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Equations of Motion for Spinning Fluids and their Deviation Equations in Finslerian Geometry

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Abstract

Finsler geometry is a natural extension of the Riemannian geometry and a good a platform used to interpret the infrastructure of physical phenomena, especially for relativistic applications. Accordingly it is worthy to study spinning fluids in the context of this geometry that would share their benefits in cosmological applications. Equations of motion of spinning fluids and their corresponding deviation equations are obtained. The problem of motion for studying a fluid with a variable mass is also obtained. The set of Equations of spinning fluids and spinning deviation fluids equations for some classes of the Finslerian geometry have been derived, using a modified type of the Bazanski Lagrangian. Due to the richness of the Finslerian geometry, a new perspective for revisiting the problem of stability is based on solving the deviation equations of spinning fluids in strong fields of gravity is performed. Such a problem has a direct application on examining the stability of accretion disk orbiting Sgr A*.

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1. Introduction

Equations of motion for spinning particles are basically good evidence to examine the behavior of objects in various gravitational fields [1]. Accordingly, the problem of stability of celestial objects have been revisited throughout the studying the motion of objects. The issue of using spinning object as a tool to examine motion is due to the reliability of its existence in nature. As it is well known that every object in space is spinning, while the case of non-spinning objects for probes to examine motion in orbits is regarded as a special case for simplicity. From this perspective many authors like Mathison, Papapetrou, Dixon and others have performed the spinning equations for objects and charged objects in gravitational fields expressed in the Riemannian geometry [2, 3]. Not only that but also, there are other approaches examining the equations of motion for spinning objects in other types of geometries like AP-space -a specific type of non-Riemannian geometry and the Finselerian geometry. Meanwhile, the reason for introducing different types of geometries is related to apply the concept made by H. Poincare who connected the feasibility of a specific geometry with its associated measurements [4].

Moreover, the problem of spinning particles has been extended to examine spinning fluids. This has been done by obtaining equations of spinning fluid and spinning deviation fluid in Riemannian geometry [5]. The stability of spinning object orbiting very strong field has been discssed by Kahil based on the use of spinning deviation tensor [6]. Yet as a step to replace spinning fluid by spinning tensor in the accretion disc orbiting SgrA*, it is mandory to obtain their corresponding spinning and spinning fluid deviation equations Spinning fluids are in fact describing the state of matter orbiting very strong gravitational fields like the material of the accretion disc orbiting the core of our galaxy Sgr A*.

Accordingly, as a step to search for an alternative type of geometry expressing the manifold which is based not only on points but also on their direction. The problem of inserting the direction is assigned to get more details about the behaviour of objects in strong fields of gravity. This may led many authors to seek for the alternative types of geometries rather than relying to the Riemannian geometry.

Finsler geometry is regarded as a wider classes of geometries based on a fundamental function L such that

$$s = \int L\, \gamma(t)\dot{\gamma(t)})dt, \qquad (1)$$

where s is the arc length of the curve γ(t) in a Finsler Space F^n , whose geodesics are defined as the solutions of the Euler-Lagrange equation

$$\frac{d}{dt}\frac{\partial L}{\partial y^{\alpha}} - \frac{\partial L}{\partial x^{\alpha}} = 0. \qquad (2)$$

From this perspective, it can be sought that Finsler spaces are more flexible than their Riemannian counterpart. These spaces have freedom of the choice of metric and connection comparing with the Riemannain ones [7], that having a unique metric and affine connection [8]. Such an advantage may give rise to implement Finsler geometry in Physics, Biology [9], and other fields such as Modeling of epidemic curves [10].

The paper is organized in the following way. In section 2, we display two different types of approaches for describing the Finslerian geometry, the Cartan-Rund and the Cartan-Finsler ones. In Section 3, we review the problem of motion in both Cartan-Rund and Cartan-Finsler approach. While in Section 4, we obtain spinning and spinning deviation equation in both Finslerian approachs. In Section 5, we perform the equations of motion for a spinning fluid with a variable mass for Cartan-Rund and Cartan-Finsler approach. Conditions for studying the stability in the presences of spinning fluid are given in Sect. 6. Section 7, we comment on the previous results and through some light on our forthcoming work.

2. Underlying Geometry: The Finsler Geometry:

A Finsler space (M, F) is an n- dimensional smooth manifold M equipped with a scalar$F(x, y)$, where the functions $\left(x(t), y(t)\right)$, with $y\left(\stackrel{\text{def}}{=} \dot{x} = \frac{dx}{dt}\right)$, defines the coordinates on the tangent bundle TM, and t is an invariant parameter. (For more details cf Ref.[11, 12, 13])

A second-order tensor $g_{\mu\nu}(x, \dot{x})$ symmetric, non-degenerate tensor and characterizes the metric of Finsler space, which is defined as

$$g_{\mu\nu} \stackrel{\text{def}}{=} \frac{\partial^2 F^2(x, \dot{x})}{\partial \dot{x}^\mu \partial \dot{x}^\nu}, \tag{3}$$

Also, this space admits a third order tensor

$$C_{(x,\dot{x})} = C_{\mu\nu\rho}\, dx^\mu \otimes dx^\nu \otimes dx^\rho, \tag{4}$$

which is known as the Cartan tensor may be defined as

$$C_{\mu\nu\rho} \stackrel{\text{def}}{=} \frac{1}{2} \frac{\partial g_{\mu\nu}}{\partial \dot{x}^\nu \partial \dot{x}^\rho}. \tag{5}$$

Using (3), one can write

$$C_{\mu\nu\rho} = \frac{1}{2} \frac{\partial^3 F^2}{\partial \dot{x}^\mu \partial \dot{x}^\nu \partial \dot{x}^\rho}. \tag{6}$$

In case of vanishing Cartan tensor $C_{\mu\nu\rho} = 0$, the Finsler space becomes a Riemannian one [14].

