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THE EXISTENCE OF MILD SOLUTION TO NON-INSTANTANEOUS IMPULSES FRACTIONAL DIFFERENTIAL EVOLUTION EQUATION WITH MEASURE OF NON COMPACTNESS

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Abstract. In this Paper, we are going to study the existence results for the non-instantaneous impulses fractional differential evolution equation by using measure of non compactness. The theory of operator semigroups, probability density function, Mönch - fixed point theorem, are the main tools of our study results for this problem . Lastly, an example is provided to illustrate the results.

1. Introduction

Fractional differential equations appear naturally in several fields such as physics, engineering, biophysics, blood flow phenomena, aerodynamics, electron-analytical chemistry, biology, and economy. For more details, we refer the readers to [4, 5, 28] and many other references therein control theory, etc. An excellent account in the study of fractional differential equations.

Impulsive effects arise from the real world and are used to describe sudden, discontinuous jumps. A differential equation with no instantaneous impulses is a generalization of the classical theory of impulsive differential equations. For some general and recent works on the theory of impulsive differential equations, we refer the readers to [24, 25, 30].

The existence of solutions of the non-instantaneous impulsive problem has been studied through some approaches, such as fixed point and analytic semigroup theories. For more information, look at. [22, 23, 31, 32]. Recently, the variational structure of non-instantaneous impulsive linear problems has been developed in [29]. Among the essential applications for fractional differential, we find control of turbines and satellite images, satellite imaging is shifting from the photo-interpreter era to one of automatic monitoring. Indeed, the vast amount of data provided by the recent constellations of satellites, performing recurrent observation of every point on the globe, can only be handled by automatic methods, controlling false detections is thus crucial. The low costs of those satellites often imply lower resolution, the fusion of multi-date images can compensate to some extent for the low resolution. Given their future role in the energetic transition and their spread over countries or continents, monitoring wind turbines is a natural candidate for such studies. This work details an algorithm for automatic, multi-date wind turbine detection on low-resolution optical satellite images. The method is based on the contrary statistical approach to provide control of false detections and exploits the geometry of wind turbines' shadows and hubs, look at. [41, 42, 43, 44].

Recently, Hernàndez and O'Regan [22], started a study on the Cauchy problem for a new type first-order evolution equation with no-instantaneous impulses of the form:

$$\begin{cases} u'(t) = Au(t) + f(t, u(t), t \in (s_i, t_{i+1}], i = 0, 1, \dots, m, \\ u(t) = g_i(t, u(t)), t \in (t_i, s_i], i = 1, 2, \dots, m, \\ u(0) = x_0 \in X, \end{cases}$$

where $A : D(A) \subset X \longrightarrow X$ is the generator of a C_0-semigroup of bounded linear operators $(T(t))t \geq 0$ defined on a Banach space $(X, \|\,.\,\|)$, $x_0 \in X, 0 = t_0 = s_0 < t_1 \leq s_1 \leq t_2 < \dots < t_N \leq s_N \leq t_N + 1 = a$ are pre-fixed numbers,

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$g_i \in C((t_i, s_i] \times X, X)$ for all $i = 1, \ldots, N$ and $f : [0, a] \times X \longrightarrow X$ is a suitable function.
Michal Feãkan, JinRong Wang, and Yong Zhou [27], have considered Periodic Solutions for Nonlinear Evolution Equations with non-instantaneous Impulses of the form:

$$\begin{cases} u'(t) = Au(t) + f(t, u(t), t \in (s_i, t_{i+1}], i = 0, 1, \ldots, \infty, \\ u(t_i^+) = g_i(t, u(t_i^-)), i = 1, 2, \ldots, \infty, \\ u(t) = g_i(t_i, u(t_i^-)), t \in (t_i, s_i], i = 1, 2, \ldots, \infty, \\ u(0) = \overline{x}, \end{cases}$$

where the fixed points s_i and t_i satisfy $0 = s_0 < t_1 \leq s_1 \leq t_2 < \ldots < t_m \leq s_m \leq t_{m+1} \leq \ldots$ with $\lim_{i \to \infty} t_i = \infty$, and $t_{i+m} = t_i + T, s_{i+m} = s_i + T, m \in N$ denoted the number of impulsive points between 0 and T. Moreover, $f : [0, \infty) \times X \longrightarrow X$ is a T-periodic, with respect to $t \in [0, \infty)$, Carathéodory function and $g_i : [t_i, s_i] \times X \longrightarrow X$ is a continuous function for all $i = 1, 2, \ldots, \infty$ with $g_{i+m} = g_i$.
Melliani et al, [34] have considered, a general class of periodic boundary value problems for controlled non-linear impulsive evolution equations on Banach spaces:

$$\begin{cases} u'(t) = Au(t) + f(t, u(t), u(\rho(t))) + B(t)c(t), t \in (s_i, t_{i+1}], i = 0, 1, 2, \ldots, m, c \in \mathcal{U}_{ad}, \\ u(t) = T(t - t_i)g_i(t, u(t)), t \in (t_i, s_i], i = 1, 2, \ldots, m, \\ u(0) = u(a) \in X. \end{cases}$$

The operator $A : D(A) : X \longrightarrow X$ is the generator of a strongly continuous semigroup $\{T(t), t \geq 0\}$ on a Banach space X with a norm $\|.\|$, and the fixed points s_i and t_i satisfying $0 = s_0 < t_1 \leq s_1 \leq t_2 < \ldots < t_m \leq s_m \leq t_{m+1} = a$ are pre-fixed numbers, $f : [0, a] \times X \times X \longrightarrow X$ is continuous, $\rho : [0, a] \longrightarrow [0, a]$ is continuou, and $g_i : [t_i, s_i] \times X \longrightarrow X$ is continuous for all $i = 1, 2, \ldots, m$.
Pradeep Kumar, Dwijendra N. Pandey, D. Bahuguna [26], have considered the following impulsive fractional differential equations in a Banach space $(H, \|.\|)$ for which impulses are no instantaneous:

$$\begin{cases} {}^C D_t^\alpha u(t) + Au(t) = f(t, u(t), u(g(t))), t \in (s_i, t_{i+1}], i = 0, 1, \ldots, N, \\ u(t) = h_i(t, u(t)), t \in (t_i, s_i], i = 1, 2, \ldots, N, \\ u(0) = u_0 \in H, \end{cases}$$

where ${}^C D_t^\alpha$ is the Caputo fractional derivative of order β, $-A$ is the infinitesimal generator of an analytic semigroup of bounded linear operators, $\{S(t), t \geq 0\}$ on a Banach space H, the impulses start suddenly at the points t_i and their action continues on the interval $[t_i, s_i]$, $0 = t_0 = s_0 < t_1 \leq s_1 \leq t_2 <, \ldots, < t_N \leq s_N \leq t_{N+1} = T_0$, the functions $h_i \in C((t_i, s_i] \times H, H)$ for each $i = 1, 2, \ldots, N, g : [0, T_0] \longrightarrow [0, T_0]$ and $f : [0, T_0] \times H \times H \longrightarrow H$ are suitable functions.
The main techniques relay on the impulsive integrodifferential equations, Mönch fixed point theorem via measure of noncompactness.
K. Malar, A. Anguraj [33], have studied the Existence Results of Abstract Impulsive Integrodifferential Systems with Measure of Non-compactness:

$$\begin{cases} u'(t) = Au(t) + f(t, u(t), \int_0^t \rho(t, s)h(t, s, u(s))ds)), t \in (s_i, t_{i+1}], i = 0, 1, \ldots, m \\ u(t) = g_i(t, u(t)), t \in (t_i, s_i], i = 1, 2, \ldots, m \\ u(0) = u_0 + k(u), \end{cases}$$

where A generate a C_0- semi group of bounded linear operator $\{T(t), t \geq 0\}$ defined on a Banach space $(X, \|.\|)$. $u_0 \in X, 0 = t_0 = s_0 < t_1 \leq s_1 \leq t_2 <, \ldots, < t_n \leq s_n \leq t_{n+1} = b$, are prefixed numbers, $k : X \longrightarrow X$, and $g_i \in C((t_i, s_i] \times X, X)$ for all $i = 1, 2, \ldots, n$, f is a given function $f : [0, b] \times X \times X \longrightarrow X$ and $h \in C(D, \mathbb{R}^+)$, $D = \{(t, s) | t, s \in [0, b], t \geq s\}$.
In this paper, we investigate the existence of a mild solution for controlled nonlinear evolution equations with non-instantaneous impulses:

$$\begin{cases} D^\alpha u(t) = Au(t) + f(t, u(t), u(\rho(t)) + B(t)c(t), t \in (s_i, t_{i+1}], i = 0, 1, \ldots, n, c \in U_{ad} \\ u(t) = S_\alpha(t - t_i) + g_i(t, u(t)), t \in (t_i, s_i], i = 1, 2, \ldots, n \\ u(0) = u_0 + k(u), \end{cases} \tag{1.1}$$

where ${}^C D_t^\alpha$ is the Caputo fractional derivative of order $\alpha \in (0, 1)$. A is the infinitesimal generator of an analytic semigroup of bounded linear operators, $\{T(t), t \geq 0\}$ on a Banach space $(X, \|.\|)$. $u_0 \in X, 0 = t_0 = s_0 < t_1 \leq s_1 \leq t_2 <, \ldots, < t_n \leq s_n \leq t_{n+1} = b$, are pre - fixed numbers, $k : X \longrightarrow X$, and $g_i \in \mathcal{C}((t_i, s_i] \times X, X)$, $B : [0, b] \longrightarrow \mathcal{L}(Y, X)$ and $I_i : X \longrightarrow X$ for all $i = 1, 2, \ldots, n$, f is a given function $f : [0, b] \times X \times X \longrightarrow X$ and $h \in \mathcal{C}(D, \mathbb{R}^+)$, $D = \{(t, s) | t, s \in [0, b], t \geq s\}$.
This paper is organized as follows: in the second section, we recall some notations and several known results.

In the third section, we present our main results on the existence of solutions of the problem above. In the fourth section, we give an application to demonstrate our main results.

2. Preliminaries

Next, we review some basic concepts, notations, and technical results that are necessary for our study. Let $\mathcal{C}([0,b],X)$ denote the Banach space of all continuous functions from $[0,b]$ into X with the norm $\|u\|_{\mathcal{C}} := \sup\{|u(t)| : t \in [0,b]\}$ for $u \in \mathcal{C}([0,b],X)$. A C_o- semi-group $T(t)$ is said to be compact if $T(t)$ is compact for $t > 0$. If the semi-group $T(t)$ is compact, then $t \longrightarrow T(t)u$ are equi-continuous at all $t > 0$ with respect to u in all bounded subset of X, that is, the semi-group $T(t)$ is equi-continuous.
We consider the space:

$$\mathcal{PC}(J,X) = \{u : J \longrightarrow X : u \in C((t_i,t_{i+1}],X), i = 0,1,\dots,n$$
$$\text{and there exist } u(t_i^-) \text{ and } u(t_i^+), i = 1,\dots,n \text{ with } u(t_i^-) = u(t_i)\},$$

with the norm $\|u\|_{\mathcal{PC}} := \sup\{|u(t)| : t \in [0,b]\}$.
Let Y be another separable reflexive Banach space where the controls c take values. Denote by $P_f(y)$ a class of nonempty closed and convex subsets of Y. We suppose that the multi-valued map $w : [0,T] \longrightarrow P_f(y)$ is measurable, $w(.) \subset E$, where E is a bounded set of Y, and the admissible control set

$$\mathcal{U}_{ad} = \{c \in L^p(E) : c(t) \in w(t), \text{ a.e.}\}, p > 1.$$

Then $\mathcal{U}_{ad} \neq \emptyset$, which can be found in [3]. Some of our results are proved using the next well-known results.

Definition 2.1. [15] *The Riemann-Liouville fractional integral of order α with the lower limit zero for a function f is defined as*

$$I^\alpha f(t) = \int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} f(s)ds, \quad \alpha > 0,$$

provided the integral exists, where Γ is the gamma function.

Definition 2.2. [15] *The Riemann-Liouville derivative of order α with the lower limit zero for a function $f : [0,\infty) \longrightarrow \mathbb{R}$ can be written as*

$$({}^L D^\alpha f)(t) = \frac{1}{\Gamma(n-\alpha)} \left(\frac{d}{dt}\right)^n \int_0^t \frac{(t-s)^{n-\alpha-1}}{\Gamma(\alpha)} f(s)ds \quad n-1 < q < n, t > 0.$$

Definition 2.3. [15] *For a function h given on the interval $[a,b]$, the Caputo fractional-order derivative of f, is defined by*

$$({}^c D_{a^+}^\alpha f)(t) = \frac{1}{\Gamma(n-\alpha)} \int_a^t \frac{(t-s)^{n-\alpha-1}}{\Gamma(\alpha)} f^{(n)}(s)ds$$

where $n = [\alpha] + 1$ and $[\alpha]$ denotes the integer part of α.

Lemma 2.1. [5] *Let $\alpha > 0$ and $x \in C(0,T) \cap L(0,T)$. Then the fractional differential equation*

$$D^\alpha x(t) = 0$$

has a unique solution

$$x(t) = k_1 t^{\alpha-1} + k_2 t^{\alpha-2} + \dots + k_n t^{\alpha-n},$$

where $k_i \in \mathbb{R}$, $i = 1,2,\dots,n$, and $n-1 < \alpha < n$.

Lemma 2.2. [5] *Let $\alpha > 0$. Then for $x \in C(0,T) \cap L(0,T)$ we have*

$$I^\alpha D^\alpha x(t) = x(t) + c_0 + c_1 t + \dots + c_{n-1} t^{n-1},$$

fore some $c_i \in \mathbb{R}$, $i = 1,2,\dots,n-1$. Where $n = [\alpha] + 1$.

Next, we introduce the Hausdorff's measure of noncompactness $\mu(.)$ defined on each bounded subset Ω of Banach space Y by

$$\mu(\Omega) = \inf\{\varepsilon > 0, \Omega \text{ has a finite } \varepsilon- \text{ net in } Y\}.$$

Some basic properties of $\mu(.)$ are given in the following Lemma:

Lemma 2.3. *([1]). Let Y be a real Banach space and $B, C \subseteq Y$ be bounded, the following properties are satisfied:*

1. *B is pre-compact if and only if $\mu(B) = 0$,*
2. *$\mu(B) = \mu(\overline{B}) = \mu(convB)$, where $\overline{B}$ and $convB$ mean the closure and convex hull of B, respectively,*
3. *$\mu(B) \leq \mu(C)$ when $B \subseteq C$,*
4. *$\mu(B + C) \leq \mu(B) + \mu(C)$, where $B + C = \{x + y, x \in B, y \in C\}$,*

(5) $\mu(B \cup C) \leq \max\{\mu(B), \mu(C)\}$,
(6) $\mu(\lambda B) = |\lambda|\mu(B)$ *for any* $\lambda \in \mathbb{R}$,
(7) *If the map* $Q : D(Q) \subseteq Y \longrightarrow Z$ *is Lipschitz continuous with constant* k, *then* $\mu(QB) \leq k\mu(B)$ *for any bounded subset* $B \subseteq D(Q)$, *where* Z *be a Banach space.*
(8) $\mu(B) = \inf\{d(B,C), C \subseteq Y$ *be pre-compact*$\} = \inf\{d(B,C), C \subseteq Y$ *be finite valued*$\}$, *where* $d(B,C)$ *means the nonsymmetric (or symmetric) Hausdorff distance between* B *and* C *in* Y.
(9) *If* $\{W_n\}_{n=1}^{\infty}$ *is a decreasing sequence of bounded closed nonempty subsets of* Y *and* $\lim_{n\to\infty}\mu(W_n) = 0$, *then* $\cap_{n=1}^{\infty} W_n$ *is nonempty and compact in* Y.
The map $Q : W \subseteq Y \longrightarrow Y$ *is said to be an* μ*-contraction if there exists a positive constant* $0 < k < 1$ *such that* $\mu(QC) \leq k\mu(C)$ *for any bounded closed subset* $C \subseteq W$, *where* Y *is a Banach space.*

Lemma 2.4. *([1]). If* $W \subseteq \mathcal{PC}([0,K], X)$ *is bounded, then* $\mu(W(t)) \leq \mu(W)_{\mathcal{PC}}$ *for all* $t \in [0,K]$, *where* $W(t) = \{u(t) : u \in W\} \subseteq X$. *Furthermore if* W *is equicontinuous on* $[0,K]$, *then* $\mu(W(t))$ *is continuous on* $[0,K]$, *and* $\mu(W)_{PC} = sup\{\mu(W(t)) : t \in [0,K]\}$.

