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UNIQUENESS OF CERTAIN DIFFERENTIAL POLYNOMIALS WITH FINITE WEIGHT

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ABSTRACT. Some fundamental terms in Nevanlinna's value distribution theory $m(r,f)$, $N(r,f)$, $T(r,f)$, etc. and let $f(z)$ and $g(z)$ be two non-constant meomorphic functions, $P(f)$ and $P(g)$ be a polynomials of degree m, whose zeros and poles are of multiplicities atleast s, where s is a positive integer, and let n, k be two positive integers with $s(n+m) > 9k+14$. If $m \geq 2$ and $\delta(\infty, f) > \dfrac{2+d}{n+m}$, if $m = 1$ and $\Theta(\infty, f) > \dfrac{2+d}{n+1}$, $[f^nP(f)]^{(k)}$ and $[g^nP(g)]^{(k)}$ share $1(1,0)$, then either $[f^nP(f)]^{(k)}[g^nP(g)]^{(k)} \equiv 1$ or $f(z)$ and $g(z)$ satisfy the algebraic equation $R(f,g) = 0$, where

$R(\omega_1,\omega_2) = \omega_1^m(a_m\omega_1^m + a_{m-1}\omega_1^{m-1} + ... + a_0) - \omega_2^m(a_m\omega_2^m + a_{m-1}\omega_2^{m-1} + ... + a_0)$.

Let $f(z)$ and $g(z)$ be two non-constant entire functions with satisfying inequality $n > 5k + 6m + 7$. The present paper deals with the study of uniqueness of certain differential polynomials with the notion of weighted sharing. The results of the paper improve and generalize the results of Rajeshwari S, Husna V and Nagarjun V [6]. We have also exhibited a series of examples satisfying our results and provided some other examples showing the sharpness of one of our results.

1. INTRODUCTION

Let $f(z)$ be a meromorphic function that is non-constant over the whole complex plane. We will employ the value distribution theory's conventional notations as follows [1],[11],[8].

$$T(r,f), m(r,f), N(r,f), \overline{N}(r,f).$$

We designate any function that satisfies $S(r,f)$ by

$$S(r,f) = O\{T(r,f)\}$$

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as $\to +\infty$, perhaps not inside a collection of finite measure. For any constant a, we define

$$\Theta(a,f) = 1 - \overline{\lim_{r\to\infty}} \frac{\overline{N}\left(r, \frac{1}{f-a}\right)}{T(r,f)}.$$

We provide the notations. Let a be a finite complex number, and k a positive integer. We designate $N_{k)}\left(r, \dfrac{1}{f-a}\right)$ the counting function for zeros of $f(z)-a$ with multiplicity not greater than k, by $\overline{N}_{k)}\left(r, \dfrac{1}{f-a}\right)$ the corresponding one for which multiplicity is not counted. Let $N_{(k}\left(r, \frac{1}{f-a}\right)$ the counting function for zeros of $f(z)-a$ with multiplicity a minimum k and $\overline{N}_{(k}\left(r, \frac{1}{f-a}\right)$ the corresponding one for which multiplicity is not counted.

Set $N_k\left(r, \frac{1}{f-a}\right) = \overline{N}\left(r, \frac{1}{f-a}\right) + \overline{N}_{(2}\left(r, \frac{1}{f-a}\right) + ... + \overline{N}_{(k}\left(r, \frac{1}{f-a}\right)$. We define

$$\delta_k(a,f) = 1 - \lim_{r\to\infty} \sup \frac{N_k\left(r, \frac{1}{f-a}\right)}{T(r,f)}.$$

Let f and g be a two non-constant meromorphic function. If $f-a$ and $g-a$, assume the same zeros with the same multiplicities, then we call that f and g share the value a CM (Counting Multiplicities), we call that f and g share the value a IM (Ignoring Multiplicity), if we do not consider the multiplicities.

In 2002, C. Y. Fang and M. L. Fann [2] proved the following result.

Theorem 1.1. [2] *Let $f(z)$ and $g(z)$ be two nonconstant entire functions, and let $n(\geq 8)$ be a positive integer. If $[f^n(z)(f(z)-1)]f'(z)$ and $[g^n(z)(g(z)-1)]g'(z)$ share 1 CM, then $f(z) \equiv g(z)$.*

The following example shows that Theorem 1.1 is not valid when f and g are two meromorphic functions.

Example 1.1. Let $f = \dfrac{(n+2)(h-h^{n+2})}{(n+1)(1-h^{n+2})}$,$g = \dfrac{(n+2)(1-h^{n+1})}{(n+1)(1-h^{n+2})}$, where $h = e^z$. Then $[f^n(z)(f(z)-1)]f'(z)$ and $[g^n(z)(g(z)-1)]g'(z)$ share 1 CM, but $f(z) \not\equiv g(z)$.

In 2002, Fang [10] proved the following result.