Also, geodesic equation in Finsler Geometry is defined as

$$\frac{d^2 x^\mu}{dS^2} + 2G^\mu = 0, \tag{7}$$

such that

$$G^\mu \stackrel{\text{def}}{=} \frac{1}{4} g^{\mu\nu} \left[\dot{x}^\sigma \frac{\partial^2 F^2}{\partial x^\nu \partial \dot{x}^\sigma} - \frac{\partial F^2}{\partial x^\nu}\right], \tag{8}$$

where G^μ is called a spray, to be defined as

$$G^\mu = \frac{1}{2} \{^{\mu}_{\nu\rho}\} \dot{x}^\nu \dot{x}^\rho, \tag{9}$$

In which $\{^{\mu}_{\nu\rho}\}$ is the Christoffel symbol as defined in the Riemannian geometry [5].

From this perspective, we are going to review two approaches of Finslerian-geometry: Cartan-Rund Approach and Finsler cartan approach [12, 13, 15].

2.1. Cartan-Rund Approach

It is well known that the Cartan-Rund Approach admits an affine connection $\tilde{\Gamma}^{\mu}_{\ \nu\rho}$defined as

$$\tilde{\Gamma}^{\mu}_{\ \nu\rho} \stackrel{\text{def}}{=} \{^{\ \mu}_{\nu\rho}\} - C^{\mu}_{\ \nu\delta}\{^{\ \delta}_{\lambda\rho}\}\dot{x}^{\lambda}, \tag{10}$$

<u>Also, its corresponding covariant derivative</u> defined due to Chern [16] to have the form

$$\frac{\underline{\delta} A^{\mu}}{\underline{\delta} S} = A^{\mu}_{\ \underline{;}\nu} \frac{dx^{\nu}}{dS}, \tag{11}$$

where

$$A^{\mu}_{\ \underline{;}\nu} = \frac{\partial A^{\mu}}{\partial x^{\nu}} + \tilde{\Gamma}^{\mu}_{\ \nu\rho} A^{\nu}\dot{x}^{\rho}, \tag{12}$$

<u>Thus, its corresponding geodesic equation</u> can be written in the following form:

$$\frac{\underline{\delta} U^{\mu}}{\underline{\delta} S} = 0, \tag{13}$$

Also, the commutation relation as defined in Cartan-Rund approach is expressed as follows

$$A^{\mu}_{\ \underline{;}\nu\rho} - A^{\mu}_{\ \underline{;}\rho\nu} = K^{\mu}_{\ \nu\rho\delta} A^{\delta}, \tag{14}$$

where $K^{\mu}_{\ \nu\rho\delta}$ is its corresponding curvature as defined in the following way

$$K^{\mu}_{\ \nu\rho\sigma} = \left(\tilde{\Gamma}^{\mu}_{\ \nu\rho,\sigma} - \frac{\partial \tilde{\Gamma}^{\mu}_{\ \nu\rho}}{\partial \dot{x}^{\delta}} \frac{\partial G^{\delta}}{\partial \dot{x}^{\sigma}}\right) - \left(\tilde{\Gamma}^{\mu}_{\ \nu\sigma,\rho} - \frac{\partial \tilde{\Gamma}^{\mu}_{\ \nu\sigma}}{\partial \dot{x}^{\delta}} \frac{\partial G^{\delta}}{\partial \dot{x}^{\rho}}\right) + \tilde{\Gamma}^{\lambda}_{\ \nu\rho}\tilde{\Gamma}^{\mu}_{\ \lambda\sigma} - \tilde{\Gamma}^{\lambda}_{\ \nu\rho}\tilde{\Gamma}^{\mu}_{\ \nu\rho}. \tag{15}$$

2.2. Finsler- cartan approach

Another version of Finsler-geometry will be displayed called the Finsler-Cartan approach. In this approach a nonlinear connection $N^{\alpha}_{\ \beta}$ has been defined and acts as gauge potential [13], to be expressed as follows:

$$N^{\alpha}_{\ \beta} = \frac{1}{2} g^{\alpha\epsilon} \left[y^{\gamma} \frac{\partial^2 F^2(x,y)}{\partial x^{\beta} \partial y^{\gamma}} - \frac{\partial F^2(x,y)}{\partial x^{\beta}} \right]. \tag{16}$$

As mentioned above, the coordinates are expressed in Finsler space within tangent space, the addition of nonlinear connection led to splitting the tangent bundle into two sub-bundles. One for the horizontal coordinates while the other assigned the vertical coordinates. The arising TM admitting the basis $(\delta_{\alpha}, \partial_{a})$, the Greek indices are used to characterize horizontal coordinates and Latin indices for vertical coordinates. This is important for describing an anisotropic gravitational field theory.

In the context of this approach, the line element is defined as following

$$ds^2 = g_{\mu\nu}(x,y)dx^{\mu}dx^{\nu} + h_{ab}(x,y)\delta y^{a}\delta y^{b}, \tag{17}$$

where δy^{a} the extended derivative as is defined in the admitted basis$(\delta_{\alpha}, \partial_{a})$ and the tensors $g_{\mu\nu}(x,y), h_{ab}(x,y)$ define (non-degenerate) metrics of horizontal and vertical coordinate systems respectively.