Lemma 2.5. *([2]). If* $\{u_n\}_{n=1}^{\infty} \subset L^1(0,K,X)$ *is uniformly integrable, then* $\mu(\{u_n\}_{n=1}^{\infty})$ *is measurable and*

$$\mu(\{\int_0^t u_n(s)ds\}_{n=1}^{\infty}) \leq 2\int_0^t \mu\{u_n(s)\}_{n=1}^{\infty} ds$$

Lemma 2.6. *([16]). If the semi-group* $T(t)$ *is equicontinuous and* $\eta \in L^1(0,K,\mathbf{R}^+)$, *then the set* $\{t \longrightarrow \int_0^t T(t - su(s)ds, u \in L^1(0,K,\mathbf{R}^+), \|u(s)\| \leq \eta(s)$, *is equicontinuous for* $s \in [0,K]\}$.

Lemma 2.7. *([8]). If* W *is bounded, then for each* $\varepsilon > 0$, *there is a sequence* $\{u_n\}_{n=1}^{\infty} \subset W$ *such that*

$$\mu(W) \leq 2\mu(\{u_n\}_{n=1}^{\infty}) + \varepsilon.$$

The following fixed point theorem, a nonlinear alternative of Mönch type, plays a key role of the problem (1.1).

Theorem 2.1. *(Mönch's Fixed Point Theorem)([11]) Let* D *be a bounded, closed and convex subset of a Banach space such that* $0 \in D$, *and let* N *be a continuous mapping of* D *into itself. If the implication:*

$$V = \overline{conv}N(V) \text{ or } V = N(V) \cup \{0\} \Longrightarrow \mu(V) = 0,$$

holds for every subset V *of* D, *then* N *has a fixed point.*

Theorem 2.2. *([2]) Let* D *be a closed convex subset of a Banach space* X *and* $0 \in D$. *Assume that* $F : D \longrightarrow X$ *is a continuous map which satisfies Mönch's condition, that is, (*$M \subseteq D$ *is countable,* $M \subseteq \overline{co}(\{0\} \cup F(M) \longrightarrow \overline{M}$ *is compact). Then* F *has a fixed point in* D.

Let us recall the following definition of mild solutions for the fractional evolution equation involving the Caputo fractional derivative.

Definition 2.4. *([19, 20]) A function* $x \in \mathcal{C}([0,b], X)$ *is said to be a mild solution of the following problem:*

$$\begin{cases} {}^cD^{\alpha}u(t) = Au(t) + y(t), t \in (0,b], \\ u(0) = u_0, \end{cases}$$

if it satisfies the integral equation

$$u(t) = \mathcal{P}_{\alpha}(t)u_0 + \int_0^t (t-s)^{\alpha-1}\mathcal{Q}_{\alpha}(t-s)y(s)ds.$$

Where

$$\mathcal{P}_{\alpha}(t) = \int_0^{\infty} \xi_{\alpha}(\theta)T(t^{\alpha}\theta)d\theta, \qquad \mathcal{Q}_{\alpha}(t) = \alpha\int_0^{\infty} \theta\xi_{\alpha}(\theta)T(t^{\alpha}\theta)d\theta, \tag{2.1}$$

$$\xi_{\alpha}(\theta) = \frac{1}{\alpha}\theta^{-1-\frac{1}{\alpha}}\overline{\omega}_{\alpha}(\theta^{-\frac{1}{\alpha}}) \geq 0,$$

$$\overline{\omega}_{\alpha}(\theta) = \frac{1}{\pi}\sum_{n=1}^{\infty}(-1)^{n-1}\theta^{-n\alpha-1}\frac{\Gamma(n\alpha+1)}{n!}\sin(n\pi\alpha), \qquad \theta \in (0,\infty), \tag{2.2}$$

and ξ_{α} *is a probability density function defined on* $(0,\infty)$ [10], *that is,*

$$\xi_{\alpha}(\theta) \geq 0, \theta \in (0,\infty), \int_0^{\infty} \xi_{\alpha}(\theta)d\theta = 1.$$

It is not difficult to verify that

$$\int_0^\infty \theta\xi_\alpha(\theta)d\theta = \frac{1}{\Gamma(1+\alpha)}.$$

Remark 2.1. *By applying the Laplace transform and probability density functions, Zhou and Jiao* [19, 20] *introduced the above definition of mild solutions for fractional evolution equations. For pioneering work on Caputo fractional evolution equations, we refer the readers to* [12, 13].

We make the following assumption on A in the whole paper.
$H(A)$: The operator A generators a strongly continuous semigroup $\{T(t) : t \geq 0\}$ in X, and there is a constant $M_A \geq 1$ such that $\sup_{t\in[0,\infty)}\{|T(t)|_{L(X)}\} \leq M_A$.

Definition 2.5. [40] *We say that a function $u \in \mathcal{PC}([0,b],X)$ is called a mild solution of Cauchy problem:*

$$\begin{cases} D^\alpha u(t) = Au(t) + f(t,u(t),u(\rho(t)) + B(t)c(t), \quad a.e. \quad t \in (s_i,t_{i+1}], i = 0,1,\ldots,n, c \in \mathcal{U}_{ad} \quad 0<\alpha<1 \\ u(t) = S_\alpha(t-t_i)g_i(t,u(t)), t \in (t_i,s_i], i = 1,2,\ldots,n \\ u(0) = u_0 + k(u), \end{cases} \tag{2.3}$$

if u satisfies

$$u(t) = \begin{cases} P_\alpha(t)(u_0+k(u)) + \int_0^t (t-s)^{\alpha-1}Q_\alpha(t-s)[f(s,u(s),u(\rho(s)) + B(s)c(s)]ds, \quad t \in [0,t_1] \\ S_\alpha(t-t_i)g_i(t,u(t)), t \in (t_i,s_i], i = 1,2,\ldots,n \\ P_\alpha(t-s_i)d_i + \int_0^t (t-s)^{\alpha-1}Q_\alpha(t-s)[f(s,u(s),u(\rho(s)) + B(s)c(s)]ds, t \in [s_i,t_{i+1}], \end{cases}$$

where, for $i = 1,2,\cdots,n$,

$$d_i \;=\; I_i(u(t_i)) + g_i(s_i,u(s_i)) - \int_0^{s_i} (s_i-s)^{\alpha-1}Q_\alpha(s_i-s)[f(s,u(s),u(\rho(s)) + B(s)c(s)]ds \tag{2.4}$$

3. The Main Results

In this section, we see the existence of solutions for problem (1.1) by applying theorem 2.2. For some real constants r, we define

$$W = \{u \in \mathcal{PC}([0,b],X), \|u(t)\|_{\mathcal{PC}} \leq r, \forall t \in [0,b]\}.$$

Now we introduce the following hypotheses:

- (H_1) (i) The C_0-semi group $T(t)$ generated by A is equicontinuous and $M_A = \sup\{|T(t)| : t \in [0,b]\}$.
  (ii) $B : [0,b] \longrightarrow \mathcal{L}(Y,X)$ is essentially bounded, $B \in \mathcal{L}^\infty([0,b],\mathcal{L}(Y,X))$.
- (H_2) (i) The functions g_i are continuous and there are positive constants L_{g_i} such that $\|g_i(t,u) - g_i(t,v)\| \leq L_{g_i}[\|u-v\|]$, for all $u,v \in X$, $t \in (t_i,s_i]$ and each $i = 0,1,2\ldots,n$.
  (ii) There are positive constants $M_i > 0$ such that $\|g_i(t,u)\| \leq M_i\|u\|$ for all $u \in X$, $t \in (t_i,s_i]$ and each $i = 0,1,\ldots,n$.
  (iii) For each bounded subset $B \subset X$ we have $\mu(g_i(t,B)) \leq M_i(\sup\mu(B(s_i)))$, $i = 0,1,\ldots,n$.
- (H_3) (i) There exists a function $m_f \in \mathcal{C}([0,b],X)$ and a non-decreasing continuous function $\Omega_f : X \longrightarrow X$ such that $\|f(t,u,v)\| \leq m_f\Omega_f(\|u\| + \|v\|)$ for all, $u \in X$ a.e $t \in [0,b]$.
  (ii) There is an integrable function $\eta : [0,b] \longrightarrow [0,+\infty]$ such that $\mu(f(t,D_1,D_2) \leq \eta(t)[\sup_{-\infty<\theta\leq 0}\mu(D_1(\theta) + \mu(D_2)]$ for a.e $t \in [0,b]$. and any bounded subsets $D_1, D_2 \subset X$ and μ is the Hausdroff measure of non-compactness. Here we let $\int_0^t \eta(s)ds \leq \zeta^*$.
- (H_4) $k : X \longrightarrow X$ is continuous and there exists positive constants c and d such that $\|k(u) - k(v)\| \leq c\|u-v\|$ and $\|k(u)\| \leq c\|u\| + d$, for all $u \in \mathcal{PC}(X)$.
- (H_5) $f : [0,b] \times X \times X \longrightarrow X$ is of Carathéodory type, that is $f(.,u,Gu)$ is measurable for all $u \in X$ and $f(t,.,.)$ is continuous for a.e $t \in [0,b]$.
- (H_6) (i) The function $h(t,s,.) : X \longrightarrow X$ is continuous for $(t,s) \in \triangle$, and for each $u \in X$, the function $h(.,.,u) : \triangle \longrightarrow X$ is measurable. Moreover, there exists a function $\nu : \triangle \longrightarrow \mathbb{R}^+$ with $\sup\limits_{t\in[0,b]}\int_0^t \nu(t,s)ds :=$ $\nu^* < \infty$ such that $\|h(t,s,u)\| \leq \nu(t,s)\|u\|, u \in X$.
  (ii) For any bounded set $D_1 \in X$ and $0 \leq s \leq t \leq b$, there exists a functions $\psi : \Delta \longrightarrow \mathbb{R}^+$ such that $\mu(h(t,s,D_1)) \leq \Psi(s,t)\mu(D_1)$ where $\sup\limits_{t\in[0,b]}\int_0^t \Psi(s,t)ds := \Psi^*$.

Remark 3.1. *From the assumptions* $(H_1)-(H_3)$ *and the definition of* $\mathcal{U}_{ad}$*, it is also easy to verify that* $Bc \in \mathcal{L}^p([0,b],X)$ *with* $p>0$ *for all* $c \in \mathcal{U}_{ad}$.
Therefore, $Bc \in \mathcal{L}^1([0,b],X)$ *and* $\|Bc\|_{\mathcal{L}^1} < \infty$.

Theorem 3.1. *Assume that conditions* (H_A) *and* $(H_1)-(H_6)$ *hold. Then there exists at least one mild solution for problem* (1.1) *provided that :*

$$\lambda^* = M_A[(L_I + L_{g_i}) + \frac{2M_A}{\Gamma(\alpha+1)}\zeta^*(t_{i+1}^{\alpha} + M_A s_i^{\alpha} + 2l\psi^* t_{i+1}^{\alpha} + 2M_A l\psi^* s_i^{\alpha})] < 1 \tag{3.1}$$

Proof. We define a mapping $\Gamma : \mathcal{PC}([0,b],X) \longrightarrow \mathcal{PC}([0,b],X)$ by:

$$\Gamma u(t) = \begin{cases} P_\alpha(t)(u_0 + k(u)) + \int_0^t (t-s)^{\alpha-1} Q_\alpha(t-s)[f(s,u(s),\int_0^s \rho(s,\tau)h(s,\tau,u(\tau))d\tau) + B(s)c(s)]ds, \quad t \in [0,t_1] \\ I_i(u(t_i)) + g_i(t,u(t)), t \in (t_i,s_i], i = 1,2,\ldots,n \\ P_\alpha(t-s_i)d_i + \int_0^t (t-s)^{\alpha-1} Q_\alpha(t-s)[f(s,u(s),\int_0^s \rho(s,\tau)h(s,\tau,u(\tau))d\tau) + B(s)c(s)]ds, t \in [s_i,t_{i+1}], \\ i = 1,2,\ldots,n, \end{cases}$$

with d_i, $i = 1,2,\ldots,n$, defined by (2.4).
for all $u \in \mathcal{PC}([0,b],X)$, and show that the operator Γ satisfies the hypothesis of theorem 2.2. The proof consists of several steps.
step 1 : We show that the operator Γ is continuous.
Let $\{u_k\}$ be a sequence such that $u_k \longrightarrow u$ in $\mathcal{PC}([0,b],X)$. Then by (H_5), we have that
$f(s,u_k(s),\int_0^s \rho(s,\tau)h(s,\tau,u_k(\tau))d\tau) \longrightarrow f(s,u(s),\int_0^s \rho(s,\tau)h(s,\tau,u(\tau))d\tau), k \longrightarrow \infty$, for all $s \in [0,b]$.
Case 1: For $t \in [0,t_1]$, we have

$$\begin{aligned} \|\Gamma u_k(t) - \Gamma u(t)\| = \Big\| & \mathcal{P}_\alpha(t)(u_0 + k(u)) + \int_0^t (t-s)^{\alpha-1}\mathcal{Q}_\alpha(t-s)[f(s,u_k(s),\int_0^s \rho(s,\tau)h(s,\tau,u_k(\tau))d\tau) \\ & + B(s)c(s)]ds - \mathcal{P}_\alpha(t)(u_0+k(u)) - \int_0^t (t-s)^{\alpha-1}\mathcal{Q}_\alpha(t-s)[f(s,u(s),\int_0^s \rho(s,\tau)h(s,\tau,u(\tau))d\tau) \\ & - B(s)c(s)]ds\Big\| \\ & \le \frac{\alpha M_A}{\Gamma(\alpha+1)} \int_0^t (t-s)^{\alpha-1}\|f(s,u_k(s),\int_0^s \rho(s,\tau)h(s,\tau,u_k(\tau))d\tau) \\ & - f(s,u(s),\int_0^s \rho(s,\tau)h(s,\tau,u(\tau))d\tau)\|_{\mathcal{PC}} ds. \end{aligned}$$

Case 2 : For $t \in (t_i,s_i]$, $i = 1,\ldots,n$, we have

$$\begin{aligned} \|\Gamma u_k(t) - \Gamma u(t)\| &= \|I_i(u_k(t_i)) + g_i(t,u_k(t))) - I_i(u(t_i)) - g_i(t,u(t)))\| \\ &\le L_I\|u_k - u\| + L_{g_i}\|u_k - u\| \\ &\le (L_I + L_{g_i})\|u_k - u\|_{\mathcal{PC}}. \end{aligned}$$

Case 3 : For $t \in (s_i,t_{i+1}]$, $i = 1,\ldots,n$, we have

$$\begin{aligned} \|\Gamma u_k(t) - \Gamma u(t)\| \le \Big\| & \mathcal{P}_\alpha(t-s_i)\Big[I_i(u_k(t_i)) - I_i(u(t_i)) \\ & + g_i(s,u_k((s_i))) - g_i(s,u(s_i))) \int_0^{s_i} (t-s)^{\alpha-1}\mathcal{Q}_\alpha(_i - s)\Big(f(s,u_k(s),\int_0^s \rho(s,\tau)h(s,\tau,u_k(\tau))d\tau) \\ & - f(s,u(s),\int_0^s \rho(s,\tau)h(s,\tau,u(\tau))d\tau)\Big)ds\Big]\Big\| \\ & + \|\int_0^t (t-s)^{\alpha-1}\mathcal{Q}_\alpha(t-s)[f(s,u_k(s),\int_0^s \rho(s,\tau)h(s,\tau,u_k(\tau))d\tau) \\ & - f(s,u(s),\int_0^s \rho(s,\tau)h(s,\tau,u(\tau))d\tau)]ds\| \\ & \le M_A\Big[(L_I + L_{g_i})\|u_k - u\|_{\mathcal{PC}} + \frac{M_A}{\Gamma(\alpha+1)} s_i^{\alpha}\|(f(s,u_k(s),\int_0^s \rho(s,\tau)h(s,\tau,u_k(\tau))d\tau) \\ & - f(s,u(s),\int_0^s \rho(s,\tau)h(s,\tau,u(\tau))d\tau))\|_{\mathcal{PC}}\Big] \\ & + \frac{M_A}{\Gamma(\alpha+1)} t_{i+1}^{\alpha}\|[f(s,u_k(s),\int_0^s \rho(s,\tau)h(s,\tau,u_k(\tau))d\tau) - f(s,u(s),\int_0^s \rho(s,\tau)h(s,\tau,u(\tau))d\tau)]\|_{\mathcal{PC}}. \end{aligned}$$

