Journal of Fractional Calculus and Applications
Vol. 14(2) July 2023, No. 2
ISSN: 2090-5858.
http://jfca.journals.ekb.eg/

SANDWICH RESULTS FOR MULTIVALENT FUNCTIONS DEFINED BY GENERALIZED SRIVASTAVA-ATTIYA OPERATOR

ADELA O. MOSTAFA, TEODOR BULBOACĂ AND MOHAMED K. AOUF

Abstract. The paper contains new results in the field of *Geometric Function Theory* of one variable functions, specially connected with the concepts of *differential subordinations* and *superordinations*, and that could be used for further investigation in this area.

We defined a new subclasses of analytic multivalent functions in the open unit disk $\mathbb{D}$ with the aid of the generalized well-known Srivastava-Attiya operator obtained by a convolution product with the general *Hurwitz-Lerch Zeta function*.

For the functions belonging to these subclasses we obtain sharp subordination and superordination results, that generalizes some previous well-known subordination properties obtained by different authors. The main results are followed by some particular cases obtained for special choices of the parameters, some of them being connected with the *Janowski type functions*. The technique used in the proofs is based on the general theory of differential subordinations and superordination initiated and developed by S.S. Miller and P. T. Mocanu.

We emphasize that these results are sharp in the sense that there are the best possible under the given assumptions of our theorems and corollaries, that is the dominants cannot be improved. These new results generalizes some previous well-known subordination properties obtained by different authors.

1. Introduction

Let $A(p)$ denote the class of functions of the form

$$f(z) = z^p + \sum_{k=1}^{\infty} a_k z^{k+p}, \tag{1}$$

2020 *Mathematics Subject Classification.* 30C80, 30C45.

Key words and phrases. Subordination, superordination, sandwich results, multivalent functions, convolution (Hadamard) product, Srivastava-Attiya operator, Briot-Bouquet differential subordination.

Submitted Dec. 21, 2022. Revised Mar. 4, 2023.

which are analytic and multivalent in the unit disc $\mathbb{D} := \{z \in \mathbb{C} : |z| < 1\}$, and let $\mathcal{A} := A(1)$.

If f and g are analytic functions in $\mathbb{D}$, we say that f is *subordinate* to g, written $f(z) \prec g(z)$, if there exists a *Schwarz function* w, which (by definition) is analytic in $\mathbb{D}$ with $w(0) = 0$, and $|w(z)| < 1$ for all $z \in \mathbb{D}$, such that $f(z) = g(w(z))$, for all $z \in \mathbb{D}$. Furthermore, if the function g is univalent in $\mathbb{D}$, then we have the following equivalence (see [7]):

$$f(z) \prec g(z) \Leftrightarrow f(0) = g(0) \text{ and } f(\mathbb{D}) \subset g(\mathbb{D}).$$

Let $H(\mathbb{D})$ denotes the class of analytic functions in the open unit disc $\mathbb{D}$, and let $H[a,p]$ denotes the subclass of the functions $f \in H(\mathbb{D})$ of the form

$$f(z) = a + a_p z^p + a_{p+1} z^{p+1} + \dots \quad (a \in \mathbb{C}).$$

Suppose that h and g are two analytic functions in $\mathbb{D}$, and let the function

$$\varphi(r,s,t;z) : \mathbb{C}^3 \times \mathbb{D} \to \mathbb{C}.$$

If h and $\varphi\left(h(z), zh'(z), z^2h''(z); z\right)$ are univalent functions in $\mathbb{D}$, and if h satisfies the second-order superordination

$$g(z) \prec \varphi\left(h(z), zh'(z), z^2h''(z); z\right), \tag{2}$$

then g is called to be a *solution of the differential superordination* (2). A function $q \in H(\mathbb{D})$ is called a *subordinant* of (2), if $q(z) \prec h(z)$ for all the functions h satisfying (2). A univalent subordinant $\widetilde{q}$ that satisfies $q(z) \prec \widetilde{q}(z)$ for all of the subordinants q of (2), is said to be the *best subordinant*.

In [8] Miller and Mocanu obtained sufficient conditions on the functions g, q and φ for which the following implication holds:

$$g(z) \prec \varphi\left(h(z), zh'(z), z^2h''(z); z\right) \Rightarrow g(z) \prec h(z).$$

Recently, Shanmugam et al. ([11], [12] and [13]) obtained the such called *sandwich results* for certain classes of analytic functions. Further subordination results can be found in [16].

For functions f given by (1) and $g \in A(p)$ given by $g(z) = z^p + \sum\limits_{k=1}^{\infty} b_k z^{k+p}$, the *Hadamard product* of f and g is defined by

$$(f*g)(z) := z^p + \sum_{k=1}^{\infty} a_k b_k z^{k+p}.$$

We begin our investigation by recalling that the general *Hurwitz-Lerch Zeta function* $\Phi(z;s,a)$ is defined by (see [15])

$$\Phi(z;s,a) := \sum_{k=0}^{\infty} \frac{z^k}{(k+a)^s},$$

where $a \in \mathbb{C} \setminus \mathbb{Z}_0^-$, $\mathbb{Z}_0^- := \{0,-1,-2,\dots\}$, with $s \in \mathbb{C}$ when $|z| < 1$, and $\operatorname{Re} s > 1$ when $|z| = 1$.

