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FRACTIONAL CHEBYSHEV DIFFERENTIAL EQUATION AND NEW FAMILY OF ORTHOGONAL POLYNOMIALS

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ABSTRACT. In this paper we consider a typical Fractional Chebyshev Differential Equation (FCDE) and we investigate the solutions, their properties and applications. For a positive real number α we prove that FCDE has solutions of the form $T_{n,\alpha}(x) = (1+x)^{\frac{\alpha}{2}} P_{n,\alpha}(x)$, where $P_{n,\alpha}(x)$ produce a family of orthogonal polynomials with respect to weight function $w_\alpha(x) = (\frac{1+x}{1-x})^{\frac{\alpha}{2}}$ on $[-1, 1]$. For integer case $\alpha = 1$ we show that these polynomials coincide with classical Chebyshev polynomials of third kind. Finally, we give some applications of $T_{n,\alpha}(x)$ in determining the solutions of some fractional order differential equations by defining a suitable integral transform.

1. INTRODUCTION

Developing classical integer order differential equations to the fractional order has been beneficial in many applications. For example it has been shown that the classical integer order oscillator circuits are only a special case of the more general fractional oscillators [1, 26]. In the recent years, it has been proved that in some applications modeling by fractional derivatives generate more accurate solutions than modeling by integer order derivatives [3, 11, 19, 22, 25, 27, 30].

It is well known that orthogonal polynomials play a fundamental role in studying ordinary and partial differential equation [9, 12, 29]. Important orthogonal polynomials such as Chebyshev, Legendre, Hermite and Laguerre are the solutions of the integer order Sturm-Lioville equation. In [13, 14, 15, 31] fractional Sturm-Liouville problems are considered and some spectral properties such as orthogonality of eigenfunctions corresponding to distinct eigenvalues are studied. Numerical solutions for fractional Sturm-Liouville problems are studied in [4, 10, 17]. Moreover, M. Klimek

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and O.P. Agrawal considered the fractional Legendre equation as a singular fractional Sturm-Liouville problem and they presented some results on the applications of Legendre polynomials in ordinary and partial differential equations [16].

The classical integer order Chebyshev equation is a second-order ordinary differential equation, where the solutions are Chebyshev polynomials of the first, second, third, and fourth kinds. For more details see [8, 20]. Moreover, Chebyshev polynomials are useful in obtaining numerical solutions of integral and differential equations by spectral method [2, 7, 18, 23]. Now the main motivation of the paper is to find the orthogonal family of polynomials on $[-1,1]$ for fractional Sturm-Liouville problems. In this paper, we define a FCDE of the following form

$$\left[{}^{c}D^{\alpha}_{1^-}(1-x^2)^{\frac{\alpha}{2}}D^{\alpha}_{-1^+}-\lambda_{n,\alpha}(1-x^2)^{\frac{-\alpha}{2}}\right]y(x)=0, \tag{1}$$

where ${}^{c}D^{\alpha}_{1^-}$ and $D^{\alpha}_{-1^+}$ are Caputo and Riemann-Liouville fractional derivatives, respectively. Note that for $\alpha=1$ the equation (1) is a classical Chebyshev differential equation of first kind, where $\lambda_{n,1}=n^2$ for Chebyshev polynomials of first kind [21]. In [5, 24] the authors applied fractional Chebyshev polynomials for solving fractional differential equations. Here we introduce a different family of orthogonal polynomials and we apply the family of orthogonal polynomials to solve some fractional differential equations. Indeed we generalized the Sturm-Liouville form of integer order Chebyshev equation to the fractional case. The paper is arranged in the following manner: In section 2, we give preliminary materials on fractional calculus. We find the solution of (1) in section 3. We prove that the solutions are $T_{n,\alpha}(x)=(1+x)^{\frac{\alpha}{2}}P_{n,\alpha}(x)$, where $P_{n,\alpha}(x)$ is a polynomial of degree n such that the coefficient of the polynomial is computed by solving a system of algebraic equations by a backward recursive formula. We prove the orthogonality of the polynomials $P_{n,\alpha}(x)$ with a specific weight function. Moreover, we show that for $\alpha=1$ the polynomials $P_{n,\alpha}(x)$ are indeed the classical Chebyshev polynomials of the third kind. In section 4, we give some applications in solving fractional differential equations.

2. Preliminaries

In this section, we give some preliminary materials of fractional calculus.