Thus we infer that $\|\Gamma u_k - \Gamma u\|_{\mathcal{PC}} \longrightarrow 0$ as $k \longrightarrow \infty$,
which implies that the mapping Γ is continuous on $\mathcal{PC}([0,b],X)$.
step 2 : The operator Γ is bounded.
We claim that $\Gamma W \subseteq W$, for any $u \in W \subseteq \mathcal{PC}([0,b],X)$, by $(H_3)(i)$, we have
Case 1: Now, for every $t \in [0,t_1]$, we have:

$$\begin{aligned}\|\Gamma u(t)\| &\leq M_A(\|u_0\| + c\|u\| + d) + \frac{\alpha M_A}{\Gamma(\alpha+1)}\|\int_0^t [f(s,u(s),\int_0^s \rho(s,\tau)h(s,\tau,u(\tau))d\tau)) + B(s)c(s)]ds\| \\ &\leq M_A(\|u_0\| + cr + d) + \frac{M_A}{\Gamma(\alpha+1)} t_1^{\alpha}[m_f(s)\Omega_f(\|u\| + l\int_0^s \nu(s,\tau)d\tau\|u\|) + \|Bc\|_{\mathcal{L}^1}] \\ &\leq M_A(\|u_0\| + cr + d) + \frac{M_A}{\Gamma(\alpha+1)} t_1^{\alpha}[m_f(s)\Omega_f r(1 + l\nu^*) + \|Bc\|_{\mathcal{L}^1}] \leq r.\end{aligned}$$

Case 2: Now, for every $t \in (t_i, s_i]$, $i = 1,2,\ldots,n$, we get

$$\begin{aligned}\|\Gamma u(t)\| &\leq \|I_i(u(t_i))\| + \|g_i(s_i,u(s_i)\| \leq \phi_I\|u\| + M_i\|u\| \\ &\leq (\phi_I + M_i)\|u\| \leq r.\end{aligned}$$

Case 3: Now, for each $t \in (s_i, t_{i+1}]$, $i = 1,2,\ldots,n$, we get

$$\begin{aligned}\|\Gamma u(t)\| \leq &\Big\|\mathcal{P}_\alpha(t-s_i)\Big[I_i(u(t_i)) + g_i(s_i,u(s_i)) + \int_0^{s_i}(s_i-s)^{\alpha-1}\mathcal{Q}_\alpha(s_i-s)(f(s,u(s),\int_0^s \rho(s,\tau)h(s,\tau,u(\tau))d\tau)) \\ &+ Bc(s))ds\Big]\Big\| + \int_0^t (t-s)^{\alpha-1}\|\mathcal{Q}_\alpha(t-s)(f(s,u(s),\int_0^s \rho(s,\tau)h(s,\tau,u(\tau))d\tau)) + B(s)c(s))\|ds \\ &\leq M_A\Big[\phi_I r + M_i r + \frac{M_A}{\Gamma(\alpha+1)} s_i^{\alpha}([m_f(s)\Omega_f r(1+l\nu^*) + \|Bc\|_{\mathcal{L}^1}])\Big] \\ &+ \frac{M_A}{\Gamma(\alpha+1)} t_{i+1}^{\alpha}([m_f(s)\Omega_f r(1+l\nu^*) + \|Bc\|_{\mathcal{L}^1}]) \\ &\leq r.\end{aligned}$$

From above, we have, $\Gamma u \in W$. Which implies that $\Gamma W \subset W$.
step 3 : $\Gamma(W)$ is equicontinuous:
Case 1: For interval $[0,t_1]$, $0 \leq \tau_1 \leq \tau_2 \leq t_1$ and for each $\Gamma \in W(u)$, we have

$$\begin{aligned}\|\Gamma u(\tau_2) - \Gamma u(\tau_1)\| \leq &\|\mathcal{P}_\alpha(\tau_2) - \mathcal{P}_\alpha(\tau_1)(\|u_0\| + c\|u\| + d) \\ &+ \|\int_0^{\tau_2}(\tau_2-s)^{\alpha-1}\mathcal{Q}_\alpha(\tau_2-s)[f(s,u(s),\int_0^s \rho(s,\tau)h(s,\tau,u(\tau))d\tau)) \\ &+ B(s)c(s)]ds - \int_0^{\tau_1}(\tau_1-s)^{\alpha-1}\mathcal{Q}_\alpha(\tau_1-s)[f(s,u(s),\int_0^s \rho(s,\tau)h(s,\tau,u(\tau))d\tau)) + B(s)c(s)]ds\| \\ &\leq \|\mathcal{P}_\alpha(\tau_2) - \mathcal{P}_\alpha(\tau_1)\|(\|u_0\| + cr + d) \\ &+ \int_0^{\tau_1}\|((\tau_2-s)^{\alpha-1}\mathcal{Q}_\alpha(\tau_2-s) - (\tau_1-s)^{\alpha-1})\mathcal{Q}_\alpha(\tau_1-s)\|[m_f(s)\Omega_f r(1+l\nu^*) \\ &+ \|Bc\|_{\mathcal{L}^1}]ds - \|\int_{\tau_1}^{\tau_2}((\tau_2-s)^{\alpha-1}\mathcal{Q}_\alpha(\tau_2-s)[m_f(s)\Omega_f r(1+l\nu^*) + \|Bc\|_{\mathcal{L}^1}]ds.\end{aligned}$$

Case 2: For interval $(t_i, s_i]$, $i = 1,2,\ldots,n$, $t_i \leq \tau_1 \leq \tau_2 \leq s_i$ we get

$$\|\Gamma u(\tau_2) - \Gamma u(\tau_1)\| \leq L_I\|u(\tau_2) - u(\tau_1)\| + L_{g_i}\|u(\tau_2) - u(\tau_1)\|.$$

Case 3: For interval $(s_i, t_{i+1}]$, $i = 1, 2, \dots, n$, $s_i \le \tau_1 \le \tau_2 \le t_{i+1}$, we get

$$\begin{aligned}
\|\Gamma u(\tau_2) - \Gamma u(\tau_1)\| \le\ & \|\mathcal{P}_\alpha(\tau_2 - s_i) - \mathcal{P}_\alpha(\tau_1 - s_i)\| [L_I \|u(\tau_2) - u(\tau_1)\| + L_{g_i} \|u(\tau_2) - u(\tau_1)\|] \\
& + \| \int_0^{\tau_2} (\tau_2 - s)^{\alpha-1} \mathcal{Q}_\alpha(\tau_2 - s)(f(s, u(s), \int_0^s \rho(s,\tau) h(s,\tau,u(\tau)) d\tau)) + B(s)c(s)) ds \\
& - \int_0^{\tau_1} (\tau_1 - s)^{\alpha-1} \mathcal{Q}_\alpha(\tau_1 - s)(f(s, u(s), \int_0^s \rho(s,\tau) h(s,\tau,u(\tau)) d\tau)) + B(s)c(s)) ds \| \\
& \le \|\mathcal{P}_\alpha(\tau_2 - s_i) - \mathcal{P}_\alpha(\tau_1 - s_i)\| [L_I \|u(\tau_2) - u(\tau_1)\| + L_{g_i} \|u(\tau_2) - u(\tau_1)\|] \\
& + \int_0^{\tau_1} (\tau_1 - s)^{\alpha-1} \|\mathcal{Q}_\alpha(\tau_2 - s) - \mathcal{Q}_\alpha(\tau_1 - s)\| [m_f(s)\Omega_f r(1 + l\nu^*) + \|Bc\|_{\mathcal{L}^1}] ds \\
& - \int_{\tau_1}^{\tau_2} (\tau_2 - s)^{\alpha-1} \|\mathcal{Q}_\alpha(\tau_2 - s)\| [m_f(s)\Omega_f r(1 + l\nu^*) + \|Bc\|_{\mathcal{L}^1}] ds.
\end{aligned}$$

step 4 : Mönch condition hold.

Suppose that $V \subseteq W$ is countable and $V \subseteq \overline{conv}(\{0\} \cup \Gamma(V))$. We show that $\mu(V) = 0$ where μ is the Hausdroff measure of noncompactness. Without loss of generality, we may assume that $V = \{u\}_{k=1}^\infty$ and we can easily verify that V is bounded and equicontinuous.