Liu [6] defined the operator $\mathrm{J}_{s,b} : A(p) \to A(p)$ by

$$\mathrm{J}_{s,b}(f)(z) = G_{p,s,b}(z) * f(z), \quad \left(b \in \mathbb{C} \setminus \mathbb{Z}_0^-,\ s \in \mathbb{C},\ p \in \mathbb{N}\right), \tag{3}$$

where

$$G_{p,s,b}(z) := (1+b)^s \left[\Phi_p(z;s,b) - b^{-s}\right]$$

and

$$\Phi_p(z;s,b) := \frac{1}{b^s} + \sum_{k=0}^{\infty} \frac{z^{k+p}}{(k+1+b)^s}. \tag{4}$$

It is easy to observe from (3) and (4) that

$$\mathrm{J}_{s,b}(f)(z) = z^p + \sum_{k=1}^{\infty} \left(\frac{1+b}{k+1+b} \right)^s a_k z^{k+p}. \tag{5}$$

For $p = 1$ the operator $\mathrm{J}_{s,b}$ reduces to *Srivastava-Attiya operator* $\mathrm{L}_{s,b}$ [14], and this last $\mathrm{L}_{s,b}$ operator contains, among its special cases, the well-known integral operators of Alexander [1], Libera [5] and Jung et al. [4].

It follows easily from (5) that

$$z \left[\mathrm{J}_{s+1,b}(f)(z) \right]' = (b+1)\, \mathrm{J}_{s,b}(f)(z) - (b+1-p)\, \mathrm{J}_{s+1,b}(f)(z). \tag{6}$$

2. Definitions and Preliminaries

To prove our results we shall need the following definition and lemmas.

The first lemma deals with the generalized *Briot-Bouquet* differential subordinations:

Lemma 2.1. [7] *Let q be univalent in the unit disc $\mathbb{D}$, and let θ and φ be analytic in a domain D containing $q(\mathbb{D})$, with $\varphi(w) \neq 0$ when $w \in q(\mathbb{D})$. Set $Q(z) = zq'(z)\varphi(q(z))$, $h(z) = \theta(q(z)) + Q(z)$ and suppose that*

$$(i) \qquad Q \text{ is a starlike function in } \mathbb{D},$$

$$(ii) \qquad \mathrm{Re}\, \frac{zh'(z)}{Q(z)} > 0, \; z \in \mathbb{D}.$$

If p is analytic in $\mathbb{D}$ with $p(0) = q(0)$, $p(\mathbb{D}) \subseteq D$ and

$$\theta(p(z)) + zp'(z)\varphi(p(z)) \prec \theta(q(z)) + zq'(z)\varphi(q(z)), \tag{7}$$

then $p(z) \prec q(z)$, and q is the best dominant of (7).

The next lemma represents a recent result about *Goluzin and Suffridge* type of differential subordinations:

Lemma 2.2. [11] *Let $\mu \in \mathbb{C}$, $\gamma \in \mathbb{C}^* = \mathbb{C} \setminus \{0\}$, and let q be a convex function in $\mathbb{D}$, with*

$$\mathrm{Re} \left(1 + \frac{zq''(z)}{q'(z)} \right) > \max \left\{ 0; -\mathrm{Re}\, \frac{\mu}{\gamma} \right\}, \; z \in \mathbb{D}.$$

If p is analytic in $\mathbb{D}$ and

$$\mu p(z) + \gamma z p'(z) \prec \mu q(z) + \gamma z q'(z), \tag{8}$$

then $p(z) \prec q(z)$, and q is the best dominant of (8).

Definition 2.1. [8] *Let $\mathcal{Q}$ be the set of all functions f that are analytic and injective on $\overline{\mathbb{D}} \setminus E(f)$, where*

$$E(f) = \left\{ \zeta \in \partial\mathbb{D} : \lim_{z \to \zeta} f(z) = \infty \right\},$$

and are such that $f'(\zeta) \neq 0$ for $\zeta \in \partial\mathbb{D} \setminus E(f)$.

Lemma 2.3. [3] *Let q be a univalent function in the unit disc $\mathbb{D}$ and let θ and φ be analytic in a domain D containing $q(\mathbb{D})$. Suppose that*

$$(i) \qquad \mathrm{Re}\, \frac{\theta'(q(z))}{\varphi(q(z))} > 0 \text{ for } z \in \mathbb{D},$$

$$(ii) \qquad h(z) = zq'(z)\varphi(q(z)) \text{ is starlike in } \mathbb{D}.$$

If $p \in H[q(0), 1] \cap \mathcal{Q}$ with $p(\mathbb{D}) \subseteq D$, $\theta(p(z)) + zp'(z)\varphi(p(z))$ is univalent in $\mathbb{D}$, and

$$\theta(q(z)) + zq'(z)\varphi(q(z)) \prec \theta(p(z)) + zp'(z)\varphi(p(z)), \tag{9}$$

then $q(z) \prec p(z)$, and q is the best subordinant of (9).

Lemma 2.4. [8] *Let q be convex in $\mathbb{D}$ and let $\gamma \in \mathbb{C}$, with $\operatorname{Re}\gamma > 0$. If $p \in H[q(0),1] \cap \mathcal{Q}$ and $p(z) + \gamma z p'(z)$ is univalent in $\mathbb{D}$, then*

$$q(z) + \gamma z q'(z) \prec p(z) + \gamma z p'(z), \tag{10}$$

implies $q(z) \prec p(z)$, and q is the best subordinant (10).