<u>In addition to</u> the non-linear connection of the Finsler-Cartan approach, there exists

$$\bar{\Gamma}^{\alpha}_{\ \mu\nu} \stackrel{\text{def}}{=} \frac{1}{2} g^{\alpha\beta}\left(\delta_{\mu} g_{\beta\nu} + \delta_{\beta} g_{\mu\nu} - \delta_{\nu} g_{\beta\mu}\right). \tag{18}$$

While, for its corresponding vertical sub-bundle, threre is by analogy another affine connection $C^{a}_{\ bc}$ defined as:

$$C^{a}_{\ bc} \stackrel{\text{def}}{=} \frac{1}{2} g^{ad}\left(\dot{\partial}_{b} g_{cd} + \dot{\partial}_{d} g_{bc} - \dot{\partial}_{c} g_{db}\right), \tag{19}$$

where, $\dot{\partial}_{b} \stackrel{\text{def}}{=} \frac{\partial}{\partial y^{b}}$

Consequently. For an arbitrary vector A^{α}, the covariant differentiation using the non-linear connection defined as:

$$\frac{\delta A^{\alpha}}{\delta x^{\mu}}, = \frac{\partial A^{\alpha}}{\partial x^{\mu}} - N^{\nu}{}_{\mu}\frac{\partial A^{\alpha}}{\partial \dot{x}^{\upsilon}}. \qquad (20)$$

The spray G^{α} is related to the non-linear connection N^{α}_{β} by the following relation

$$G^{\alpha} = \frac{1}{4} N^{\alpha}_{\beta} y^{\beta}, \qquad (21)$$

Accordingly, it's easy to show that

$$N^{\alpha}_{\beta} = 2\frac{\partial G^{\alpha}}{\partial y^{\beta}}. \qquad (22)$$

Geodesic equation: Consequently, we have two equations for geodesic one defined for the horizontal coordinate, which can be written as:

$$\frac{\nabla U^{\mu}}{\nabla s} = 0, \qquad (23)$$

where $U^{\mu} \stackrel{\text{def}}{=} \frac{\delta x^{\mu}}{\delta s}$ is the unit tangent vector, and

$$\frac{\nabla U^{\mu}}{\nabla s} \stackrel{\text{def}}{=} \frac{\delta U^{\mu}}{\delta s} + \Gamma^{\mu}_{\nu\sigma} U^{\upsilon} U^{\sigma}.$$

While for the vertical coordinate, the geodesic have the form:

$$\frac{DV^{a}}{Ds} = 0, \qquad (24)$$

where

$$\frac{DV^{a}}{Ds} \stackrel{\text{def}}{=} \frac{\partial V^{a}}{\partial s} + C^{a}{}_{bd} V^{b} V^{d},$$

as $V^{a} \stackrel{\text{def}}{=} \frac{\partial y^{a}}{\partial s}$.

3. From Geodesic Equation to Spinning Equation in Finsler Geometry:

3.1. Cartan-Rund Approach

Equation of geodesic and geodesic deviation in the Finslerian geometry based on the Cartan -Rund approach may be expressed based on the following Lagrangian function [17]

$$L = g_{\alpha\beta}(x, \dot{x}) U^{\alpha} \frac{\underline{\delta} \Psi^{\beta}}{\underline{\delta} S}. \qquad (25)$$

Where U^{α} is its four vector velocity, Ψ^{β} its corresponding deviation vector defined as follows

$$\Psi^{\beta} = \epsilon \frac{\partial x^{\beta}}{\partial \epsilon} |_{\epsilon=0}$$

Thus, by taking the variation with respect to Ψ^{β} one gets

$$\frac{\underline{\delta} U^{\mu}}{\underline{\delta} S} = 0. \qquad (26)$$

Also, the relationship between the two parameters between S and τ becomes [18]:

$$\frac{\underline{\delta} U^{\mu}}{\underline{\delta} S} = \frac{\underline{\delta} \Psi^{\mu}}{\underline{\delta} \tau}. \qquad (27)$$

Using the commutation relation as defined in (14) together with the condition (27) in a similar way as in [16], one gets

$$\frac{\underline{\delta}^2 \Psi^{\mu}}{\underline{\delta} S^2} = K^{\mu}_{\ \nu\rho\sigma}\, U^{\nu}\, U^{\rho}\, \Psi^{\sigma}. \qquad (28)$$

Accordingly, the spinning equations may be obtained due to the following relation

$$\tilde{U}^{\mu} = U^{\mu} + \beta \frac{\underline{\delta} \Psi^{\mu}}{\underline{\delta} S},$$

Such that $\tilde{U}^{\mu}$ is a vector defined as combination between the ususal four unit vector velocity and its corresponding deviation vector

$$\frac{\underline{\delta} \tilde{U}^{\mu}}{\underline{\delta} \rho} = \frac{\underline{\delta}}{\underline{\delta} S}\left(U^{\mu} + \beta \frac{\underline{\delta} \Psi^{\mu}}{\underline{\delta} S}\right)\frac{dS}{d\rho}, \qquad (29)$$

which ρ is a parameter associated with spinning motion, provided that the spin tensor $S^{\mu\nu}$ is defined as(cf. [6])

$$S^{\mu\nu} \stackrel{\text{def}}{=} \sigma(U^{\mu}\Psi^{\nu} - U^{\nu}\Psi^{\mu}), \qquad (30)$$

Thus, the spinning equation in Finsler Equation takes the form [13]:

$$\frac{\underline{\delta} \tilde{U}^{\mu}}{\underline{\delta} \rho} = \frac{1}{2m} K^{\mu}_{\ \nu\rho\sigma}\, S^{\rho\sigma} U^{\nu}, \qquad (31)$$

3.2. Finsler- Cartan approach

Equation of geodesic and geodesic deviation in the context of Finsler-Cartan approach can be obtained using the following Lagrangian function [13]

$$L = g_{\alpha\beta}(x,y)U^{\alpha}U^{\beta} + h_{ab}(x,y)V^{a}V^{b}. \qquad (32)$$

The deviation equation for the h-derivative can be obtained using the commutation relation as defined in (14) together with the condition (27) in a similar way as in, one gets

$$\frac{\nabla^2 \Psi^{\mu}}{\nabla S} = \underline{R}^{\mu}_{\ \nu\alpha\sigma}\, U^{\nu}\, U^{\alpha}\, \Psi^{\sigma}, \qquad (33)$$

where, $\underline{R}^{\mu}_{\ \nu\alpha\sigma}$represents the curvature tensor, have the following definition

$$\underline{R}^{\alpha}_{\ \mu\nu\sigma} = \bar{\Gamma}^{\alpha}_{\ \mu\sigma||\nu} - \bar{\Gamma}^{\alpha}_{\ \mu\nu||\sigma} + \bar{\Gamma}^{\epsilon}_{\ \mu\sigma}\bar{\Gamma}^{\alpha}_{\ \epsilon\nu} - \bar{\Gamma}^{\epsilon}_{\ \mu\nu}\bar{\Gamma}^{\alpha}_{\ \epsilon\sigma}$$