Case 1: For each $t \in [0, t_1]$ we get

$$(\Gamma u)(t) = \mathcal{P}_\alpha(t)[u_0 - k(u)] + \int_0^t (t-s)^{\alpha-1} \mathcal{Q}_\alpha(t-s)[f(s, u(s), \int_0^s \rho(s,\tau) h(s,\tau,u(\tau)) d\tau) + B(s)c(s)] ds,$$

$$(\Gamma u)(t) = (\Gamma_1 u)(t) + (\Gamma_2 u)(t),$$

with

$$(\Gamma_1 u)(t) = \mathcal{P}_\alpha(t)[u_0 - k(u)]$$

$$(\Gamma_2 u)(t) = \int_0^t (t-s)^{\alpha-1} \mathcal{Q}_\alpha(t-s)[f(s, u(s), \int_0^s \rho(s,\tau) h(s,\tau,u(\tau)) d\tau) + B(s)c(s)] ds.$$

Moreover, $\Gamma_1 : V \longrightarrow \mathcal{PC}([0,b], X)$ is Lipschitz continuous with constant $M_A c$ due to the conditions (H_1) and (H_4). In fact $u, v \in V$, we have

$$\|(\Gamma_1 u)(t) - (\Gamma_1 v)(t)\| \le \sup_{t\in[0,t_1]} \|\mathcal{P}_\alpha(t)[k(u) - k(v)]\| \le M_A c \|u - v\|_{\mathcal{PC}}.$$

So, from lemma 2.3 - 2.5, 2.7 and hypotheses $(H_3)(ii)$, $(H_6)(ii)$, we get:

$$\begin{aligned}
\mu(\{\Gamma u_k\}_{k=1}^\infty) &\le \mu(\{\Gamma_1 u_k\}_{k=1}^\infty) + \mu(\{\Gamma_2 u_k\}_{k=1}^\infty) \\
&\le M_A c \mu(\{u_k\}_{k=1}^\infty) + \mu\Big(\int_0^t (t-s)^{\alpha-1} \mathcal{Q}_\alpha(t-s)[f(s, u(s), \int_0^s \rho(s,\tau) h(s,\tau,u(\tau)) d\tau) + B(s)c(s)] ds \Big) \\
&\le M_A c \mu(\{u_k\}_{k=1}^\infty) + \frac{2\alpha M_A}{\Gamma(\alpha+1)} \int_0^t \mu\Big((t-s)^{\alpha-1} [f(s, u(s), \int_0^s \rho(s,\tau) h(s,\tau,u(\tau)) d\tau) + B(s)c(s)] ds \Big) \\
&\le M_A c \mu(\{u_k\}_{k=1}^\infty) + \frac{2\alpha M_A}{\Gamma(\alpha+1)} \int_0^t (t-s)^{\alpha-1} \eta(t) \Big[\sup_{0\le s\le t_1} \mu(\{u_k(s)\}_{k=1}^\infty) + \mu(\int_0^s \rho(s,\tau) h(s,\tau,\{u_k(\tau)\}_{k=1}^\infty) d\tau) \Big] ds \\
&\le M_A c \mu(\{u_k\}_{k=1}^\infty) + \frac{2\alpha M_A}{\Gamma(\alpha+1)} \int_0^t (t-s)^{\alpha-1} \eta(t) [\sup_{0\le s\le t_1} \mu(\{u_k(s)\}_{k=1}^\infty) + 2l \int_0^s \psi(s,\tau) d\tau \mu(\{u_k(\tau)\}_{k=1}^\infty)] ds \\
&\le M_A c \mu(\{u_k\}_{k=1}^\infty) + \frac{2\alpha M_A}{\Gamma(\alpha+1)} \int_0^t (t-s)^{\alpha-1} \eta(t) [\sup_{0\le s\le t_1} \mu(\{u_k(s)\}_{k=1}^\infty) + 2l\psi^* \mu(\{u_k(\tau)\}_{k=1}^\infty)] ds \\
&\le M_A c \mu(\{u_k\}_{k=1}^\infty) + \frac{2M_A t_1^\alpha}{\Gamma(\alpha+1)} \zeta^*(1 + 2l\psi^*)(\sup_{0\le s\le t_1} \mu_{\mathcal{PC}}(V(s))) \\
&\le M_A [c + \frac{2t_1^\alpha}{\Gamma(\alpha+1)} \zeta^*(1 + 2l\psi^*)](\sup_{0\le s\le t_1} \mu_{\mathcal{PC}}(V(s))).
\end{aligned}$$

Case 2: For each $t \in (t_i, s_i]$, $i = 1, 2, \dots, n$, we get:

$$\begin{aligned}
\mu(\{\Gamma u\}_{k=1}^\infty) &\le \mu(I_i(u_i(s_i))) + \mu(g_i(s_i, u(s_i))) \\
&\le (\phi_I + M_i)(\sup_{0\le t_i\le s_i} \mu_{\mathcal{PC}}(V(s))).
\end{aligned}$$

Case 3: For each $t \in (s_i, t_{i+1}]$, $i = 1, 2, \dots, n$, we get:

$$\begin{aligned}(\Gamma u)(t) &= \mathcal{P}_\alpha(t-s_i)\Big[I_i(u(t_i)) + g_i(s_i, u(s_i)) \\ &\quad - \int_0^{s_i} (s_i - s)^{\alpha-1}\mathcal{Q}_\alpha(s_i - s)[f(s, u(s), \int_0^s \rho(s,\tau)h(s,\tau,u(\tau))d\tau) + B(s)c(s)]\Big] \\ &\quad + \int_0^t (t-s)^{\alpha-1}\mathcal{Q}_\alpha(t-s)[f(s,u(s), \int_0^s \rho(s,\tau)h(s,\tau,u(\tau))d\tau) + B(s)c(s)], ds\end{aligned}$$

$$(\Gamma u)(t) = (\Gamma_1 u)(t) + (\Gamma_2 u)(t),$$

with

$$(\Gamma_1 u)(t) = \mathcal{P}_\alpha(t-s_i)(I_i(u(t_i)) + g_i(s_i, u(s_i)),$$

$$\begin{aligned}(\Gamma_2 u)(t) &= \int_0^t (t-s)^{\alpha-1}\mathcal{Q}_\alpha(t-s)[f(s,u(s), \int_0^s \rho(s,\tau)h(s,\tau,u(\tau))d\tau) + B(s)c(s)]ds \\ &\quad - \mathcal{P}_\alpha(t-s_i)\int_0^{s_i}(s_i-s)^{\alpha-1}\mathcal{Q}_\alpha(s_i-s)[f(s,u(s), \int_0^s \rho(s,\tau)h(s,\tau,u(\tau))d\tau) + B(s)c(s)]ds.\end{aligned}$$

Moreover, $\Gamma : V \longrightarrow \mathcal{PC}([0,b], X$ is Lipschitz continuous with constant $M_A(L_I + L_{g_i})$ due to the conditions (H_1) and $(H_2)(i)$. In fact $u, v \in V$, we have

$$\begin{aligned}\|(\Gamma_1 u)(t) - (\Gamma_1 v)(t)\| &= \|\mathcal{P}_\alpha(t-s_i)\|(\|I_i(u(t_i)) - I_i(u(t_i))\| + \|g_i(s_i, u(s_i) - g_i(s_i, v(s_i)\|) \\ &\le M_A(L_I + L_{g_i})\|u - v\|_{\mathcal{PC}}.\end{aligned}$$