Theorem 1.2. [10] *Let $f(z)$ and $g(z)$ be two non-constant entire functions, and let n, k be two positive integers with $n > 2k+8$. If $[f^n(z)(f(z)-1)]f'(z)$ and $[g^n(z)(g(z)-1)]g'(z)$ share $1CM$, then $f(z) \equiv g(z)$.*

In 2004, Lin and Yi [12] generalized the above result.

Theorem 1.3. [12] *Let $f(z)$ and $g(z)$ be two nonconstant meromorphic functions with $\Theta(\infty, f) > \dfrac{2}{n+1}$, and let $n(\geq 12)$ be a positive integer. If $[f^n(z)(f(z)-1)]f'(z)$ and $[g^n(z)(g(z)-1)]g'(z)$ share 1 CM, then $f(z) \equiv g(z)$.*

In 2007, Bhoosnurmath and Dyavanal [4] proved the following results.

Theorem 1.4. [4] *Let $f(z)$ and $g(z)$ be two nonconstant meromorphic functions satisfying $\Theta(\infty, f) > \dfrac{3}{n+1}$, and let n, k be two positive integer with $n > 3k+13$. If $[f^n(z)(f(z)-1)]^{(k)}$ and $[g^n(z)(g(z)-)]^{(k)}$ share 1 CM, then $f(z) \equiv g(z)$.*

In 2008, L. Liu [3] for some general differential polynomials such as $[f^n(f-1)^m]^{(k)}$, proved the following result.

Theorem 1.5. [3] *Let $f(z)$ and $g(z)$ be two nonconstant entire functions, and let n, m, k be three positive integer such that $n > 5k+4m+9$. If $[f^n(z)(f(z)-1)^m]^{(k)}$ and $[g^n(z)(g(z)-1)^m]^{(k)}$ share 1 IM, then either $f(z) \equiv g(z)$ or f and g satisfy the algebraic equation $R(f,g) \equiv 0$, where $R(\omega_1, \omega_2) = \omega_1^n(\omega_1-1)^m - \omega_2^n(\omega_2-1)^m$.*

In 2011, Jin-Dong Li [9] we improve the above results.

Theorem 1.6. [9] *Let $f(z)$ and $g(z)$ be two nonconstant mermomorphic functions, and let n, k be two positive integers with $n > 3k+11$. If $\Theta(\infty, f) > \frac{2}{n}$, $[f^n(z)(f(z)-1)]^k$, and $[g^n(z)(g(z)-1)]^k$ share $1(1,2)$, then $f(z) \equiv g(z)$ or $[f^n(z)(f(z)-1)]^k.[g^n(z)(g(z)-1)]^k \equiv 1$.*

Theorem 1.7. [9] *Let $f(z)$ and $g(z)$ be two nonconstant mermomorphic functions, and let n, k be two positive integers with $n > 5k+14$. If $\Theta(\infty, f) > \frac{2}{n}$, $[f^n(z)(f(z)-1)]^k$, and $[g^n(z)(g(z)-1)]^k$ share $1(1,1)$, then $f(z) \equiv g(z)$ or $[f^n(z)(f(z)-1)]^k.[g^n(z)(g(z)-1)]^k \equiv 1$.*

In 2022, Rajeshwari S, Husna V and Nagarjun V. [6] proving the following theorem.

Theorem 1.8. [6] *Let $f(z)$ and $g(z)$ be two non-constant meromorphic functions. $P(f)$ and $P(g)$ be a polynomials of degree m and let n, k be two positive integers with $t(n+m) > 3k+8$. If $\Theta(\infty, f) > \frac{2}{n+m}$, $[f^nP(f)]^{(k)}$ and $[g^nP(g)]^{(k)}$ share $1(1,2)$, then either $[f^nP(f)]^{(k)}[g^nP(g)]^{(k)} \equiv 1$ or $f(z)$ and $g(z)$ satisfy the algebraic equation $R(f,g) = 0$ where*

$$R(\omega_1, \omega_2) = \omega_1^m(a_m\omega_1^m + a_{m-1}\omega_1^{m-1} + ... + a_0) - \omega_2^m(a_m\omega_2^m + a_{m-1}\omega_2^{m-1} + ... + a_0).$$

Theorem 1.9. [6] *Let $f(z)$ and $g(z)$ be two non-constant meromorphic functions. $P(f)$ and $P(g)$ be a polynomials of degree m and let n, k be two positive integers with $t(n+m) > 5k+10$. If $\Theta(\infty, f) > \frac{2}{n+m}$, $[f^nP(f)]^{(k)}$ and $[g^nP(g)]^{(k)}$ share $1(1,1)$, then either $[f^nP(f)]^{(k)}[g^nP(g)]^{(k)} \equiv 1$.*

For certain difference polynomial of meromorphic functions and its certain properties, we refer to the article [[20]]. For recent developments in difference polynomials and different aspects of it, we refer to the articles [[21], [22], [23], [24]].

Now the following question come naturally.

Question 1.1. If we consider the sharing value 1(1,0) in Theorem 1.8 or Theorem 1.9, then what happens?

Question 1.2. Can we take non-constant meromorphic functions in place of non-constant entire functions in Theorem 1.8 or Theorem 1.9 ?