This last lemma gives us a necessary and sufficient condition for the univalence of a special function, necessary in some particular cases:

Lemma 2.5. [10] *The function $q(z) = (1-z)^{-2ab}$ is univalent in $\mathbb{D}$ if and only if $|2ab-1| \le 1$ or $|2ab+1| \le 1$.*

3. Subordination results for analytic functions

Unless otherwise mentioned, we assume throughout this paper that $b \in \mathbb{C} \setminus \mathbb{Z}_0^-$, $s \in \mathbb{C}$ and $p \in \mathbb{N}$.

Theorem 3.1. *Let q be univalent in $\mathbb{D}$, with $q(0) = 1$, and let $\lambda \in \mathbb{C}$. For $\lambda \in \mathbb{C}^*$ suppose, in addition, that*

$$\operatorname{Re}\left(1 + \frac{zq''(z)}{q'(z)}\right) > \max\left\{0; -p \operatorname{Re}\frac{b+1}{\lambda}\right\},\ z \in \mathbb{D}. \tag{11}$$

If $f \in A(p)$ satisfies the subordination

$$\frac{\lambda}{p}\left(\frac{\mathrm{J}_{s,b}(f)(z)}{z^p}\right) + \frac{p-\lambda}{p}\left(\frac{\mathrm{J}_{s+1,b}(f)(z)}{z^p}\right) \prec q(z) + \frac{\lambda z q'(z)}{p(b+1)}, \tag{12}$$

then

$$\frac{\mathrm{J}_{s+1,b}(f)(z)}{z^p} \prec q(z),$$

and q is the best dominant of (12).

Proof. If we let

$$g(z) := \frac{\mathrm{J}_{s+1,b}(f)(z)}{z^p},$$

then, by differentiating g and using the identity (6), we have

$$\frac{\mathrm{J}_{s,b}(f)(z)}{z^p} = g(z) + \frac{zg'(z)}{b+1}.$$

A simple computation shows that

$$\frac{\lambda}{p}\left(\frac{\mathrm{J}_{s,b}(f)(z)}{z^p}\right) + \frac{p-\lambda}{p}\left(\frac{\mathrm{J}_{s+1,b}(f)(z)}{z^p}\right) = g(z) + \frac{\lambda z g'(z)}{p(b+1)},$$

hence the subordination (12) is equivalent to

$$g(z) + \frac{\lambda z g'(z)}{p(b+1)} \prec q(z) + \frac{\lambda z q'(z)}{p(b+1)}.$$

Now, applying Lemma 2.2 with $\mu = 1$ and $\gamma = \dfrac{\lambda}{p(b+1)}$, the proof is completed. □

Taking $q(z) = \dfrac{1+Az}{1+Bz}$ in Theorem 3.1, where $-1 \le B < A \le 1$, the condition (11) becomes

$$\operatorname{Re}\frac{1-Bz}{1+Bz} > \max\left\{0; -p \operatorname{Re}\frac{b+1}{\lambda}\right\},\ z \in \mathbb{D}. \tag{13}$$

It is easy to check that the function $\varphi(\zeta)=\dfrac{1-\zeta}{1+\zeta}$, $|\zeta|<|B|\le 1$, is convex in $\mathbb{D}$, and since $\varphi(\overline{\zeta})=\overline{\varphi(\zeta)}$ for all $|\zeta|<|B|$, it follows that the image $\varphi(\mathbb{D})$ is a convex domain symmetric with respect to the real axis, hence

$$\inf\left\{\operatorname{Re}\frac{1-Bz}{1+Bz}:z\in\mathbb{D}\right\}=\frac{1-|B|}{1+|B|}\ge 0. \tag{14}$$

Then, the inequality (13) is equivalent to

$$p\operatorname{Re}\frac{b+1}{\lambda}\ge\frac{|B|-1}{|B|+1},$$

hence we obtain the following result:

Corollary 3.0. *Let $-1\le B<A\le 1$, $b\in\mathbb{C}\setminus\mathbb{Z}_0^-$, and let $\lambda\in\mathbb{C}$. For $\lambda\in\mathbb{C}^*$ suppose, in addition, that*

$$\frac{1-|B|}{1+|B|}\ge\max\left\{0;-p\operatorname{Re}\frac{b+1}{\lambda}\right\}.$$

If $f\in A(p)$, and

$$\frac{\lambda}{p}\left(\frac{\mathrm{J}_{s,b}(f)(z)}{z^p}\right)+\frac{p-\lambda}{p}\left(\frac{\mathrm{J}_{s+1,b}(f)(z)}{z^p}\right)\prec\frac{1+Az}{1+Bz}+\frac{\lambda}{p(b+1)}\frac{(A-B)z}{(1+Bz)^2}, \tag{15}$$

then

$$\frac{\mathrm{J}_{s+1,b}(f)(z)}{z^p}\prec\frac{1+Az}{1+Bz},$$

and $\dfrac{1+Az}{1+Bz}$ is the best dominant of (15).

