}=-\frac{\partial^2 \ln L}{\partial \alpha_2^2}\bigg|_{\hat{\alpha}_1,\hat{\alpha}_2,\hat{\alpha}_3,\hat{\lambda}}=\frac{n_2}{(\hat{\alpha}_2+\hat{\alpha}_3)^2}+\frac{n_1}{\hat{\alpha}_2^2},$$

$$a_{33}=-\frac{\partial^2 \ln L}{\partial \alpha_3^2}\bigg|_{\hat{\alpha}_1,\hat{\alpha}_2,\hat{\alpha}_3,\hat{\lambda}}=\frac{n_1}{(\hat{\alpha}_1+\hat{\alpha}_3)^2}+\frac{n_2}{(\hat{\alpha}_2+\hat{\alpha}_3)^2},$$

$$a_{12}=-\frac{\partial^2 \ln L}{\partial \alpha_1\partial \alpha_2}\bigg|_{\hat{\alpha}_1,\hat{\alpha}_2,\hat{\alpha}_3,\hat{\lambda}}=0,$$

$$a_{13}=-\frac{\partial^2 \ln L}{\partial \alpha_1\partial \alpha_3}\bigg|_{\hat{\alpha}_1,\hat{\alpha}_2,\hat{\alpha}_3,\hat{\lambda}}=\frac{n_1}{(\hat{\alpha}_1+\hat{\alpha}_3)^2},$$

$$a_{23}=-\frac{\partial^2 \ln L}{\partial \alpha_2\partial \alpha_3}\bigg|_{\hat{\alpha}_1,\hat{\alpha}_2,\hat{\alpha}_3,\hat{\lambda}}=\frac{n_2}{(\hat{\alpha}_2+\hat{\alpha}_3)^2},$$

$$a_{14} = -\left.\frac{\partial^2 \ln L}{\partial\alpha_1 \partial\lambda}\right|_{\hat{\alpha}_1,\hat{\alpha}_2,\hat{\alpha}_3,\hat{\lambda}} = \sum_{i=1}^{n_1} \frac{x_{1i}\, e^{-\lambda x_{1i}}}{1-e^{-\lambda x_{1i}}} + \sum_{i=1}^{n_2} \frac{x_{1i}\, e^{-\lambda x_{1i}}}{1-e^{-\lambda x_{1i}}},$$

$$a_{24} = -\left.\frac{\partial^2 \ln L}{\partial\alpha_2 \partial\lambda}\right|_{\hat{\alpha}_1,\hat{\alpha}_2,\hat{\alpha}_3,\hat{\lambda}} = \sum_{i=1}^{n_2} \frac{x_{2i}\, e^{-\lambda x_{2i}}}{1-e^{-\lambda x_{2i}}} + \sum_{i=1}^{n_2} \frac{x_{2i}\, e^{-\lambda x_{2i}}}{1-e^{-\lambda x_{2i}}},$$

$$a_{34} = -\left.\frac{\partial^2 \ln L}{\partial\alpha_3 \partial\lambda}\right|_{\hat{\alpha}_1,\hat{\alpha}_2,\hat{\alpha}_3,\hat{\lambda}} = \sum_{i=1}^{n_1} \frac{x_{1i}\, e^{-\lambda x_{1i}}}{1-e^{-\lambda x_{1i}}} + \sum_{i=1}^{n_2} \frac{x_{2i}\, e^{-\lambda x_{2i}}}{1-e^{-\lambda x_{2i}}},$$