Definition 1 Let α be a positive real number and $f(x)\in L_1(a,b)$. Left-sided and right-sided Riemann-Liouville integrals of order α are defined as ([15])

$$I^{\alpha}_{a^+}f(x)=\frac{1}{\Gamma(\alpha)}\int_a^x\frac{f(s)}{(x-s)^{1-\alpha}}ds,\ x>a,$$

$$I^{\alpha}_{b^-}f(x)=\frac{1}{\Gamma(\alpha)}\int_x^b\frac{f(s)}{(s-x)^{1-\alpha}}ds,\ x<b.$$

Definition 2 Let α be a positive real number and $m-1<\alpha<m$ where m is an integer. Then left-sided and right-sided Riemann-Liouville fractional derivatives of order α are defined as ([15])

$$(D^{\alpha}_{a^+}f)(x)=D^m(I^{m-\alpha}_{a^+}f)(x),\ x>a,$$

$$(D^{\alpha}_{b^-}f)(x)=(-D)^m(I^{m-\alpha}_{b^-}f)(x),\ x<b,$$

Analogous formulas yield the left-sided and right-sided Caputo derivatives of order α :

$$({}^{c}D^{\alpha}_{a^+}f)(x)=(I^{m-\alpha}_{a^+}D^mf)(x),\ x>a,$$

$$({}^{c}D^{\alpha}_{b^-} f)(x) = (I^{m-\alpha}_{b^-}(-D)^m f)(x),\ x < b.$$

In our study of fractional Sturm–Liouville problems we shall apply the fractional version of integration by parts formula presented below.

Lemma 1 The following properties are satisfied ([15])

$$\int_a^b f(x) I^{\alpha}_{a^+} g(x) dx = \int_a^b g(x) I^{\alpha}_{b^-} f(x) dx,$$

$$\int_a^b f(x) D^{\alpha}_{b^-} g(x) dx = \int_a^b g(x) {}^{c}D^{\alpha}_{a^+} f(x) dx + \sum_{k=0}^{m-1} (-1)^{m-k} f^{(k)}(x) D^{m-k-1} I^{m-\alpha}_{b^-} g(x)|^b_{x=a}, \quad (2)$$

$$\int_a^b f(x) D^{\alpha}_{a^+} g(x) dx = \int_a^b g(x) {}^{c}D^{\alpha}_{b^-} f(x) dx + \sum_{k=0}^{m-1} (-1)^{k} f^{(k)}(x) D^{m-k-1} I^{m-\alpha}_{a^+} g(x)|^b_{x=a}. \quad (3)$$

Proof. The first property is easily proved by changing the order of integration. The second property is proved by using the first property and the relation between Caputo and Riemann-Liouville derivatives. The proof of the third property is similar to the second property. □

Property 1 If $m - 1 < \alpha < m$ and $\beta > \alpha$ and $f \in C[a, b]$, then the following identities are hold ([15])

(i) $D^{\alpha}_{a^+} I^{\alpha}_{a^+} f(x) = f(x),\ D^{\alpha}_{b^-} I^{\alpha}_{b^-} f(x) = f(x),$

(ii) $D^{\alpha}_{a^+} I^{\beta}_{a^+} = I^{\beta-\alpha}_{a^+} f(x),\ D^{\alpha}_{b^-} I^{\beta}_{b^-} = I^{\beta-\alpha}_{b^-} f(x),$

(iii) ${}^{c}D^{\alpha}_{a^+} I^{\alpha}_{a^+} f(x) = f(x),\ {}^{c}D^{\alpha}_{b^-} I^{\alpha}_{b^-} f(x) = f(x),$

(iv) $I^{\alpha}_{a^+} I^{\beta}_{a^+} f(x) = I^{\alpha+\beta}_{a^+} f(x),\ I^{\alpha}_{b^-} I^{\beta}_{b^-} f(x) = I^{\alpha+\beta}_{b^-} f(x).$

3. Fractional Chebyshev differential equation

In this section, we define FCDE and obtain main results of this paper. It is well known that the Chebyshev polynomials of the first kind are the solutions of the following second order classical differential equation:

$$\left[D(1-x^2)^{\frac{1}{2}} D + n^2 (1-x^2)^{-\frac{1}{2}}\right] y(x) = 0,$$

Extending this differential equation to fractional form we define the Fractional Chebyshev differential equation of the form (1), where $\lambda_{n,\alpha}$ is a constant number that will be computed. Here we use the following identity which is the result of integration by parts (2) and (3),

$$\int_{-1}^{1} \left[g(x) {}^{c}D^{\alpha}_{1^-} (1-x^2)^{\frac{\alpha}{2}} D^{\alpha}_{-1^+} f(x) - f(x) {}^{c}D^{\alpha}_{1^-} (1-x^2)^{\frac{\alpha}{2}} D^{\alpha}_{-1^+} g(x)\right] dx = 0. \quad (4)$$

Now we are ready to state one of the main results of this section.