Theorem 3.2. *Let q be univalent in $\mathbb{D}$, with $q(0)=1$ and $q(z)\neq 0$ for all $z\in\mathbb{D}$. Let $\gamma,\mu\in\mathbb{C}^*$ and $\nu,\eta\in\mathbb{C}$ with $\nu+\eta\neq 0$. Let $f\in A(p)$ and suppose that f and q satisfy the conditions:*

$$\frac{\nu\,\mathrm{J}_{s+1,b}(f)(z)+\eta\,\mathrm{J}_{s,b}(f)(z)}{(\nu+\eta)z^p}\neq 0,\ z\in\mathbb{D}, \tag{16}$$

and

$$\operatorname{Re}\left(1+\frac{zq''(z)}{q'(z)}-\frac{zq'(z)}{q(z)}\right)>0,\ z\in\mathbb{D}. \tag{17}$$

If

$$1+\gamma\mu\left[\frac{\nu z\left[\mathrm{J}_{s+1,b}(f)(z)\right]'+\eta z\left[\mathrm{J}_{s,b}(f)(z)\right]'}{\nu\,\mathrm{J}_{s+1,b}(f)(z)+\eta\,\mathrm{J}_{s,b}(f)(z)}-p\right]\prec 1+\gamma\frac{zq'(z)}{q(z)}, \tag{18}$$

then

$$\left[\frac{\nu\,\mathrm{J}_{s+1,b}(f)(z)+\eta\,\mathrm{J}_{s,b}(f)(z)}{(\nu+\eta)z^p}\right]^{\mu}\prec q(z),$$

and q is the best dominant of (18). *(The power is the principal one.)*

Proof. Letting

$$g(z):=\left[\frac{\nu\,\mathrm{J}_{s+1,b}(f)(z)+\eta\,\mathrm{J}_{s,b}(f)(z)}{(\nu+\eta)z^p}\right]^{\mu},$$

from (16) it follows that g is analytic in $\mathbb{D}$ and $g(0)=1$. Differentiating g logarithmically with respect to z, we get

$$\frac{zg'(z)}{g(z)}=\mu\left[\frac{\nu z[\mathrm{J}_{s+1,b}(f)(z)]'+\eta z[\mathrm{J}_{s,b}(f)(z)]'}{\nu\,\mathrm{J}_{s+1,b}(f)(z)+\eta\,\mathrm{J}_{s,b}(f)(z)}-p\right].$$

Now, using Lemma 2.1 with $\theta(w)=1$ and $\varphi(w)=\dfrac{\gamma}{w}$, then θ is analytic in $\mathbb{C}$ and $\varphi(w)\neq 0$ is analytic in $\mathbb{C}^*$. Also, if we let

$$Q(z)=zq'(z)\varphi(q(z))=\gamma\frac{zq'(z)}{q(z)},$$

and

$$h(z)=\theta(q(z))+Q(z)=1+\gamma\frac{zq'(z)}{q(z)},$$

then $Q(0)=0$ and $Q'(0)\neq 0$, and the assumption (17) yields that Q is a starlike function in $\mathbb{D}$. From (17) we have

$$\operatorname{Re}\frac{zh'(z)}{Q(z)}=\operatorname{Re}\left(1+\frac{zq''(z)}{q'(z)}-\frac{zq'(z)}{q(z)}\right)>0,\ z\in\mathbb{D},$$

and then, by using Lemma 2.1 we deduce that the assumption (18) implies $g(z)\prec q(z)$, and the function q is the best dominant of (18). □

Taking $\nu=0$, $\eta=\gamma=1$ and $q(z)=\dfrac{1+Az}{1+Bz}$ in Theorem 3.2, the assumption (17) holds whenever $-1\leq A<B\leq 1$, hence we obtain the next result:

Corollary 3.0. *Let $-1\leq A<B\leq 1$ and $\mu\in\mathbb{C}^*$. Let $f\in A(p)$ and suppose that*

$$\frac{\mathrm{J}_{s,b}(f)(z)}{z^p}\neq 0,\ z\in\mathbb{D}.$$

If

$$1+\mu\left[\frac{z\left[\mathrm{J}_{s,b}(f)(z)\right]'}{\mathrm{J}_{s,b}(f)(z)}-p\right]\prec 1+\frac{(A-B)z}{(1+Az)(1+Bz)},\tag{19}$$

then

$$\left[\frac{\mathrm{J}_{s,b}(f)(z)}{z^p}\right]^{\mu}\prec\frac{1+Az}{1+Bz},$$

and $\dfrac{1+Az}{1+Bz}$ is the best dominant of (19). (The power is the principal one.)

Putting $\nu=0$, $\eta=p=1$, $s=0$, $\gamma=\dfrac{1}{\alpha\beta}$ $(\alpha,\beta\in\mathbb{C}^*)$, $\mu=\alpha$, and $q(z)=(1-z)^{-2\alpha\beta}$ in Theorem 3.2, and combining this together with Lemma 2.5 we obtain the next result due to Obradović et al.:

Corollary 3.0. [9, Theorem 1] *Let $\alpha,\beta\in\mathbb{C}^*$, such that $|2\alpha\beta-1|\leq 1$ or $|2\alpha\beta+1|\leq 1$. Let $f\in\mathcal{A}$ and suppose that $\dfrac{f(z)}{z}\neq 0$ for all $z\in\mathbb{D}$. If*

$$1+\frac{1}{\beta}\left(\frac{zf'(z)}{f(z)}-1\right)\prec\frac{1+z}{1-z},\tag{20}$$

then

$$\left(\frac{f(z)}{z}\right)^{\alpha}\prec(1-z)^{-2\alpha\beta},$$

and $(1-z)^{-2\alpha\beta}$ is the best dominant of (20). (The power is the principal one.)