In this paper we try to solve Question 1.1 and Question 1.2 prove the following theorems.

2. Main Results

Theorem 2.1. *Let $f(z)$ and $g(z)$ be two non-constant meomorphic functions, $P(f)$ and $P(g)$ be a polynomials of degree m, whose zeros and poles are of multiplicities atleast s, where s is a positive integer. and let n, k be two positive integers with $s(n+m) > 9k+14$. If $m \geq 2$ and $\delta(\infty, f) > \frac{2+d}{n+m}$, if $m = 1$ and $\Theta(\infty, f) > \frac{2+d}{n+1}$, $[f^nP(f)]^{(k)}$ and $[g^nP(g)]^{(k)}$ share $1(1,0)$, then either $[f^nP(f)]^{(k)}[g^nP(g)]^{(k)} \equiv 1$ or $f(z)$ and $g(z)$ satisfy the algebraic equation $R(f,g) = 0$, where*

$$R(\omega_1, \omega_2) = \omega_1^m(a_m\omega_1^m + a_{m-1}\omega_1^{m-1} + ... + a_0) - \omega_2^m(a_m\omega_2^m + a_{m-1}\omega_2^{m-1} + ... + a_0).$$

Theorem 2.2. *Let $f(z)$ and $g(z)$ be two non-constant entire functions, $P(f)$ and $P(g)$ be a polynomials of degree m and let n, k be two positive integers with $n > 5k+6m+7$. If $[f^nP(f)]^{(k)}$ and $[g^nP(g)]^{(k)}$ share $1(1,0)$, then either $[f^nP(f)]^{(k)}[g^nP(g)]^{(k)} \equiv 1$ or $f(z)$ and $g(z)$ satisfy the algebraic equation $R(f,g) = 0$, where*

$$R(\omega_1, \omega_2) = \omega_1^m(a_m\omega_1^m + a_{m-1}\omega_1^{m-1} + ... + a_0) - \omega_2^m(a_m\omega_2^m + a_{m-1}\omega_2^{m-1} + ... + a_0).$$

Example 2.1. $P(z) = z^5 - 1$, $f(z) = \frac{\pi^2}{sin^2 \pi z}$, $g(z) = \frac{\pi^2}{cos^2 \pi z}$, $k = 0$, and $s = 1$. It is easy to see that $n + m > 14$ and $P(f(z))f^n(z) = P(g(z))g^n(z)$. Therefore $P(f(z))f^n(z)$ and $P(g(z))g^n(z)$ share $1(1,0)$. It is also clear that though f and g satisfy $R(f,g) = 0$, where $R(\omega_1, \omega_2) = P(\omega_1)\omega_1(z) - P(\omega_2)\omega_2(z)$

Example 2.2. A polynomial $P(z) = a_mz^m + ... + a_0$ with $a_m \neq 0$ has a pole of order m at infinity. In fact, conversely, ever entire function $P(z)$ with a pole of order m at infinity is a polynomial of degree m.

3. Auxiliary definitions

Definition 3.1. [7] *A meromorphic function $b(z)$ ($\not\equiv 0, \infty$) defined in $\mathbb{C}$ is called a "small function" with respect to $f(z)$ if $T(r, b(z)) = S(r, f)$.*

Definition 3.2. [7] *Let k be a positive integer, for any constant a in the complex plane $\mathbb{C}$.*
We denote
(i) by $N_{k)}\left(r, \frac{1}{f-a}\right)$ the counting function of a-pints of $f(z)$ with multiplicity $\geq k$.
(ii)by $N_{(k}\left(r, \frac{1}{f-a}\right)$ the counting function of a-pints of $f(z)$ with multiplicity $\leq k$.

Definition 3.3. *Let a be an any value in the extended complex plane and let k be an arbitrary non-negative integer. we define*

$$\Theta(a, f) = 1 - \lim_{r\to\infty} \sup \frac{\overline{N}\left(r, \frac{1}{f-a}\right)}{T(r,f)},$$

$$\delta_k(a, f) = 1 - \lim_{r\to\infty} \sup \frac{N_k\left(r, \frac{1}{f-a}\right)}{T(r,f)},$$

where

$$N_k\left(r, \frac{1}{f-a}\right) = \overline{N}\left(r, \frac{1}{f-a}\right) + \overline{N}_{(2}\left(r, \frac{1}{f-a}\right) + ... + \overline{N}_{(k}\left(r, \frac{1}{f-a}\right).$$

Remark 3.1. *By Definition 1.3 we have*

$$0 \leq \delta_k(a,f) \leq \delta_{k-1}(a,f) \leq \delta_1(a,f) \leq \theta(a,f) \leq 1.$$