Similarly, one can obtain the deviation equation for the v-derivative

$$\frac{\nabla^2 \Phi^a}{\nabla S} = B^a{}_{bcd}\, V^b\, V^c\, \Phi^d, \qquad (34)$$

where;

$$B^a{}_{bcd} = C^a{}_{bd|c} - C^a{}_{bc|d} + C^l{}_{bd} C^a{}_{lc} - C^l{}_{bc} C^a{}_{ld}.$$

In order to derive the equation of motion for spinning fluids in the context of Finsler-Cartan geometry, we have to introduce the following functions

$$\underline{U}^\mu = U^\mu + \beta \frac{\nabla \Psi^\mu}{\nabla S},$$

$$\underline{V}^\alpha = V^\alpha + \beta \frac{D\Phi^\alpha}{DS},$$

to express horizontal and the vertical components each for its corresponding coordinate systems defined in the two sub-bundles.

Accordingly, equation of motion for spinning fluids for the h-derivative has the form

$$\frac{\nabla \underline{U}^\mu}{\nabla S} = \frac{1}{2m} \underline{R}^\mu{}_{\nu\alpha\sigma}\, U^\nu\, S^{\alpha\sigma}. \qquad (35)$$

While, for the v-derivative

$$\frac{D\underline{V}^\alpha}{DS} = \frac{1}{2m} B^a{}_{bcd} V^b\, S^{cd}, \qquad (36)$$

Where; $\frac{\nabla}{\nabla S}, \frac{D}{DS}$ characterizes horizontal and the vertical derivatives, respectively.

4. Spinning and Spinning Deviation Equations in Finsler Geometry:

The aim of the present section is to derive equation of motion of spinning fluid in the context of both approach of Finslerian geometry that mentioned above. Those equations have a vital role to describe extended object.

4.1. Cartan-Rund Approach

In this section, we are going to drive equation of motion of spinning fluid in the context of Cartan-Rund approach.

Accordingly, the Weyssenhoff spin tensor is expressed as follows (cf. [19]):

$$S^{\alpha\beta\gamma} = S^{\beta\gamma} U^\alpha \qquad (37)$$

(i) In case $P^\alpha = mU^\alpha$:

The Lagrangian of spinning fluid take the form [13]:

$$L = g_{\alpha\beta}(x,y) U^\alpha \frac{\delta \Psi^\beta}{\underline{\delta} S} + S_{\alpha\beta} \frac{\delta \Psi^{\alpha\beta}}{\underline{\delta} S} + \frac{1}{2m} K_{\mu\nu\alpha\beta}\, S^{\alpha\beta} U^\nu \Psi^\mu. \qquad (38)$$

Applying the Euler-Largrange equation with respect to deviation vectors becomes:

$$\frac{d}{dS}\frac{\partial L}{\partial \dot{\Psi}^{\alpha}} - \frac{\partial L}{\partial \Psi^{\alpha}} = 0, \qquad (39)$$

to obtain

$$\frac{\underline{\delta} U^{\mu}}{\underline{\delta} S} = \frac{1}{2m} K^{\mu}_{\ \nu\rho\sigma}\, S^{\rho\sigma} U^{\nu}. \qquad (40)$$

Similarly, the Euler Lagrange equation with respect to spinning deviation tensor becomes

$$\frac{d}{dS}\frac{\partial L}{\partial \dot{\Psi}^{\alpha\beta}} - \frac{\partial L}{\partial \Psi^{\alpha\beta}} = 0. \qquad (41)$$

We get,

$$\frac{\underline{\delta} S^{\mu\nu}}{\underline{\delta} S} = 0. \qquad (42)$$

Accordingly, the equation of motion of spinning fluid will have the form:

$$\frac{\underline{\delta} S^{\alpha\beta\gamma}}{\underline{\delta} S} = \frac{1}{2m} K^{\alpha}_{\ \mu\nu\sigma} S^{\nu\sigma}\, S^{\beta\gamma} U^{\mu}. \qquad (43)$$

Consequently, to obtain its corresponding deviation equations, must satisfy the following condition[18]

$$\frac{\underline{\delta} U^{\beta}}{\underline{\delta} S} = \frac{\underline{\delta} \Psi^{\beta}}{\underline{\delta} \tau}, \qquad (44)$$

to get the following relation

$$\left(\frac{\underline{\delta}}{\underline{\delta} S}\frac{\underline{\delta}}{\underline{\delta} \tau} - \frac{\underline{\delta}}{\underline{\delta} \tau}\frac{\underline{\delta}}{\underline{\delta} S}\right) A^{\alpha} = A^{\mu}\, K^{\alpha}_{\ \mu\rho\sigma}\, U^{\rho}\, \Psi^{\sigma}. \qquad (45)$$

Thus, one can get,

$$\frac{\underline{\delta}^{2} \Psi^{\mu}}{\underline{\delta} S^{2}} = K^{\mu}_{\ \nu\rho\sigma}\, U^{\nu}\, U^{\rho}\, \Psi^{\sigma} + \frac{1}{2m}\left(K^{\mu}_{\ \nu\alpha\sigma}\, U^{\nu}\, S^{\alpha\sigma}\right)_{\underline{;}\rho} \Psi^{\rho}, \qquad (46)$$

and

$$\frac{\underline{\delta}^{2} \Psi^{\mu\nu}}{\underline{\delta} S^{2}} = S^{\mu[\alpha} K^{\nu]}_{\ \alpha\rho\sigma}\, U^{\rho}\, \Psi^{\sigma}. \qquad (47)$$