So, from lemma 2.1 - 2.3, 2.5 and hypotheses $(H_3)(ii)$, $(H_6)(ii)$, we have

$$\begin{aligned}\mu(\{\Gamma u_k\}_{k=1}^\infty) &\le \mu(\{\Gamma_1 u_k\}_{k=1}^\infty) + \mu(\{\Gamma_2 u_k\}_{k=1}^\infty) \\
&\le M_A(L_I + L_{g_i})\mu(\{u_k\}_{k=1}^\infty) + \frac{2\alpha M_A}{\Gamma(\alpha+1)}\int_0^t (t-s)^{\alpha-1}\mu([f(s,u(s), \int_0^s \rho(s,\tau)h(s,\tau,u(\tau))d\tau) \\
&\quad + B(s)c(s)]ds) \\
&\quad + \frac{2\alpha M_A^2}{\Gamma(\alpha+1)}\int_0^{s_i}(s_i-s)^{\alpha-1}\mu([f(s,u(s), \int_0^s \rho(s,\tau)h(s,\tau,u(\tau))d\tau) + B(s)c(s)]ds) \\
&\le M_A(L_I + L_{g_i})\mu(\{u_k\}_{k=1}^\infty) + \frac{2\alpha M_A}{\Gamma(\alpha+1)}\int_0^t (t-s)^{\alpha-1}\eta(s)[\sup_{s_i \le s \le t_{i+1}} \mu(\{u_k(s)\}_{k=1}^\infty) \\
&\quad + \mu(\int_0^s \rho(s,\tau)h(s,\tau,\{u_k(\tau)\}_{k=1}^\infty)d\tau))]ds \\
&\quad + \frac{2\alpha M_A^2}{\Gamma(\alpha+1)}\int_0^{s_i}(s_i-s)^{\alpha-1}\eta(s)[\sup_{s_i \le s \le t_{i+1}}\mu(\{u_k(s)\}_{k=1}^\infty) + \mu(\int_0^s \rho(s,\tau)h(s,\tau,\{u_k(\tau)\}_{k=1}^\infty)d\tau))]ds \\
&\le M_A(L_I + L_{g_i})\mu(\{u_k\}_{k=1}^\infty) + \frac{2\alpha M_A}{\Gamma(\alpha+1)}\int_0^t (t-s)^{\alpha-1}[\eta(s)[\sup_{s_i \le s \le t_{i+1}}\mu(\{u_k(s)\}_{k=1}^\infty) \\
&\quad + 2l\int_0^s \psi(s,\tau)d\tau\mu(\{u_k(\tau)\}_{k=1}^\infty)]ds \\
&\quad + \frac{2\alpha M_A^2}{\Gamma(\alpha+1)}\int_0^{s_i}(s_i-s)^{\alpha-1}\eta(s)[\sup_{s_i \le s \le t_{i+1}}\mu(\{u_k(s)\}_{k=1}^\infty) + 2l\int_0^s \psi(s,\tau)d\tau\mu(\{u_k(\tau)\}_{k=1}^\infty)]ds \\
&\le M_A(L_I + L_{g_i})\mu(\{u_k\}_{k=1}^\infty) + \frac{2\alpha M_A}{\Gamma(\alpha+1)}\int_0^t (t-s)^{\alpha-1}[\eta(s)[\sup_{s_i \le s \le t_{i+1}}\mu(\{u_k(s)\}_{k=1}^\infty) + 2l\psi^*\mu(\{u_k(\tau)\}_{k=1}^\infty)]ds \\
&\quad + \frac{2\alpha M_A^2}{\Gamma(\alpha+1)}\int_0^{s_i}(s_i-s)^{\alpha-1}\eta(s)[\sup_{s_i \le s \le t_{i+1}}\mu(\{u_k(s)\}_{k=1}^\infty) + 2l\psi^*\mu(\{u_k(\tau)\}_{k=1}^\infty)] \\
&\le M_A(L_I + L_{g_i})\mu(\{u_k\}_{k=1}^\infty) + \frac{2M_A t_{i+1}^\alpha}{\Gamma(\alpha+1)}\zeta^*[\sup_{s_i \le s \le t_{i+1}}\mu(\{u_k(s)\}_{k=1}^\infty) \\
&\quad + 2l\psi^*\mu(\{u_k(\tau)\}_{k=1}^\infty)] + \frac{2M_A^2 s_i^\alpha}{\Gamma(\alpha+1)}\zeta^*[\sup_{s_i \le s \le t_{i+1}}\mu(\{u_k(s)\}_{k=1}^\infty) + 2l\psi^*\mu(\{u_k(\tau)\}_{k=1}^\infty)] \\
&\le M_A[(L_I + L_{g_i}) + \frac{2M_A}{\Gamma(\alpha+1)}\zeta^*(t_{i+1}^\alpha + M_A s_i^\alpha + 2l\psi^* t_{i+1}^\alpha + 2M_A l\psi^* s_i^\alpha)]\sup_{s_i \le s \le t_{i+1}}\mu_{\mathcal{PC}}(V(s)).\end{aligned}$$

From above, we have

$$\mu_{\mathcal{PC}}(\Gamma(v)) \leq M_A[(L_I + L_{g_i}) + \frac{2M_A}{\Gamma(\alpha+1)}\zeta^*(t_{i+1}^{\alpha} + M_A s_i^{\alpha} + 2l\psi^* t_{i+1}^{\alpha} + 2M_A l\psi^* s_i^{\alpha})] \sup_{s_i \leq s \leq t_{i+1}} \mu_{\mathcal{PC}}(V)$$
$$\leq \lambda^* \mu_{\mathcal{PC}}(V).$$

Where $\lambda^* = M_A[(L_I + L_{g_i}) + \frac{2M_A}{\Gamma(\alpha+1)}\zeta^*(t_{i+1}^{\alpha} + M_A s_i^{\alpha} + 2l\psi^* t_{i+1}^{\alpha} + 2M_A l\psi^* s_i^{\alpha})] < 1$. Thus from Mönch condition we get

$$\mu_{\mathcal{PC}}(V) \leq \mu_{\mathcal{PC}}((\overline{conv}\{0\}) \cup \Gamma(V)) = \mu_{\mathcal{PC}}(\Gamma(V)) \leq \lambda^* \mu_{\mathcal{PC}}(V).$$

which implies that $\mu_{\mathcal{PC}}(V) = 0$ Hence using Theorem 2.2, there is a fixed point of u of Γ in W. Which is a mild solution of (1.1). This completes the proof. □

4. Application

In this section, $X = L^2([0,\pi], \mathbb{R})$ and define the operator A by $Ax = x''$ with the domain

$$D(A) = \{x \in X : x'' \in X, x(0) = x(\pi) = 0\}.$$

It is well known that A is the bounded linear operator of a compact semigroup $\{T(t), t \geq 0\}$ on X and that $\|T(t)\| \leq e^{-t}$ for all $t \geq 0$. Consider a nonlocal problem of impulsive differential equations given by,

$$\begin{cases} D^{\frac{1}{2}} u(t,w) = \frac{\partial^2}{\partial w^2} u(t,w) + \int_0^t h(t,s,u(s,w))ds + F(t,u(t,w)) + c(t,w) \\ (t,w) \in \cup_{n=1}^{n} [s_i, t_{i+1}] \times [0,\pi] \\ u(t,0) = u(t,\pi), t \in [0,b] \\ u(0,w) + \sum_{j=1}^{n} c_j u(t_j, w) = u_0(w), w \in [0,\pi] \\ u(t,w) = I_i(u(t_i)) + G_i(t, u(t,w)), w \in [0,\pi], t \in [t_i, s_i], \end{cases} \tag{4.1}$$

with $0 = t_0 = s_0 < t_1 \leq s_1 \leq t_2 < \ldots < t_n \leq s_n \leq t_{n+1} = b$ are fixed real numbers, $u_0 \in X$, $F \in C([0,b] \times \mathbb{R}, \mathbb{R})$, $c(t,w) = B(t)c(t)w$ and $G_i \in C((t_i, s_i] \times \mathbb{R}, \mathbb{R})$ for all $i = 1, 2, \ldots, n$.
To represent the problem (4.1) in the abstract form (1.1), we assume that

(i) $f : [0,b] \times X \longrightarrow X$ defined by $f(t,x)(w) = \int_0^t h(t,s,u(s,w))ds + F(t,u(t,w))$ for $t \in [0,b], w \in [0,\pi]$.
(ii) $k : PC([0,b], X) \longrightarrow X$ is continuous function defined by $k(u)(w) = u_0(w) - \sum_{j=1}^{n} c_j u(t_j)(w)$, $t \in [0,b]$, $w \in [0,\pi]$, where $u(t)(w) = u(t,w), t \geq 0, w \in [0,\pi]$
(iii) . $g_i : (t_i, s_i] \times X \longrightarrow X$ defined by $g_i(t,x)(w) = G_i(t, x(w))$.

Now, we say that $u \in PC(X)$ is a mild solution of (4.1) if $u(.)$ is a mild solution of the associated abstract problem (1.1).

Conclusion

The existence of mild solutions of a non-instantaneous impulses fractional differential evolution equations is largely studied in several above works . Our contribution in this present work is the study of existence of mild solutions of a non-instantaneous impulses fractional differential evolution equations by means of the Mönch - fixed point theorem combined with theory of operator semigroups, probability density function. In the future work, we will work on the large applications of fractional calculus which among them is the control of turbines and satellite images .