4. Lemmas

Lemma 4.1. [1] *Let $a_n \neq 0$ and the $a_0, a_1, ..., a_n$ be finite complex number, and Let $f(z)$ be a non-constant mermorphic function. Then*

$$T(r, a_n f^n + a_{n-1} f^{n-1} + ... + a_0) = nT(r,f) + S(r,f).$$

Lemma 4.2. [1]*Let k be a positive integer, c a non-zero finite complex number, and $f(z)$ be a non-constant meromorphic function. Then*

$$\begin{aligned} T(r,f) \leq& \overline{N}(r,f) + N\left(r, \frac{1}{f}\right) + N\left(r, \frac{1}{f^{(k)} - c}\right) - N\left(r, \frac{1}{f^{(k+1)}}\right) + S(r,f). \\ \leq& \overline{N}(r,f) + N_{k+1}\left(r, \frac{1}{f}\right) + \overline{N}\left(r, \frac{1}{f^{(k)} - c}\right) - N_0\left(r, \frac{1}{f^{(k+1)}}\right) + S(r,f). \end{aligned}$$

where $N_0\left(r, \frac{1}{f^{(k+1)}}\right)$ is the counting function which only counts those points such that $f^{(k+1)} = 0$, but note that $f(f^{(k+1)} - c) \neq 0$.

Lemma 4.3. [3]*Let k be a positive integer and let $f(z)$ be a non-constant meromorphic function. If $f^{(k)} \not\equiv 0$ is true, then*

$$N\left(r, \frac{1}{f^{(k)}}\right) \leq N\left(r, \frac{1}{f}\right) + k\overline{N}(r,f) + S(r,f).$$

Lemma 4.4. [5]*Let t, k be any two positive integers and let $f(z)$ be a non-constant meromorphic function.*

$$N_s\left(r, \frac{1}{f^{(k)}}\right) \leq k\overline{N}(r,f) + N_{t+s}\left(r, \frac{1}{f}\right) + S(r,f).$$

Clearly, $\overline{N}\left(r, \frac{1}{f^{(k)}}\right) = N_1\left(r, \frac{1}{f^{(k)}}\right)$.

Lemma 4.5. [1]*Let $f(z)$ be a transcendental meromorphic function and let $b_1(z)$, $b_2(z)$ be two meromorphic functions such that $T(r, b_j) = S(r,f), j = 1, 2, ..., n$. Then*

$$T(r,f) \leq \overline{N}(r,f) + \overline{N}\left(r, \frac{1}{f - b_1}\right) + \overline{N}\left(r, \frac{1}{f - b_2}\right).$$

Lemma 4.6. *Let $f(z)$ and $g(z)$ be two non-constant meromorphic functions, let $k \geq 1$, $l \geq 0$ be two positive integers. Suppose that $f^{(k)}$ and $g^{(k)}$ share $(1,l)$, if $l = 0$ and*

$$\Delta = (2k+4)\Theta(\infty,f) + (2k+3)\Theta(\infty,g) + \Theta(0,f) + \Theta(0,g) + 3\delta_{k+1}(0,f) + 2\delta_{k+1}(0,g) > 4k+13. \tag{1}$$

Then either $f^{(k)} g^{(k)} \equiv 1$ or $f(z) \equiv g(z)$

Proof. Let

$$h(z) = \frac{f^{(k+2)}}{f^{(k+1)}} - 2\frac{f^{(k+1)}}{f^{(k)} - 1} - \frac{g^{(k+2)}}{g^{(k+1)}} + 2\frac{g^{(k+1)}}{g^{(k)} - 1}. \tag{2}$$

Assume that $h \not\equiv 0$.
By replacing their Taylaor series at z_0, if z_0 is common simple 1-point of $f^{(k)}$ and

$g^{(k)}$, equation (2). We observe that z_0 is an integer zero of $h(z)$.
As a result, we have

$$N_{11}\left(r, \frac{1}{f^{(k)}-1}\right) = N_{11}\left(r, \frac{1}{g^{(k)}-1}\right) \leq \overline{N}\left(r, \frac{1}{h}\right) \leq T(r,f)+0(1) \leq N(r,h)+S(r,f)+S(r,g) \tag{3}$$

According to our hypothesis, $h(z)$ have poles exclusively at the zeros of $f^{(k+1)}$ and $g^{(k+1)}$, poles of f and g and those 1-point of $f^{(k)}$ and $g^{(k)}$ whose multiplicities are different from the multiplicities of correspond to 1-point of $f^{(k)}$ and $g^{(k)}$, respectively, we draw conclusion from

$$\begin{aligned} N(r,h) \leq & \overline{N}(r,f) + \overline{N}(r,g) + \overline{N}\left(r, \frac{1}{f}\right) + \overline{N}\left(r, \frac{1}{g}\right) + N_0\left(r, \frac{1}{f^{(k+1)}}\right) \\ & + N_0\left(r, \frac{1}{g^{(k+1)}}\right) + \overline{N}_L\left(r, \frac{1}{f^{(k)}-1}\right) + \overline{N}_L\left(r, \frac{1}{g^{(k)}-1}\right). \end{aligned} \tag{4}$$