Putting $\nu=0$, $\eta=p=\gamma=1$, $s=0$ and $q(z)=(1+Bz)^{\frac{\mu(A-B)}{B}}$ $(-1\leq B<A\leq 1,\ B\neq 0)$ in Theorem 3.2 and using Lemma 2.5, we get the next corollary:

Corollary 3.0. *Let $-1\leq B<A\leq 1$, with $B\neq 0$, and suppose that $\left|\dfrac{\mu(A-B)}{B}-1\right|\leq 1$ or $\left|\dfrac{\mu(A-B)}{B}+1\right|\leq 1$. Let $f\in\mathcal{A}$ such that $\dfrac{f(z)}{z}\neq 0$ for all $z\in\mathbb{D}$, and let $\mu\in\mathbb{C}^*$. If*

$$1+\mu\left(\frac{zf'(z)}{f(z)}-1\right)\prec\frac{1+\left[B+\mu(A-B)\right]z}{1+Bz},\tag{21}$$

then

$$\left(\frac{f(z)}{z}\right)^{\mu}\prec(1+Bz)^{\frac{\mu(A-B)}{B}},$$

and $(1+Bz)^{\frac{\mu(A-B)}{B}}$ is the best dominant of (21). (The power is the principal one.)

Taking $\nu = 0,\ \eta = p = 1,\ s = 0,\ \gamma = \dfrac{e^{i\theta}}{\alpha\beta\cos\varsigma}$ $\left(\alpha, \beta \in \mathbb{C}^*,\ |\theta| < \dfrac{\pi}{2}\right),\ \mu = \alpha$ and $q(z) = (1-z)^{-2\alpha\beta\cos\theta e^{-i\theta}}$ in Theorem 3.2, we obtain the next special case due to Aouf et al. [2]:

Corollary 3.0. [2] *Let $\alpha, \beta \in \mathbb{C}^*$ and $|\theta| < \frac{\pi}{2}$, and suppose that $\left|2\alpha\beta\cos\theta e^{-i\theta} - 1\right| \le 1$ or $\left|2\alpha\beta\cos\theta e^{-i\theta} + 1\right| \le 1$. Let $f \in \mathcal{A}$ such that $\dfrac{f(z)}{z} \neq 0$ for all $z \in \mathbb{D}$. If*

$$1 + \frac{e^{i\theta}}{\beta\cos\theta}\left(\frac{zf'(z)}{f(z)} - 1\right) \prec \frac{1+z}{1-z}, \tag{22}$$

then

$$\left(\frac{f(z)}{z}\right)^{\alpha} \prec (1-z)^{-2\alpha\beta\cos\theta e^{-i\theta}},$$

and $(1-z)^{-2\alpha\beta\cos\theta e^{-i\theta}}$ is the best dominant of (22). (The power is the principal one.)

Theorem 3.3. *Let q be univalent in $\mathbb{D}$ with $q(0) = 1$, let $\mu, \gamma \in \mathbb{C}^*$, and let $\sigma, \nu, \eta \in \mathbb{C}$ with $\nu + \eta \neq 0$. Let $f \in A(p)$ and suppose that f and q satisfy the next two conditions:*

$$\frac{\nu\,\mathrm{J}_{s+1,b}(f)(z) + \eta\,\mathrm{J}_{s,b}(f)(z)}{(\nu+\eta)z^p} \neq 0,\ z \in \mathbb{D}, \tag{23}$$

and

$$\operatorname{Re}\left(1 + \frac{zq''(z)}{q'(z)}\right) > \max\left\{0; -\operatorname{Re}\frac{\sigma}{\gamma}\right\},\ z \in \mathbb{D}. \tag{24}$$

If

$$\psi(z) := \left[\frac{\nu\,\mathrm{J}_{s+1,b}(f)(z) + \eta\,\mathrm{J}_{s,b}(f)(z)}{(\nu+\eta)z^p}\right]^{\mu} \cdot \left[\sigma + \gamma\mu\left(\frac{\nu z\left[\mathrm{J}_{s+1,b}(f)(z)\right]' + \eta z\left[\mathrm{J}_{s,b}(f)(z)\right]'}{\nu\,\mathrm{J}_{s+1,b}(f)(z) + \eta\,\mathrm{J}_{s,b}(f)(z)} - p\right)\right] \tag{25}$$

and

$$\psi(z) \prec \sigma q(z) + \gamma z q'(z), \tag{26}$$

then

$$\left[\frac{\nu\,\mathrm{J}_{s+1,b}(f)(z) + \eta\,\mathrm{J}_{s,b}(f)(z)}{(\nu+\eta)z^p}\right]^{\mu} \prec q(z),$$

and q is the best dominant of (26). (All the powers are the principal ones.)