To obtain the spinning density deviation equation for a spinning fluid, we will apply the following commutation relation

$$\left(\frac{\underline{\delta}}{\underline{\delta} S}\frac{\underline{\delta}}{\underline{\delta} \tau} - \frac{\underline{\delta}}{\underline{\delta} \tau}\frac{\underline{\delta}}{\underline{\delta} S}\right) S^{\alpha\beta\gamma} = S^{\epsilon[\beta\gamma} K^{\alpha]}_{\ \epsilon\rho\sigma}\, U^{\rho}\, \Psi^{\sigma}, \qquad (48)$$

To gether with condition (44), we get

$$\frac{\underline{\delta}^{2} \Psi^{\alpha\beta\gamma}}{\underline{\delta} S^{2}} = S^{\rho[\beta\gamma} K^{\alpha]}_{\ \rho\nu\sigma}\, U^{\nu}\, \Psi^{\sigma} + \frac{1}{2m}\left(K^{\alpha}_{\ \mu\nu\sigma}\, U^{\mu}\, S^{\nu\sigma}\, S^{\beta\gamma}\right)_{\underline{;}\rho} \Psi^{\rho}. \qquad (49)$$

(ii) In case $P^{\alpha} \neq mU^{\alpha}$:

In this case the Lagrangian of spinning fluid have the form:

$$L = g_{\alpha\beta}(x,y) U^{\alpha} \frac{\underline{\delta} \Psi^{\beta}}{\underline{\delta} S} + S_{\alpha\beta}\, \frac{\underline{\delta} \Psi^{\alpha\beta}}{\underline{\delta} S} + \frac{1}{2m} K_{\mu\nu\alpha\beta}\, S^{\alpha\beta} U^{\nu} \Psi^{\mu} + 2P_{\alpha} U_{\beta} \Psi^{\alpha\beta}. \qquad (50)$$

Then, by taking the variation with respect to Ψ^{μ} as given by (39), we get

$$\frac{\underline{\delta} P^{\mu}}{\underline{\delta} S} = \frac{1}{2m} K^{\mu}_{\ \nu\rho\sigma} S^{\rho\sigma} U^{\nu}. \tag{51}$$

Also, by operating the variation with respect to $\Psi^{\mu\nu}$ as given by (41), we obtain

$$\frac{\underline{\delta} S^{\mu\nu}}{\underline{\delta} S} = 2P^{\mu}U^{\nu}. \tag{52}$$

To obtain the equation of motion for spinning fluid, we take in consideration the Weyssenhoff tensor written in the following form:

$$S^{\alpha\beta\gamma} = S^{\beta\gamma}P^{\alpha} \tag{53}$$

Using (51) and(52), we get

$$\frac{\underline{\delta} S^{\alpha\beta\gamma}}{\underline{\delta} S} = 2P^{\alpha}P^{\beta}U^{\gamma} + \frac{1}{2m} K^{\alpha}_{\ \mu\nu\sigma} S^{\nu\sigma}\, S^{\beta\gamma} U^{\mu}. \tag{54}$$

Consequently, its corresponding spin density deviation tensor equation can be obtained, by applying the commutation relations as given in (48) and the condition (44), to have the form:

$$\frac{\underline{\delta}^{2}\Psi^{\alpha\beta\gamma}}{\underline{\delta} S^{2}} = S^{\rho[\beta\gamma} K^{\alpha]}_{\ \ \rho\nu\sigma}\, U^{\nu}\, \Psi^{\sigma} + \left(2P^{\alpha}P^{\beta}U^{\gamma} + \frac{1}{2m} K^{\alpha}_{\ \mu\nu\sigma} S^{\nu\sigma}\, S^{\beta\gamma} U^{\mu}\right)_{\underline{;}\lambda} \Psi^{\lambda}. \tag{55}$$

4.2. Finsler-Cartan Approach:

For such a tendency the chosen Lagrangian describe the spinning motion in both horizontal and the vertical coordinate coordinates. Thus by taking in consideration the terms $g_{\mu\nu}(x), \Psi^{\nu},\ \Psi^{\mu\nu}$describe coefficients of metric and deviation tensors for the horizontal coordinate system. Beside additive terms $h_{ab}(y),\ \Phi^{b}, \Phi^{ab}$ and $\underline{P}^{a}$ describe coefficients of metric, deviation tensors and momentum vector for the vertical coordinate system.

(i) Case $P^{\alpha} = mU^{\alpha}$ and $\underline{P}^{a} = mV^{a}$

$$L = g_{\mu\nu}(x)U^{\mu}\frac{\nabla\Psi^{\nu}}{\nabla S} + S_{\mu\nu}\ \frac{\nabla\Psi^{\mu\nu}}{\nabla S} + h_{ab}(y)V^{a}\frac{D\Phi^{b}}{DS} + \underline{S}_{ab}\ \frac{D\Phi^{ab}}{DS} + \frac{1}{2m}\underline{R}_{\mu\nu\alpha\beta}\, S^{\alpha\beta}U^{\nu}\Psi^{\mu} + \frac{1}{2m}B_{abcd}\,\underline{S}^{cd}V^{b}\Phi^{a}. \tag{56}$$

Taking the variation for the above Lagrangian w.r.to Ψ^{α} and $\Psi^{\alpha\beta}$, we obtain the following equations

$$\frac{\nabla \underline{U}^{\mu}}{\nabla S} = \frac{1}{2m}\underline{R}^{\mu}_{\ \nu\alpha\sigma}\, U^{\nu}\, S^{\alpha\sigma}, \tag{57}$$

and,

$$\frac{\nabla S^{\alpha\beta}}{\nabla S} = 0. \tag{58}$$

Consequently, the equation of motion of spinning fluid will have the form, using the Weyssenhoff tensor

$$\frac{\nabla S^{\alpha\beta\gamma}}{\nabla S} = \frac{1}{2m}\underline{R}^{\alpha}_{\ \mu\nu\sigma} S^{\nu\sigma}\, S^{\beta\gamma} U^{\mu}. \tag{59}$$