Here $N_0\left(r, \frac{1}{f^{(k+1)}}\right)$ has the same meaning as in Lemma 4.2
By Lemma 4.2 we have

$$T(r,f) \leq \overline{N}(r,f) + N_{k+1}\left(r, \frac{1}{f}\right) + \overline{N}\left(r, \frac{1}{f^{(k)}-1}\right) - N_0\left(r, \frac{1}{f^{(k+1)}}\right) + S(r,f). \tag{5}$$

$$T(r,g) \leq \overline{N}(r,g) + N_{k+1}\left(r, \frac{1}{g}\right) + \overline{N}\left(r, \frac{1}{g^{(k)}-1}\right) - N_0\left(r, \frac{1}{g^{(k+1)}}\right) + S(r,g). \tag{6}$$

Since $f^{(k)}$ and $g^{(k)}$ share $(1,0)$, we get

$$\begin{aligned} \overline{N}\left(r, \frac{1}{f^{(k)}-1}\right) + \overline{N}\left(r, \frac{1}{g^{(k)}-1}\right) = & 2N_{11}\left(r, \frac{1}{f^{(k)}-1}\right) + 2\overline{N}_L\left(r, \frac{1}{f^{(k)}-1}\right) \\ & + 2\overline{N}_L\left(r, \frac{1}{g^{(k)}-1}\right) + 2N_E^{(2)}\left(r, \frac{1}{f^{(k)}-1}\right). \end{aligned} \tag{7}$$

By (3)-(7), we have

$$\begin{aligned} T(r,f) + T(r,g) \leq & N_{11}\left(r, \frac{1}{f^{(k)}-1}\right) + 2N_E^{(2)}\left(r, \frac{1}{f^{(k)}-1}\right) + N_0\left(r, \frac{1}{f^{(k+1)}}\right) \\ & + N_0\left(r, \frac{1}{g^{(k+1)}}\right) + 3\overline{N}_L\left(r, \frac{1}{f^{(k)}-1}\right) + 3\overline{N}_L\left(r, \frac{1}{g^{(k)}-1}\right) \\ & + 2\overline{N}(r,f) + 2\overline{N}(r,g) + \overline{N}\left(r, \frac{1}{f}\right) + \overline{N}\left(r, \frac{1}{g}\right) + S(r,f) + S(r,g). \end{aligned} \tag{8}$$

Since

$$\begin{aligned} N\left(r, \frac{1}{g^{(k)}-1}\right) \leq & T(r, g^{(k)}) + S(r,f) = m(r, g^{(k)}) + N(r,f) + S(r,g) \\ \leq & m(r,g) + m\left(r, \frac{g^{(k)}}{g}\right) + N(r,f) + k\overline{N}(r,g) + S(r,f) \\ \leq & T(r,g) + k\overline{N}(r,g) + S(r,g). \end{aligned} \tag{9}$$

By lemma 4.3, we have

$$\begin{aligned}\overline{N}_L\left(r,\frac{1}{f^{(k)}-1}\right) \leq & N\left(r,\frac{1}{f^{(k)}-1}\right)+\overline{N}\left(r,\frac{1}{f^{(k)}-1}\right)\leq N\left(r,\frac{f^{(k)}}{f^{(k+1)}}\right)\\ \leq & N\left(r,\frac{f^{(k+1)}}{f^{(k)}}\right)+S(r,f)\\ \leq & \overline{N}\left(r,\frac{1}{f^{(k)}}\right)+\overline{N}(r,f)+S(r,f)\\ \leq & (k+1)\overline{N}(r,f)+N_{k+1}\left(r,\frac{1}{f}\right)+S(r,f).\end{aligned} \tag{10}$$

Similarly (10), we have

$$\overline{N}_L\left(r,\frac{1}{g^{(k)}-1}\right)\leq (k+1)\overline{N}(r,g)+N_{k+1}\left(r,\frac{1}{g}\right)+S(r,g). \tag{11}$$

If $l=0$, it is easy to see that

$$\begin{aligned}& N_{11}\left(r,\frac{1}{f^{(k)}-1}\right)+2N_E^{2)}\left(r,\frac{1}{f^{(k+1)}}\right)+\overline{N}_L\left(r,\frac{1}{f^{(k)}-1}\right)+2\overline{N}_L\left(r,\frac{1}{g^{(k)}-1}\right)\\ & \leq N\left(r,\frac{1}{g^{(k)}-1}\right)+S(r,f)+S(r,g).\end{aligned} \tag{12}$$

From (5), (6), (8) and (9)-(11), we get

$$\begin{aligned}T(r,f)\leq & 3N_{k+1}\left(r,\frac{1}{f}\right)+2N_{k+1}\left(r,\frac{1}{g}\right)+\overline{N}\left(r,\frac{1}{f}\right)+\overline{N}\left(r,\frac{1}{g}\right)\\ & +(2k+4)\overline{N}(r,f)+(2k+3)\overline{N}(r,g)+S(r,f)+S(r,g).\end{aligned}$$