Proof. Letting

$$g(z) := \left[\frac{\nu\,\mathrm{J}_{s+1,b}(f)(z) + \eta\,\mathrm{J}_{s,b}(f)(z)}{(\nu+\eta)z^p}\right]^{\mu}, \tag{27}$$

then from (23) it follows that g is analytic in $\mathbb{D}$, and $g(0) = 1$. Differentiating (27) logarithmically with respect to z, we have

$$\frac{zg'(z)}{g(z)} = \mu\left[\frac{\nu z\left[\mathrm{J}_{s+1,b}(f)(z)\right]' + \eta z\left[\mathrm{J}_{s,b}(f)(z)\right]'}{\nu\,\mathrm{J}_{s+1,b}(f)(z) + \eta\,\mathrm{J}_{s,b}(f)(z)} - p\right],$$

hence

$$zg'(z) = \mu g(z)\left[\frac{\nu z\left[\mathrm{J}_{s+1,b}(f)(z)\right]' + \eta z\left[\mathrm{J}_{s,b}(f)(z)\right]'}{\nu\,\mathrm{J}_{s+1,b}(f)(z) + \eta\,\mathrm{J}_{s,b}(f)(z)} - p\right].$$

Now, let

$$\theta(w) = \sigma w,\ \varphi(w) = \gamma,\ w \in \mathbb{C},$$
$$Q(z) = zq'(z)\varphi(q(z)) = \gamma z q'(z)\ z \in \mathbb{D},$$

and

$$h(z) = \theta(q(z)) + Q(z) = \sigma q(z) + \gamma z q'(z),\ z \in \mathbb{D}.$$

Using (24), we see that Q is starlike in $\mathbb{D}$ and

$$\operatorname{Re}\frac{zh'(z)}{Q(z)}=\operatorname{Re}\left(\frac{\sigma}{\gamma}+1+\frac{zq''(z)}{q'(z)}\right)>0,\ z\in\mathbb{D},$$

hence, by applying Lemma 2.1 the proof is completed. □

Taking $q(z)=\dfrac{1+Az}{1+Bz}$ in Theorem 3.3, where $-1\le B<A\le 1$ and according to (14), the condition (24) becomes

$$\max\left\{0;-\operatorname{Re}\frac{\sigma}{\gamma}\right\}\le\frac{1-|B|}{1+|B|},$$

and for the special case $\nu=\gamma=1$, $\eta=0$, the above result reduces to:

Corollary 3.0. *Let $-1\le B<A\le 1$ and let $\sigma\in\mathbb{C}$ with*

$$\max\{0;-\operatorname{Re}\sigma\}\le\frac{1-|B|}{1+|B|}.$$

Let $f\in A(p)$ and suppose that $\dfrac{\mathrm{J}_{s+1,b}(f)(z)}{z^p}\neq 0$ for all $z\in\mathbb{D}$, and let $\mu\in\mathbb{C}^*$. If

$$\left[\frac{\mathrm{J}_{s+1,b}(f)(z)}{z^p}\right]^{\mu}\left[\sigma+\mu\left(\frac{z\left[\mathrm{J}_{s,b}(f)(z)\right]'}{\mathrm{J}_{s,b}(f)(z)}-p\right)\right]\prec\sigma\frac{1+Az}{1+Bz}+z\frac{(A-B)}{(1+Bz)^2},\tag{28}$$

then

$$\left[\frac{\mathrm{J}_{s+1,b}(f)(z)}{z^p}\right]^{\mu}\prec\frac{1+Az}{1+Bz},$$

and $\dfrac{1+Az}{1+Bz}$ is the best dominant of (28). (All the powers are the principal ones.)

Another special case of Theorem 3.3 may be obtained for $\eta=p=\gamma=1$, $\nu=s=0$ and $q(z)=\dfrac{1+z}{1-z}$:

Corollary 3.0. *Let $f\in\mathcal{A}$ such that $\dfrac{f(z)}{z}\neq 0$ for all $z\in\mathbb{D}$, let $\mu\in\mathbb{C}^*$, and $\sigma\in\mathbb{C}$ with $\operatorname{Re}\sigma\ge 0$. If*

$$\left[\frac{f(z)}{z}\right]^{\mu}\left[\sigma+\mu\left(\frac{zf'(z)}{f(z)}-1\right)\right]\prec\sigma\frac{1+z}{1-z}+\frac{2z}{(1-z)^2},\tag{29}$$

then

$$\left[\frac{f(z)}{z}\right]^{\mu}\prec\frac{1+z}{1-z},$$

and $\dfrac{1+z}{1-z}$ is the best dominant of (29). (All the powers are the principal ones.)

4. Superordination and sandwich results

Theorem 4.4. *Let q be convex in $\mathbb{D}$ with $q(0)=1$, and $\lambda\in\mathbb{C}^*$ with $\operatorname{Re}\dfrac{\lambda}{b+1}>0$. Let $f\in A(p)$ and suppose that $\dfrac{\mathrm{J}_{s,b}(f)(z)}{z^p}\in H[q(0),1]\cap\mathcal{Q}$. If the function*

$$\frac{\lambda}{p}\left(\frac{\mathrm{J}_{s,b}(f)(z)}{z^p}\right)+\frac{p-\lambda}{p}\left(\frac{\mathrm{J}_{s+1,b}(f)(z)}{z^p}\right)$$

is univalent in the unit disc $\mathbb{D}$, and

$$q(z)+\frac{\lambda zq'(z)}{p(b+1)}\prec\frac{\lambda}{p}\left(\frac{\mathrm{J}_{s,b}(f)(z)}{z^p}\right)+\frac{p-\lambda}{p}\left(\frac{\mathrm{J}_{s+1,b}(f)(z)}{z^p}\right),\tag{30}$$

then

$$q(z)\prec\frac{\mathrm{J}_{s+1,b}(f)(z)}{z^p},$$

and q is the best subordinant of (30).