$$a_{44} = -\left.\frac{\partial^2 \ln L}{\partial\lambda^2}\right|_{\hat{\alpha}_1,\hat{\alpha}_2,\hat{\alpha}_3,\hat{\lambda}} = (\hat{\alpha}_1+\hat{\alpha}_3-1)\sum_{i=1}^{n_1} \frac{x_{1i}^2\, e^{-\lambda x_{1i}}}{(1-e^{-\lambda x_{1i}})^2} + (\hat{\alpha}_1-1)\sum_{i=1}^{n_2} \frac{x_{1i}^2\, e^{-\lambda x_{1i}}}{(1-e^{-\lambda x_{1i}})^2}$$
$$+(\hat{\alpha}_2+\hat{\alpha}_3-1)\sum_{i=1}^{n_2} \frac{x_{2i}^2\, e^{-\lambda x_{2i}}}{(1-e^{-\lambda x_{2i}})^2} + (\hat{\alpha}_2-1)\sum_{i=1}^{n_1} \frac{x_{2i}^2\, e^{-\lambda x_{2i}}}{(1-e^{-\lambda x_{2i}})^2} + \frac{2(n_1+n_2)}{\lambda^2}.$$

Now, to obtain the asymptotic confidence intervals of $\lambda, \alpha_1, \alpha_2$ and α_3. The asymptotic normality results can be stated as follows

$$\sqrt{n}\,[(\hat{\lambda}-\lambda),(\hat{\alpha}_1-\alpha_1),(\hat{\alpha}_2-\alpha_2),(\hat{\alpha}_2-\alpha_2)] \to N_4(0\,,\, I(\Theta)^{-1}) \quad \text{as} \quad n\to\infty \qquad (4.4)$$

where $I^{-1}(\Theta)$ is the variance-covariance matrix, $\hat{\Theta} = (\hat{\alpha}_1, \hat{\alpha}_2, \hat{\alpha}_3, \hat{\lambda})$ and $\Theta = (\alpha_1, \alpha_2, \alpha_3, \lambda)$ .Since Θ is unknown in (4.4), $I^{-1}(\Theta)$ is estimated by $I^{-1}(\hat{\Theta})$.

5 Simulation Results

In this Section, a simulation experiment is presented in which the estimation of the parameters of the BBBGE distribution are evaluated . The simulations were performed using the Mathcad program, the number of the replications $R = 1000$.

The evaluation of the point estimation was performed based on the following quantities for each sample size: the Average Estimates (AE), the Mean Squared Error, (MSE) are estimated from R replications and the coverage rate of the 95% confidence interval for $\lambda, \alpha_1, \alpha_2$ and α_3, the sample size is chosen at $n = 20,40,60$ and 100, and considered some values for the parameters $\lambda, \alpha_1, \alpha_2$ and α_3.

It can be seen from Table 1 that the estimates are slightly positively biased and that the MSE decreases as the sample size increases, as expected. The estimates are close to the true values. Also the coverage probabilities are close to the nominal level. These results indicate that the proposed model and the asymptotic approximation work well under the situation where no censoring occurs.

Table 1: The average estimates (AE), the mean squared errors (MSE), and the coverage percentages (CI) of $\alpha_1, \alpha_2, \alpha_3$ and λ for BBBGE distribution

n	parameters	AE	MSE	95% CI Coverage
20	α_1	0.725	0.055	0.94
	α_2	0.828	0.010	0.95
	α_3	0.522	0.123	0.91
	λ	0.0559	0.018	0.95
40	α_1	1.541	0.001	0.95
	α_2	3.024	0.009	0.91
	α_3	2.125	0.0815	0.92
	λ	0.0530	0.001	0.97
60	α_1	1.113	0.001	0.97
	α_2	2.471	0.004	0.95
	α_3	3.441	0.031	0.93
	λ	0.0525	0.002	0.98
100	α_1	1.501	0.0007	0.99
	α_2	2.657	0.0005	0.97
	α_3	2.701	0.002	0.96
	λ	0.0528	0.00003	0.98

6. Conclusion

In this paper the absolutely continuous bivariate model following the approach of Block and Basu (1974) has been introduced. That obtained from the Marshal – Olkin bivariate generalized exponential model by removing the singular part. The BBBGE model has an absolutely continuous probability density function. The moment generating function for the BBBGE distribution has been obtained. The exact forms for the distribution of sum; ratio and product of dependent variables follows the BBBGE distribution have been derived. The maximum likelihood estimates for the four unknown parameters and their approximate variance- covariance matrix have been obtained. Finally some a numerical illustration has been performed to see the performances of the MLEs.

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