The spin density deviation tensor equation can be obtained, by applying the commutation relations as given in (48) and the condition (44), as follows:

$$\frac{\nabla^2 \Psi^{\alpha\beta\gamma}}{\nabla S^2} = S^{\rho[\beta\gamma} \underline{R}^{\alpha]}{}_{\rho\nu\sigma} U^\nu \Psi^\sigma + \frac{1}{2m} \left(\underline{R}^\alpha{}_{\mu\nu\sigma} S^{\nu\sigma} S^{\beta\gamma} U^\mu \right)_{||\lambda} \Psi^\lambda. \quad (60)$$

Meanwhile, for the V-components, by applying the variation for the above Lagrangian w.r.to Φ^a and Φ^{ab} , we get

$$\frac{D\underline{V}^a}{DS} = \frac{1}{2m} B^a{}_{bcd} V^b \underline{S}^{cd}, \quad (61)$$

$$\frac{D\underline{S}^{ab}}{DS} = 0. \quad (62)$$

Then, the equations of motion for spinning fluid and spin density deviation will have the form

$$\frac{DS^{abc}}{DS} = \frac{1}{2m} B^a{}_{def} \underline{S}^{bc} \underline{S}^{ef} V^d. \quad (63)$$

$$\frac{D^2 \Psi^{abc}}{DS^2} = \underline{S}^{e[bc} B^{a]}{}_{ebc} V^b \Phi^c + \frac{1}{2m} \left(B^a{}_{def} \underline{S}^{bc} \underline{S}^{ef} V^d \right)_{|l} \Phi^l. \quad (64)$$

(ii) Case $P^\alpha \neq mU^\alpha$ and $\underline{P}^a \neq mV^a$:

$$L = g_{\mu\nu}(x) P^\mu \frac{\nabla \Psi^\nu}{\nabla S} + S_{\mu\nu} \frac{\nabla \Psi^{\mu\nu}}{\nabla S} + h_{\mu\nu}(y) \underline{P}^a \frac{D\Phi^b}{DS} + \underline{S}_{ab} \frac{D\Phi^{ab}}{DS} + \frac{1}{2m} \underline{R}_{\mu\nu\alpha\beta} S^{\alpha\beta} U^\nu \Psi^\mu + \frac{1}{2m} B_{abcd} \underline{S}^{cd} U^b \Phi^a + 2P_\mu U_\nu \Psi^{\mu\nu} + 2\underline{P}_a V_b \Phi^{ab}. \quad (65)$$

Thus, by taking the variation w.r.to $\Psi^\alpha$and Φ^a, we get

$$\frac{\nabla P^\mu}{\nabla S} = \frac{1}{2m} \underline{R}^\mu{}_{\nu\alpha\sigma} U^\nu S^{\alpha\sigma}, \quad (66)$$

and

$$\frac{D\underline{P}^a}{DS} = \frac{1}{2m} B^a{}_{bcd} V^b \underline{S}^{cd}, \quad (67)$$

Also, taking the variation for Lagrangian (64) w.r.to $\Psi^{\alpha\beta}$and Φ^{ab}, we obtain

$$\frac{\nabla S^{\alpha\beta}}{\nabla S} = 2P^\alpha U^\beta, \quad (68)$$

and

$$\frac{D\underline{S}^{ab}}{DS} = 2P^a V^b. \quad (69)$$

Accordingly, the equation of motion of spinning equation will have the form

$$\frac{\nabla^2 S^{\alpha\beta\gamma}}{\nabla S^2} = 2P^\alpha P^\beta U^\gamma + \frac{1}{2m} \underline{R}^\alpha{}_{\mu\nu\sigma} S^{\nu\sigma} S^{\beta\gamma} U^\mu, \quad (70)$$

and

$$\frac{DS^{abc}}{DS} = 2P^a P^b V^c + \frac{1}{2m} B^a{}_{def} V^d \underline{S}^{cd} \underline{S}^{ef}, \quad (71)$$

The spin density deviation tensor equations can be obtained, by applying the commutation relations as given in (48) and the condition (44), as follows:

$$\frac{\nabla^2 \Psi^{\alpha\beta\gamma}}{\nabla S^2} = S^{\rho[\beta\gamma} \underline{R}^{\alpha]}{}_{\rho\nu\sigma} U^\nu \Psi^\sigma + \left(2P^\alpha P^\beta U^\gamma + \frac{1}{2m} \underline{R}^\alpha{}_{\mu\nu\sigma} S^{\nu\sigma} S^{\beta\gamma} U^\mu \right)_{||\rho} \Psi^\rho, (72)$$

and

$$\frac{D^2\Phi^{abc}}{DS^2} = S^{e[bc}B^{a]}{}_{edf}\, V^d\, \Phi^f + \left(2P^aP^bV^c + \frac{1}{2m}B^a{}_{def}\, V^d\, \underline{S}^{cd}\underline{S}^{ef}\right)_{|\rho} \Phi^\rho. \quad (73)$$

5. Equations of Motion for a Spinning Fluid with a Variable Mass

In case of a variable mass the Weyssenhoff tensor will be written as (cf. [4])

$$\hat{S}^{\alpha\beta\gamma} = m(s)S^{\beta\gamma}U^\alpha, \quad (74)$$

where $m(s)$ is chosen to represents variable mass which is a function of the parameter s.

In what follows, we are going to derive spinning density tensor and spinning density deviation tensor equations for a fluid with a variable mass in the context of Finsler geometry, Cartan-Rund Approach and Finsler-Cartan Approach.