References

[1] J. Banas, K.Goebel, Measure of noncompactness in Banach space Lecture Notes in Pure and Applied Mathematics, Vol. 60,Marcel Dekker, New york 1980.
[2] H.Mönch, Boundary value problems for nonlinear ordinary differential equations of second order in Banach spaces, Nonlinear Anal. 4, 985-999 , 1980.
[3] V.Lakshmikantham, D.D. Bainov, P.S. Simeonov, Theory of Impulsive Differential Equations, World Scientific, Singapore 1989.
[4] K.S. Miller, B. Ross, An Introduction to the Fractional Calculus and Fractional Differential Equations, Wiley, New York, 1993.
[5] I. Podlubny, Fractional Differential Equations, Academic Press, New York, 1993.
[6] S.K. Ntouyas and P. Tsamatos, Global existence for second order semilinear ordinary and delay integro differential with nonlocal conditions, Applicable Anal.Vol.67, no. 3-4, pp. 245-257, 1997.
[7] I. Podlubny, Fractional differential equations, Academic Press, 1999.
[8] D.Bothe, Multivalued perturbation of m-accretive differential inclusions, J Math. 108, 109-138, 1998.
[9] I. Podlubny, Fractional differential equations, Academic Press, 1999.
[10] F. Mainardi, P. Paradisi, R. Gorenflo, Probability distributions generated by fractional diffusion equations, In: Kertesz, J,Kondor, I (eds.) Econophysics: An Emerging Science. Kluwer Academic, Dordrecht, 2000.
[11] Agarwal, R. P. Meehan, M. O'Regan, Fixed Point Theory and Applications, Cambridge University Press, Cambridge, 2001.
[12] M.M. El-Borai, Some probability densities and fundamental solutions of fractional evolution equations, Chaos Solitons Fractals 14, 433-440, 2002.

[13] M.M. El-Borai, The fundamental solutions for fractional evolution equations of parabolic type, J. Appl. Math. Stoch.Anal. 3, 197-211, 2004.
[14] J. Liang, J.H. Liu, T.J.Xiao, Nonlocal Cauchy problems governed by compact operator families, Nonlinear Anal. 57, 183-189, 2004.
[15] A.A. Kilbas, H.M. Srivastava, J.J. Trujillo, Theory and Applications of Fractional Differential Equations, North-Holland Mathematics Studies, vol. 204. Elsevier, Amsterdam 2006.
[16] X. Xue, Semilinear nonlocal differential equations with measure of noncompactness in Banach spaces, J.Nanjing Univ. Mayh. Big. 24, 264-276, 2007.
[17] Q. Dong, G.Li, Existence of solutions for semilinear differential equations with nonlocal conditions in Banach spaces, Electron.J.Qual.Theory Differ. Equ. 2009, 47, 2009.
[18] V. Lakshmikantham, S. Leela and J. Devi Vasundhara, Theory of fractional dynamic systems, Cambridge Scientific Publishers, 2009.
[19] Y. Zhou, F. Jiao, Existence of mild solutions for fractional neutral evolution equations, Comput. Math. Appl. 59, 1063-1077, 2010.
[20] J.R. Wang, Y.Zhou, Existence and controllability results for fractional semilinear differential inclusions, Nonlinear Anal.12(6), 3642-3653, 2011.
[21] Mallika Arjunan, M.Kavitha, V.S. Selvi, Existence results for impulsive differential equations with nonlocal conditions via measures of noncompactness, J.Nonlinear Sci.Appl., Vol. 5, 195-205, 2012.
[22] E. Hernàndez, D. O'Regan, On a new class of abstract impulsive differential equations, Proc. Amer. Math. Soc., 141, 1641-1649, 2013.
[23] M. Pierri, D. O'Regan, V. Rolnik, Existence of solutions for semi-linear abstract differential equations with not instantaneous impulses, Appl. Math. Comput. 219, 6743-6749 ,2013.
[24] G. A. Afrouzi, A. Hadjian, V. D. Radulescu, Variational approach to fourth-order impulsive differential equations with two control parameters, Results Math.65, 371-384, 2014.
[25] G. Bonanno, R. Rodriguez-Lopez, S. Tersian, Existence of solutions to boundary value problem for impulsive fractional differential equations, Fract. Calc. Appl. Anal. 17, 717-744, 2014.
[26] P. Kumar, D. N. Pandey, D. Bahuguna, On a new class of abstract impulsive functional differential equations of fractional order, J. Nonlinear Sci. Appl. 7, 102-114, 2014.
[27] M. Feckan, J. Wang, and Y. Zhou, Periodic Solutions for Nonlinear Evolution Equations with Non-instantaneous Impulses, Nonauton. Dyn. Syst. 1:93-101, 2014.
[28] Yong Zhou, Basic theory of fractional differential equations, Xiangtan University, China, 2014.
[29] R. Rodriguez-Lopez, S. Tersian, Multiple solutions to boundary value problem for impulsive fractional differential equations, Fract. Calc. Appl. Anal, 17, 1016-1038, 2014.
[30] G. A. Afrouzi, A. Hadjian, S. Shokooh, Infinitely many solutions for a Dirichlet boundary value problem with impulsive condition, Politehn. Univ. Bucharest Sci. Bull. Ser. A Appl. Math. Phys. 77, 9-22, 2015.
[31] V. Colao, L. Muglia, H.K. Xu, Existence of solutions for a second-order differential equation with non-instantaneous impulses and delay, Ann. Mat. Pura Appl. 195, $(2016), 697-716$.
[32] M. Pierri, H. R. Henriquez, A. Prokopczyk, Global solutions for abstract differential equations with non-instantaneous impulses, Mediterr. J. Math. 13, $(2016), 1685-1708$.
[33] K. Malar.A. Anguraj, Existence Results of Abstract Impulsive Integrodifferential Systems with Measure of Non-compactness, Journal of Statistical Science and Application, April Vol. 4, No. 03-04, 108-117, 2016.
[34] S. Melliani, A. El Allaoui and L. S. Chadli, A general class of periodic boundary value problems for controlled nonlinear impulsive evolution equations on Banach spaces, Advances in Difference Equations 2016.
[35] M. Hannabou and K. Hilal, Investigation of a Mild Solution to Coupled Systems of Impulsive Hybrid Fractional Differential Equations, Hindawi Publishing Corporation International Journal of Differential Equations. Volume 2019 , Article ID 2618982, 2019.
[36] Mohamed Hannabou and Khalid Hilal, Existence Results for a System of Coupled Hybrid Differential Equations with Fractional Order, Hindawi Publishing Corporation International Journal of Differential Equations Volume 2020, Article ID 3038427, 2020.
[37] S. Saidi, Some results associated to first-order set-valued evolution problems with subdifferentials, J. Nonlinear Var. Anal. 5, 227-250, 2021.
[38] H. Sun, H. Gu and L. Ma, An averaging result for a class of impulsive fractional neutral stochastic evolution equations, Journal of Nonlinear Functional Analysis, Vol.2021, Article ID 30, 2021.
[39] Xi Fu, Xiao you Liu and Bowen Lu, On a new class of impulsive fractional evolution equations, Advances in Difference Equations 2015:227, 2015.
[40] M. Kostić, Abstract Volterra Integro-Differential Equations, CRC Pess, Bocan Raton, FL, 2015.
[41] Manuel Henriques, Duarte Valério, and Rui Melicio, Fractional Order Processing of Satellite Images, Appl. Sci. 11, 5288. 2021.
[42] Xiaolei Li, Guanghui Sun, Cong Xue, Fractional-order deployment control of space tethered satellite via adaptive super-twisting sliding mode, Aerospace Science and Technology Volume 121, 107390 . 2022.
[43] Babbar, J. Rathee, N. Satellite Image Analysis: A Review. In Proceedings of the IEEE International Conference on Electronics, Communication and Computing Technologies (ICECCT), Coimbatore, India, 20-22 February 2019.
[44] N. Mandroux, S. Drouyer, R. Grompone von Gioi, MULTI-DATE WIND TURBINE DETECTION ON OPTICAL SATELLITE IMAGES, ISPRS Annals of the Photogrammetry, Remote Sensing and Spatial Information Sciences, Volume V.2, 2022.

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