Without loss of generality, we suppose that there exists a set I with infinite measure such that
$T(r,g)\leq T(r,f)$ for $r\in I$:

$$\begin{aligned}& T(r,f)\{[4k+14-(2k+4)\Theta(\infty,f)-(2k+3)\Theta(\infty,g)-\Theta(0,f)-\Theta(0,g)\\ & \quad -3\delta_{k+1}(0,f)-2\delta_{k+1}(0,g)]+\epsilon\}T(r,f)+S(r,f)\end{aligned}$$

For $r\in I$ and

$$0<\epsilon<(2k+4)\Theta(\infty,f)+(2k+3)\Theta(\infty,g)+\Theta(0,f)+\Theta(0,g)+3\delta_{k+1}(0,f)+2\delta_{k+1}(0,g)-4k-13.$$

Thus we obtain from (1), that $T(r,f)\leq S(r,g)$ for $r\in I$, by a contradiction.
Hence, we get $h(z)\equiv 0$, that is

$$\frac{f^{(k+2)}}{f^{(k+1)}}-2\frac{f^{(k+1)}}{f^{(k)}-1}=\frac{g^{(k+2)}}{g^{(k+1)}}-2\frac{g^{(k+1)}}{g^{(k)}-1}.$$

By solving this equation, we obtain

$$\frac{1}{f^{(k)}-1}=\frac{bg^{(k)}+a-b}{g^{(k)}-1}.$$

Where a,b are two constants. By using the argument of as in [4], we can obtain $f^{(k)}g^{(k)}\equiv 1$ or $f\equiv g$, we here omit the detail.
The proof the Lemma 4.6 is completed. □

Let f and g be an entire function; we have $\Theta(\infty, f) = 1$ and $\Theta(\infty, g) = 1$ Using the same argument as above Lemma 4.6, we can easily obtain the following lemma.

Lemma 4.7. *Let $f(z)$ and $g(z)$ be a two non-constant entire functions, let $k \geq 1$, $l \geq 1$ be two positive integers. Suppose that $f^{(k)}$ and $g^{(k)}$ share $(1, l)$, if $l = 0$ and*

$$\Delta = \Theta(0, f) + \Theta(0, g) + 3\delta_{k+1}(0, f) + 2\delta(0, g) > 6.$$

Then either $f^{(k)}g^{(k)} \equiv 1$ or $f(z) = g(z)$.

5. Main Results Proof

Theorem 2.1.

Proof. Let $F(z) = f^nP(f)$ and $G(z) = g^nP(g)$. We have from Lemma 4.6

$$\Delta = (2k+4)\Theta(\infty, f)+(2k+3)\Theta(\infty, g)+\Theta(0, f)+\Theta(0, g)+3\delta_{k+1}(0, f)+2\delta_{k+1}(0, g). \tag{13}$$

$$\overline{N}\left(r, \frac{1}{F}\right) = \overline{N}\left(r, \frac{1}{f^nP(f)}\right) \leq \frac{1}{s(n+m)}N\left(r, \frac{1}{F}\right) \leq \frac{1}{s(n+m)}(T(r, F) + O(1)). \tag{14}$$

Since

$$\begin{aligned}\Theta(0, F) =& 1 - \overline{\lim_{r\to\infty}}\frac{\overline{N}\left(r, \frac{1}{F}\right)}{T(r, F)} = 1 - \overline{\lim_{r\to\infty}}\frac{\overline{N}\left(r, \frac{1}{f^nP(f)}\right)}{s(n+m)T(r, f)}\\ \geq& 1 - \overline{\lim_{r\to\infty}}\frac{\overline{N}\left(r, \frac{1}{f^n}\right) + \overline{N}\left(r, \frac{1}{P(f)}\right)}{s(n+m)T(r, f)}\end{aligned}$$

i.e,

$$\Theta(0, F) \geq 1 - \frac{1}{s(n+m)}. \tag{15}$$

Similarly

$$\Theta(0, G) \geq 1 - \frac{1}{s(n+m)}. \tag{16}$$

$$\begin{aligned}\Theta(\infty, F) =& 1 - \overline{\lim_{r\to\infty}}\frac{\overline{N}(r, F)}{T(r, F)} = 1 - \overline{\lim_{r\to\infty}}\frac{\overline{N}(r, f^nP(f))}{s(n+m)T(r, f)}\\ \geq& 1 - \overline{\lim_{r\to\infty}}\frac{T(r, f)}{s(n+m)T(r, f)}\end{aligned}$$

i.e,

$$\Theta(\infty, F) \geq 1 - \frac{1}{s(n+m)}. \tag{17}$$

Similarly

$$\Theta(\infty, G) \geq 1 - \frac{1}{s(n+m)}. \tag{18}$$

Moreover

$$\begin{aligned}\delta_{k+1}(0, F) =& 1 - \overline{\lim_{r\to\infty}}\frac{N_{k+1}\left(r, \frac{1}{F}\right)}{T(r, f)}\\ \geq& 1 - \frac{k+1}{s(n+m)}.\end{aligned} \tag{19}$$