5.1. Cartan-Rund Approach

The Lagrangian for spinning variable mass can be written as:

$$L = m(s)g_{\alpha\beta}(x,y)U^\alpha \frac{\underline{\delta}\Psi^\beta}{\underline{\delta}s} + S_{\alpha\beta}\,\frac{\underline{\delta}\Psi^{\alpha\beta}}{\underline{\delta}s} + \left(m(s)_{,\sigma} + \frac{1}{2m(s)}K_{\sigma\nu\alpha\beta}\,S^{\nu\alpha\beta}\right)\Psi^\sigma. (75)$$

By applying the variation for Lagrangian (74) with respect to Ψ^μ and $\Psi^{\mu\nu}$, we get

$$\frac{\underline{\delta}U^\mu}{\underline{\delta}s} = \frac{m(s)_{,\nu}}{m(s)}(g^{\mu\nu} - U^\mu U^\nu) + \frac{1}{2m(s)}K^\mu{}_{\nu\rho\sigma}\,S^{\nu\rho\sigma}. \quad (76)$$

and

$$\frac{\underline{\delta}S^{\mu\nu}}{\underline{\delta}s} = 0. \quad (77)$$

Accordingly, the equation of motion of spinning fluid for a variable mass has the form:

$$\frac{\underline{\delta}\hat{S}^{\alpha\beta\gamma}}{\underline{\delta}s} = \left(\frac{m(s)_{,\nu}}{m(s)}(g^{\alpha\nu} - U^\alpha U^\nu) + \frac{1}{2m(s)}K^\alpha{}_{\nu\rho\sigma}\,S^{\nu\rho\sigma}\right) S^{\beta\gamma}. \quad (78)$$

Following the same steps for deriving the spinning deviation equation mentioned in the above section, and then we can obtain the spinning deviation equation for a variable mass to have the form

$$\frac{\underline{\delta}^2\Psi^{\alpha\beta\gamma}}{\underline{\delta}s^2} = S^{\rho[\beta\gamma}K^{\alpha]}{}_{\rho\nu\sigma}\,U^\nu\,\Psi^\sigma + \left(\frac{m(s)_{,\nu}}{m(s)}(g^{\alpha\nu} - U^\alpha U^\nu)S^{\beta\gamma} + \frac{1}{2m(s)}K^\alpha{}_{\mu\nu\sigma}\,S^{\mu\nu\sigma}\,S^{\beta\gamma}\right)_{\underline{;}\rho} \Psi^\rho. \quad (79)$$

5.2. Finsler-Cartan Approach

We suggest the Lagrangian for spinning variable mass in the context of Finsler-Cartan Approach to have the form:

$$L = m(s) g_{\mu\nu}(x) U^{\mu} \frac{\nabla \Psi^{\nu}}{\nabla S} + S_{\mu\nu} \frac{\nabla \Psi^{\mu\nu}}{\nabla S} + m(s) h_{ab}(y) V^{a} \frac{D\Phi^{b}}{DS} + \underline{S}_{ab} \frac{D\Phi^{ab}}{DS} + \left(m(s)_{,\mu} + \frac{1}{2} \underline{R}_{\mu\nu\alpha\beta} S^{\nu\alpha\beta}\right) \Psi^{\mu} + \left(m(s)_{,a} + \frac{1}{2} B_{abcd} \underline{S}^{bcd}\right) \Phi^{a}. \quad (80)$$

By operating the variation for the above Lagrangian (79) w.r.to Ψ^{α} and $\Psi^{\alpha\beta}$, we get the following equations

$$\frac{\nabla \underline{U}^{\mu}}{\nabla s} = \frac{m(s)_{,\nu}}{m(s)} \left(g^{\mu\nu}(x) - U^{\mu} U^{\nu}\right) + \frac{1}{2m(s)} \underline{R}^{\mu}_{\ \nu\alpha\sigma} S^{\nu\alpha\sigma}, \quad (81)$$

and,

$$\frac{\nabla S^{\alpha\beta}}{\nabla s} = 0. \quad (82)$$

Using the Weyssenhoff tensor for variable mass given by (73), then, the equation of motion for spinning fluid can be written as,

$$\frac{\nabla \hat{S}^{\alpha\beta\gamma}}{\nabla s} = \left(\frac{m(s)_{,\nu}}{m(s)} \left(g^{\mu\nu}(x) - U^{\mu} U^{\nu}\right) + \frac{1}{2m(s)} \underline{R}^{\alpha}_{\ \mu\nu\sigma} S^{\mu\nu\sigma}\right) S^{\beta\gamma}. \quad (83)$$

The spin density deviation tensor equation will have the form:

$$\frac{\nabla^{2} \Psi^{\alpha\beta\gamma}}{\nabla s^{2}} = S^{\rho[\beta\gamma} \underline{R}^{\alpha]}_{\ \rho\nu\sigma} U^{\nu} \Psi^{\sigma} + \left(\left(\frac{m(s)_{,\nu}}{m(s)} \left(g^{\mu\nu}(x) - U^{\mu} U^{\nu}\right) + \frac{1}{2m(s)} \underline{R}^{\alpha}_{\ \mu\nu\sigma} S^{\mu\nu\sigma}\right) S^{\beta\gamma} \right)_{||\lambda} \Psi^{\lambda}. \quad (84)$$

While, for the V-components, we perform variation for Lagrangian (79) w.r.to Φ^{a} and Φ^{ab} , to obtain

$$\frac{D \underline{V}^{a}}{Ds} = \frac{m(s)_{,b}}{m(s)} \left(h^{ab}(y) - V^{a} V^{b}\right) + \frac{1}{2m(s)} B^{a}_{\ bcd} S^{bcd}, \quad (85)$$

and

$$\frac{D \underline{S}^{ab}}{Ds} = 0. \quad (86)$$

Consequently, the equations of motion for spinning fluid and spin density deviation will have the form

$$\frac{D \hat{S}^{abc}}{Ds} = \left(\frac{m(s)_{,d}}{m(s)} \left(h^{ad}(y) - V^{a} V^{d}\right) + \frac{1}{2m(s)} B^{a}_{\ efd} S^{efd}\right) \underline{S}^{bc}. \quad (87)$$

$$\frac{D^{2} \Psi^{abc}}{Ds^{2}} = \underline{S}^{e[bc} B^{a]}_{\ ebc} V^{b} \Phi^{c} + \left(\left(\frac{m(s)_{,d}}{m(s)} \left(h^{ad}(y) - V^{a} V^{d}\right) + \frac{1}{2m(s)} B^{a}_{\ efd} S^{efd}\right) \underline{S}^{bc} \right)_{|l} \Phi^{l}. \quad (88)$$

6. Condition of Stability for Spinning Fluid

In 1995, Wanas and Bakry used the deviation vector as an indicator of stability of gravitating systems. This method has a vital issue to become independent of any types of coordinate systems [21, 22].