Proof. If we let
$$g(z) := \frac{\mathrm{J}_{s,b}(f)(z)}{z^p}, \tag{31}$$
from the assumption of the theorem, the function g is analytic in $\mathbb{D}$. Differentiating (31) and according to (6), we have
$$g(z) + \frac{\lambda z g'(z)}{p(b+1)} = \frac{\lambda}{p}\left(\frac{\mathrm{J}_{s+1,b}(f)(z)}{z^p}\right) + \frac{p-\lambda}{p}\left(\frac{\mathrm{J}_{s,b}(f)(z)}{z^p}\right),$$
and then, by using Lemma 2.4 the proof is completed. □

Taking $q(z) = \dfrac{1+Az}{1+Bz}$ in Theorem 4.4, where $-1 \le B < A \le 1$, we obtain the next corollary:

Corollary 4.0. *Let $\lambda \in \mathbb{C}^*$ with $\operatorname{Re} \dfrac{\lambda}{b+1} > 0$. Let $f \in A(p)$ and suppose that $\dfrac{\mathrm{J}_{s,b}(f)(z)}{z^p} \in H[1,1] \cap \mathcal{Q}$. If the function*
$$\frac{\lambda}{p}\left(\frac{\mathrm{J}_{s,b}(f)(z)}{z^p}\right) + \frac{p-\lambda}{p}\left(\frac{\mathrm{J}_{s+1,b}(f)(z)}{z^p}\right)$$
is univalent in $\mathbb{D}$, and
$$\frac{1+Az}{1+Bz} + \frac{\lambda(A-B)z}{p(b+1)(1+Bz)^2} \prec \frac{\lambda}{p}\left(\frac{\mathrm{J}_{s,b}(f)(z)}{z^p}\right) + \frac{p-\lambda}{p}\left(\frac{\mathrm{J}_{s+1,b}(f)(z)}{z^p}\right), \tag{32}$$
then
$$\frac{1+Az}{1+Bz} \prec \frac{\mathrm{J}_{s+1,b}(f)(z)}{z^p},$$
and $\dfrac{1+Az}{1+Bz}$ is the best subordinant of (32), where $-1 \le B < A \le 1$.

Using arguments similar to those used in the proof of Theorem 3.3, and then by applying Lemma 2.3, we obtain the following result:

Theorem 4.5. *Let q be convex in $\mathbb{D}$ with $q(0) = 1$, let $\mu, \gamma \in \mathbb{C}^*$, and let $\sigma, \Omega, \nu, \eta \in \mathbb{C}$ with $\nu+\eta \neq 0$ and $\operatorname{Re} \dfrac{\sigma}{\gamma} > 0$. Let $f \in A(p)$ and suppose that f satisfies the next conditions:*
$$\frac{\nu\,\mathrm{J}_{s+1,b}(f)(z) + \eta\,\mathrm{J}_{s,b}(f)(z)}{(\nu+\eta)z^p} \neq 0,\ z \in \mathbb{D},$$
and
$$\left[\frac{\nu\,\mathrm{J}_{s+1,b}(f)(z) + \eta\,\mathrm{J}_{s,b}(f)(z)}{(\nu+\eta)z^p}\right]^{\mu} \in H[q(0),1] \cap \mathcal{Q}.$$
If the function ψ given by (25) is univalent in $\mathbb{D}$, and
$$\sigma q(z) + \gamma z q'(z) \prec \psi(z), \tag{33}$$
then
$$q(z) \prec \left[\frac{\nu\,\mathrm{J}_{s+1,b}(f)(z) + \eta\,\mathrm{J}_{s,b}(f)(z)}{(\nu+\eta)z^p}\right]^{\mu},$$
and q is the best subordinant of (33). (All the powers are the principal ones.)

Combining Theorem 3.1 with Theorem 4.4 and Theorem 3.3 with Theorem 4.5, we deduce respectively the following *sandwich results*:

Theorem 4.6. *Let q_1 and q_2 be two convex functions in $\mathbb{D}$ with $q_1(0) = q_2(0) = 1$, let $\lambda \in \mathbb{C}^*$ with $\operatorname{Re} \dfrac{\lambda}{b+1} > 0$. Let $f \in A(p)$ and suppose that $\dfrac{\mathrm{J}_{s,b}(f)(z)}{z^p} \in H[1,1] \cap \mathcal{Q}$. If the function*
$$\frac{\lambda}{p}\left(\frac{\mathrm{J}_{s,b}(f)(z)}{z^p}\right) + \frac{p-\lambda}{p}\left(\frac{\mathrm{J}_{s+1,b}(f)(z)}{z^p}\right)$$

is univalent in the unit disc $\mathbb{D}$, *and*

$$q_1(z)+\frac{\lambda z q_1'(z)}{p(b+1)} \prec \frac{\lambda}{p}\left(\frac{\mathrm{J}_{s,b}(f)(z)}{z^p}\right)+\frac{p-\lambda}{p}\left(\frac{\mathrm{J}_{s+1,b}(f)(z)}{z^p}\right) \prec q_2(z)+\frac{\lambda z q_2'(z)}{p(b+1)}, \quad (34)$$

then

$$q_1(z) \prec \frac{\mathrm{J}_{s,b}(f)(z)}{z^p} \prec q_2(z),$$

and q_1 *and* q_2 *are, respectively, the best subordinant and the best dominant of* (34).