Similarly

$$\delta_{k+1}(0,G) = 1 - \frac{k+1}{s(n+m)}. \tag{20}$$

From the inequalities (15)-(20), we get,

$$\Delta \geq (2k+4)\left(1-\frac{1}{s(n+m)}\right)+(2k+3)\left(1-\frac{1}{s(n+m)}\right)+2\left(1-\frac{1}{s(n+m)}\right)+5\left(1-\frac{k+1}{s(n+m)}\right)$$

On simplyfying, the above expression, we get

$$\Delta \geq 4k + 14 - \frac{9k+14}{n+m}.$$

Since $n > 9k+14$, we get $\Delta \geq 4k+13$. Considering that $F^{(k)}$ and $G^{(k)}$ share (1,0), then by Lemma 4.6, we deduce that either $F^{(k)}G^{(k)} \equiv 1$ or $F \equiv G$.
Next we consider the following two cases.
Case 1. $F^{(k)}G^{(k)} \equiv 1$, that is

$$[f^n P(f)][g^n P(g)] \equiv 1. \tag{21}$$

Case 2. $F \equiv G$, that is

$$f^n P(f) = g^n P(g). \tag{22}$$

Suppose that $f \equiv g$, then we consider following two cases:

i . Let $h = \dfrac{f}{g}$ be a constant. Then from (22), we get

$$f^n[a_m f^m + a_{m-1} f^{m-1} + ... + a_1 z] = g^n[a_m g^m + a_{m-1} g^{m-1} + ... + a_1 z].$$

i.e,

$$[a_m g^{n+m}(h^{m+n} - 1) + a_{m-1} g^{m+n-1}(h^{m+n-1} - 1) + ... + a_1 g^n(h^n - 1) = 0]. \tag{23}$$

If follow that, $h^n \neq 1$, $h^{n+m} \neq 1$, $h^{m+n-1} \neq 1$ and $a_m g^{n+m}(h^{n+m} - 1) + ... + a_1 g^n(h^n - 1) = 0$.
Which implies, that $h^{d_1} = 1$.
Where $d_1 = GCD(n+m, n+m-i, ..., n)$, $a_{m-i} \neq 0$ for $i = 0, 1, 2, ..., m$.

ii . Let $h = \dfrac{f}{g}$ be a not constant.
Eq (23), given as

$$g^{n+m}(h^{n+m} - 1) = -g^n(h^n - 1). \tag{24}$$

Assume that h is a non-constant meromorphic function that is not constant. By (24), we have

$$g^m = -\frac{h^n - 1}{h^{n+m} - 1} \tag{25}$$

If $h \not\equiv 1$, if $d = gcd(n,m)$. Then clearly $h^d = 1$ is the common factor of $h^n - 1$ and $h^{n+m} - 1$.
As result (25), we have

$$g^m = -\frac{1 + h + ... + h^{n-d}}{1 + h + ... + h^{n+m-d}}. \tag{26}$$

Then substituting $f = hg$, if $m \geq 2$, then from above, we get that every poles of

$$f^m = -\frac{(1 + h + ... + h^{n-d})h^m}{1 + h + ... + h^{n+m-d}}.$$

If follow that,

$$T(r,f)=\frac{n+m}{m}T(r,h)+S(r,f).$$

On the other hand, every poles of f of order p must be a zero $h^{n+m}-1$ of order mp. Hence

$$\overline{N}(r,f)=\frac{1}{m}\sum_{i=1}^{N}\overline{N}\left(r,\frac{1}{h-\lambda_i}\right)\geq\frac{1}{m}[n+m-d-2]T(r,h)+S(r,f).$$

As $r\to\infty$. Here $\lambda_1,\lambda_2,.....,\lambda_{n+m-d}$ are $(n+m-d)$ distinct finite complex numbers satisfying $\lambda_i\neq 1$ and $\lambda_i^{n+m-d}=1$ for $1\leq i\leq n+m-d$. We have

$$\begin{aligned}\delta(\infty,f)=1-\overline{\lim_{r\to\infty}}\frac{\overline{N}(r,f)}{T(r,f)}\leq&1-\overline{\lim_{r\to\infty}}\frac{\frac{n+m-d-2}{m}T(r,h)+S(r,f)}{\frac{n+m}{m}T(r,h)+S(r,f)}\\ \leq&1-\frac{n+m-d-2}{n+m}\\ \leq&\frac{2+d}{n+m}.\end{aligned}$$

Which contradicts the assumption $\delta(\infty,f)>\dfrac{2+d}{n+m}$.
If $m=1$ (26), we get

$$g=\frac{1+h+.....+h^{n-d}}{1+h+.....+h^{n+1-d}}$$

From $f=hg$, we have

$$f=\frac{(1+h+.....+h^{n-d})h}{1+h+.....+h^{n+1-d}}$$

It follow that $T(r,f)=T(r,gh)=(n+1-d)T(r,h)+S(r,f)$.
On the hand, by the second fundamental theorem we have

$$\overline{N}(r,f)=\sum_{j=1}^{N}\overline{N}\left(r,\frac{1}{h-\lambda_j}\right)\geq(n-d-1)T(r,h)+S(r,f).$$