Inspiring by the idea that the deviation vector can reflects the reaction of the system under perturbation. Their criterion to study the stability of any gravitating system was determined by evaluating the following quanitity

$$A \stackrel{\text{def}}{=} \sqrt{\Psi^{\mu}\Psi_{\mu}}\,, \tag{89}$$

where, $\Psi^{\mu}(s)$ is the deviation vector in a given interval $[a,b]$. The system under consideration was unstable if $A \to \infty$, otherwise it is stable.

This idea has been extended to study spinning objects with precession [23]. The criteria of stability for study spinning objects with precession are

$$A \stackrel{\text{def}}{=} \sqrt{\Psi^{\mu}\Psi_{\mu}}\,, \tag{90}$$

and

$$\underline{A} \stackrel{\text{def}}{=} \sqrt{\Psi^{\mu\nu}\Psi_{\mu\nu}}\,, \tag{91}$$

since, $\Psi^{\mu\nu}(s)$ is the deviation tensor. Thus, the system under consideration was unstable if $A \to \infty$ and $\underline{A} \to \infty$, otherwise it is stable. But if each of A& $\underline{A}$ vanishes this is the indicator for strong stability.

In what follows, we will suggest a general formula which can be used to study the stability of spinning fluid being in strong gravitating systems geometrically.

6.1. Cartan-Rund Approach

We suggest that the condition of stability for spinning fluid using the spin density deviation tensor $\Psi^{\mu\nu\alpha}(s)$, as follows:

$$\tilde{A} \stackrel{\text{def}}{=} \sqrt{\Psi^{\mu\nu\alpha}\Psi_{\mu\nu\alpha}}\,, \tag{92}$$

In order to study stability for spinning fluid implies determining both quantities for each $\underline{A}$& $\widetilde{A}$. Accordingly, the assigned gravitating system is becoming unstable if $\underline{A} \to \infty$ *or* $\widetilde{A} \to \infty$, otherwise it is stable. While the condition of strong stability implies that

$$\sqrt{\Psi^{\mu\nu\alpha}\Psi_{\mu\nu\alpha}} = 0, \qquad \& \qquad \sqrt{\Psi^{\mu\nu}\Psi_{\mu\nu}} = 0,$$

provided that $\sqrt{\Psi^{\mu}\Psi_{\mu}} = 0.$

6.2. Finsler-Cartan Approach

We suggest that the stability condition in this approach is connected to the magnitude of spin density deviation tensor horizontal and vertical coordinate, as follows:

$$\tilde{A} \stackrel{\text{def}}{=} \sqrt{\Psi^{\mu\nu\alpha}\Psi_{\mu\nu\alpha}}\,, \tag{93}$$

and

$$\underline{\tilde{A}} \stackrel{\text{def}}{=} \sqrt{\Phi^{abc}\Phi_{abc}}\,. \tag{94}$$

To study stability for spinning fluid implies determining both quantities $\underline{A}, \tilde{A}$ *and* $\underline{\tilde{A}}$. The gravitating system is unstable if $\underline{A} \to \infty$ *or* $\underline{\tilde{A}} \to \infty$, otherwise the system is stable. Also, the condition of strong stability can be written as:

$$\sqrt{\Psi^{\mu\nu\alpha}\Psi_{\mu\nu\alpha}} = 0, \qquad \& \qquad \sqrt{\Psi^{\mu\nu}\Psi_{\mu\nu}} = 0,$$

provided that $$\sqrt{\Psi^{\mu}\Psi_{\mu}} = 0.$$

And the following conditions

$$\sqrt{\Phi^{abc}\Phi_{abc}} = 0 \qquad \& \qquad \sqrt{\Phi^{ab}\Phi_{ab}} = 0$$

provided that $$\sqrt{\Phi^{a}\Phi_{a}} = 0.$$

7. Concluding Remarks

In the present article, we review briefly the main features of two different classes of Finsler geometry. This can be found by obtaining the problem of motion for objects defined in these presented classes. Accordingly, we obtain the spinning and spinning deviation equations for both Finslerian approaches. Such a vital result has been introduced by developing the equations of motion for a spinning fluid with a variable mass. These sets of equations have a vital role in studying dark matter, which may contribute as one of candidates of comprehending its meaning. Not only that but also in examining the mass production for particles nearby strong fields.

Such a tendency of work may through some light on the way to accumulate mass nearby strong fields of gravity, as well as becoming one of methods to explain the problem of dark matter within different geometries apart from the Riemannian ones. Such an issue is quite interesting to be examined in Finsler geometry.

Finally, we have extended the scheme of testing the stability problem of a fluid orbiting such a strong gravitational field. Such a type of work has been inspired by using the Wanas-Bakry method [21, 22] and discussed by Kahil in [6] and [23].

Yet, it is very essential to implement the Finslerian geometry to define a specific theory of gravity able to describe strong fields. As we know that the orthodox general theory of relativity is uncapable to examine such regions. These conditions will lead us to solve the stability problem for different objects orbiting strong field by means of obtaining their deviation vectors and tensors respectively. Such a process will be assigned in our future work.

8. Recommendations

The present work may be extended to examine different versions of bi-metric theories of gravity in the presence some classes of the Finslerian geometry to solve the problem of motion for different objects to get a better accuracy to reveal the possible discrepancies in the Universe. Such an approach will be considered in our future work.

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