Theorem 4.7. *Let* q_1 *and* q_2 *be two convex functions in* $\mathbb{D}$ *with* $q_1(0)=q_2(0)=1$, *let* $\mu,\gamma\in\mathbb{C}^*$, *and let* $\sigma,\Omega,\nu,\eta\in\mathbb{C}$ *with* $\nu+\eta\neq 0$ *and* $\operatorname{Re}\dfrac{\sigma}{\gamma}>\ 0$. *Let* $f\in A(p)$ *and suppose that* f *satisfies the next conditions:*

$$\frac{\nu\,\mathrm{J}_{s+1,b}(f)(z)+\eta\,\mathrm{J}_{s,b}(f)(z)}{(\nu+\eta)z^p}\neq 0,\ z\in\mathbb{D},$$

and

$$\left[\frac{\nu\,\mathrm{J}_{s+1,b}(f)(z)+\eta\,\mathrm{J}_{s,b}(f)(z)}{(\nu+\eta)z^p}\right]^{\mu}\in H[1,1]\cap\mathcal{Q}.$$

If the function ψ *given by* (25) *is univalent in* $\mathbb{D}$, *and*

$$\sigma q_1(z)+\gamma z q_1'(z)\prec\psi(z)\prec\sigma q_2(z)+\gamma z q_2'(z), \quad (35)$$

then

$$q_1(z)\prec\left[\frac{\nu\,\mathrm{J}_{s+1,b}(f)(z)+\eta\,\mathrm{J}_{s,b}(f)(z)}{(\nu+\eta)z^p}\right]^{\mu}\prec q_2(z),$$

and q_1 *and* q_2 *are, respectively, the best subordinant and the best dominant of* (35). *(All the powers are the principal ones.)*

References

[1] Alexander, J. W. Functions which map the interior of the unit circle upon simple regions, Ann. Math. Ser. 2, 17 (1915), 12–22.

[2] Aouf M. K.; Al-Oboudi F. M.; Haidan M. M. On some results for λ-spirallike and λ-Robertson functions of complex order, Publ. Institute Math. Belgrade, 77(91) (2005), 93–98.

[3] Bulboacă, T. Classes of first order differential superordinations, Demonstratio Math., 35(2) (2002), 287-292.

[4] Jung, I. B.; Kim, Y. C.; Srivastava, H. M. The Hardy space of analytic functions associated with certain one-parameter families of integral operators, J. Math. Anal. Appl., 176 (1993), 138–147.

[5] Libera, R. J. Some classes of regular univalent functions, Proc. Amer. Math. Soc., 16 (1969), 755–758.

[6] Liu, J.-L. Subordinations for certain multivalent analytic functions associated with the generalized Srivastava-Attiya operator, Integral Transforms Spec. Funct., 19(12) (2008), 893–901.

[7] Miller, S. S.; Mocanu, P.T. Differential Subordination. Theory and Applications, Series on Monographs and Textbooks in Pure and Applied Mathematics, Vol. 225, Marcel Dekker Inc., New York and Basel, 2000.

[8] Miller, S. S.; Mocanu, P.T. Subordinates of differential superordinations, Complex Variables, 48(1)(2003), 815–826.

[9] Obradović, M.; Aouf, M. K.; Owa, S. On some results for starlike functions of complex order, Publ. Inst. Math. Belgrade, 46(60) (1989), 79–85.

[10] Royster, W. C. On the univalence of a certain integral, Michigan Math. J., 12 (1965), 385–387.

[11] Shanmugam, T. N.; Sivasubramanian, S.; Silverman, H. On sandwich theorems for some classes of analytic functions, Internat. J. Math. Math. Sci., Vol. 2006, Article ID 29684, 1–13.

[12] Shanmugam, T. N.; Ravichandran, V.; Sivasubramanian, S. Differantial sandwich theorems for some subclasses of analytic functions, J. Austr. Math. Anal. Appl., 3(1) (2006), art. 8, 1–11.

[13] Shanmugan, T. N.; Ramachandran, C.; Darus, M.; Sivasubbramanian, S. Differantial sandwich theorems for some subclasses of analytic functions involving a linear operator, Acta Math. Univ. Comenianae, 74(2) (2007), 287–294.

[14] Srivastava, H. M.; Attiya, A. A. An integral operator associated with the Hurwitz-Lerch Zeta function and differential subordination, Integral Transforms Spec. Funct., 18(3) (2007), 207–216.

[15] Srivastava, H. M.; Choi, J. Series associated with the Zeta and related functions, Kluwer Academic Publishers, Dordrecht, Boston, London, 2001.

[16] Srivastava, H. M.; Lashin, A. Y. Some applications of the Briot-Bouquet differential subordination, J. Ineq. Pure Appl. Math., 6(2)(2005), article 41, 1–7.

A. O. Mostafa,, Department of Mathematics, Faculty of Science, Mansoura University, Mansoura 35516, Egypt.

Email address: adelaeg254@yahoo.com

T. Bulboacă, Faculty of Mathematics and Computer Science, Babeş-Bolyai University, 400084 Cluj-Napoca, Romania

Email address: bulboaca@math.ubbcluj.ro

M. K. Aouf, Department of Mathematics, Faculty of Science, Mansoura University, Mansoura 35516, Egypt.

Email address: mkaouf127@yahoo.com