As $r\to\infty$. Here $\lambda_1,\lambda_2,.....,\lambda_{n+1-d}$ are $(n+1-d)$ distinct finite complex numbers satisfying $\lambda_j\neq 1$ and $\lambda_j^{n+1-d}=1$ for $1\leq j\leq n+1-d$. We have

$$\begin{aligned}\Theta(\infty,f)=1-\overline{\lim_{r\to\infty}}\frac{\overline{N}(r,f)}{T(r,f)}\leq&1-\overline{\lim_{r\to\infty}}\frac{(n-d-1)T(r,f)+S(r,f)}{(n+1)T(r,f)+S(r,f)}\\ \leq&1-\frac{n-d-1}{n+1}\\ \leq&\frac{2+d}{n+1}.\end{aligned}$$

Which contradicts to the assumption that $\Theta(\infty,f)>\dfrac{2+d}{n+1}$
Thus $h\equiv 1$, that is, $F\equiv G$. Hence the proof of Theorem 2.1.

□

Theorem 2.2.

Proof. Let $F(z) = f^nP(f)$ and $G(z) = g^nP(g)$.
where F and G are two entire functions. We have from Lemma 4.7

$$\Delta = \Theta(0, f) + \Theta(0, g) + 3\delta_{k+1}(0, f) + 2\delta_{k+1}(0, g). \tag{27}$$

Since

$$\begin{aligned}\Theta(0, F) =& 1 - \overline{\lim_{r\to\infty}} \frac{\overline{N}\left(r, \frac{1}{F}\right)}{T(r, F)} = 1 - \overline{\lim_{r\to\infty}} \frac{\overline{N}\left(r, \frac{1}{f^nP(f)}\right)}{(n+m)T(r, f)} \\ \geq& 1 - \overline{\lim_{r\to\infty}} \frac{\overline{N}\left(r, \frac{1}{f^n}\right) + \overline{N}\left(r, \frac{1}{P(f)}\right)}{(n+m)T(r, f)}\end{aligned}$$

i.e,

$$\Theta(0, F) \geq 1 - \frac{m+1}{n+m}. \tag{28}$$

Similarly

$$\Theta(0, G) \geq 1 - \frac{m+1}{n+m}. \tag{29}$$

Moreover,

$$\begin{aligned}\delta_{k+1}(0, F) =& 1 - \overline{\lim_{r\to\infty}} \frac{N_{k+1}\left(r, \frac{1}{F}\right)}{T(r, f)} \\ \geq& \frac{(k+1)\overline{N}\left(r, \frac{1}{f^n}\right) + N_{k+1}\left(r, \frac{1}{P(f)}\right)}{(n+m)T(r, f)}\end{aligned}$$

i.e,

$$\delta_{k+1} \geq 1 - \frac{k+1+m}{n+m}. \tag{30}$$

Similarly

$$\delta_{k+1}(0, G) = 1 - \frac{k+1+m}{n+m} \tag{31}$$

From the inequalities (28)-(31), we get,

$$\Delta \geq 2\left(1 - \frac{m+1}{n+m}\right) + 5\left(1 - \frac{k+1+m}{n+m}\right)$$

On simplyfying, the above expression, we get

$$\Delta \geq 7 - \frac{5k+7m+7}{n+m}$$

Since $n > 5k + 6m + 7$, we get $\Delta \geq 6$. Considering that $F^{(k)}$ and $G^{(k)}$ share (1,0), then by Lemma 4.7 we deduce that either $F^{(k)}G^{(k)} \equiv 1$ or $F \equiv G$.
Next we consider the following two cases.
Case 1. $F^{(k)}G^{(k)} \equiv 1$, that is

$$[f^nP(f)][g^nP(g)] \equiv 1.$$

Case 2. $F \equiv G$, that is

$$f^nP(f) = g^nP(g), \tag{32}$$

we also say

$$f^n[a_mf^m + a_{m-1}f^{m-1} + ... + a_1z] = g^n[a_mg^m + a_{m-1}g^{m-1} + ... + a_1z]. \quad (33)$$

Let $h = \frac{f}{g}$. If h is a constant, then substituting $f = gh$ into (33) we deduce

$$a_mg^{n+m}(h^{m+n} - 1) + a_{m-1}g^{m+n-1}(g^{m+n-1} - 1) + ... + a_1g^n(h^n - 1) = 0.$$

Which implies, that $h^{d_1} = 1$.

Where $d_1 = GCD(n + m, n + m - i, ..., n)$, $a_{m-i} \neq 0$ for $i = 0, 1, 2, ..., m$. Thus $f(z) \equiv g(z)$. If h is not a constant, then we know by (32) that f and g satisfy the algebraic equation $R(f, g) = 0$, where

$$R(\omega_1, \omega_2) = \omega_1^m(a_m\omega_1^m + a_{m-1}\omega_1^{m-1} + ... + a_0) - \omega_2^m(a_m\omega_2^m + a_{m-1}\omega_2^{m-1} + ... + a_0).$$

This completes the proof of Theorem 2.2. □

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