Theorem 1 Let α be a positive real number. Then the Fractional Chebyshev differential equation (1) has solutions of the form $T_{n,\alpha}(x) = (1 + x)^{\frac{\alpha}{2}} P_{n,\alpha}(x),\ n = 0, 1, 2, \cdots$, where

$$\lambda_{n,\alpha} = \left[\frac{\Gamma(n + \frac{\alpha}{2} + 1)}{\Gamma(n - \frac{\alpha}{2} + 1)}\right]^2, \quad (5)$$

and

$$P_{n,\alpha}(x)=\sum_{k=0}^{n} a_k(1+x)^k. \tag{6}$$

The coefficients a_k are computed by the following recursive formula

$$a_k=-\frac{1}{\left[\frac{\Gamma(k+\frac{\alpha}{2}+1)}{\Gamma(k-\frac{\alpha}{2}+1)}\right]^2-\lambda_{\alpha,n}}\sum_{i=k+1}^{n}2^{i-k}\binom{i}{k}\left[\frac{\Gamma(i+\frac{\alpha}{2}+1)}{\Gamma(i-\frac{\alpha}{2}+1)}.\frac{\Gamma(k+\frac{\alpha}{2}+1)}{\Gamma(k-\frac{\alpha}{2}+1)}-\lambda_{n,\alpha}\right]a_i,$$
$$k=n-1,n-2,...,1,0, \tag{7}$$

and $a_n=(2\alpha)^n$.

Proof. First, we show that $T_{n,\alpha}(x)$ is a solution of FCDE (1). The second term in the left-hand side of the equation is computed as follows

$$\begin{aligned}(1-x^2)^{\frac{-\alpha}{2}}T_{n,\alpha}(x) &= (1-x)^{\frac{-\alpha}{2}}\sum_{k=0}^{n}a_k(1+x)^k\\ &= (1-x)^{\frac{-\alpha}{2}}\sum_{k=0}^{n}a_k(2-(1-x))^k\\ &= \sum_{k=0}^{n}\sum_{j=0}^{k}a_k\binom{k}{j}(-1)^j2^{k-j}(1-x)^{j-\frac{\alpha}{2}},\end{aligned}$$

and the first term in the left-hand side of the equation is computed as follows:

$$\begin{aligned}(1-x^2)^{\frac{\alpha}{2}}D^{\alpha}_{-1^+}T_{n,\alpha}(x) &= (1-x^2)^{\frac{\alpha}{2}}D^{\alpha}_{-1^+}\sum_{k=0}^{n}a_k(1+x)^{k+\frac{\alpha}{2}}\\ &= (1-x)^{\frac{\alpha}{2}}\sum_{k=0}^{n}a_k\frac{\Gamma(k+\frac{\alpha}{2}+1)}{\Gamma(k-\frac{\alpha}{2}+1)}(1+x)^k\\ &= (1-x)^{\frac{\alpha}{2}}\sum_{k=0}^{n}a_k\frac{\Gamma(k+\frac{\alpha}{2}+1)}{\Gamma(k-\frac{\alpha}{2}+1)}(2-(1-x))^k\\ &= \sum_{k=0}^{n}\sum_{j=0}^{k}a_k\binom{k}{j}(-1)^j2^{k-j}\frac{\Gamma(k+\frac{\alpha}{2}+1)}{\Gamma(k-\frac{\alpha}{2}+1)}(1-x)^{j+\frac{\alpha}{2}}.\end{aligned} \tag{8}$$

By taking the right Caputo derivative of the equation (8) we find

$$\begin{aligned}{}^cD^{\alpha}_{1^-}(1-x^2)^{\frac{\alpha}{2}}D^{\alpha}_{-1^+}T_{n,\alpha}(x) &= {}^cD^{\alpha}_{1^-}\sum_{k=0}^{n}\sum_{j=0}^{k}a_k\binom{k}{j}(-1)^j2^{k-j}\frac{\Gamma(k+\frac{\alpha}{2}+1)}{\Gamma(k-\frac{\alpha}{2}+1)}(1-x)^{j+\frac{\alpha}{2}}\\ &= \sum_{k=0}^{n}\sum_{j=0}^{k}a_k\binom{k}{j}(-1)^j2^{k-j}\frac{\Gamma(k+\frac{\alpha}{2}+1)}{\Gamma(k-\frac{\alpha}{2}+1)}\frac{\Gamma(j+\frac{\alpha}{2}+1)}{\Gamma(j-\frac{\alpha}{2}+1)}(1-x)^{j-\frac{\alpha}{2}}.\end{aligned}$$

Substituting in (1) we obtain:

$$\begin{aligned}&\sum_{k=0}^{n}\sum_{j=0}^{k}a_k\binom{k}{j}(-1)^j2^{k-j}\frac{\Gamma(k+\frac{\alpha}{2}+1)}{\Gamma(k-\frac{\alpha}{2}+1)}\frac{\Gamma(j+\frac{\alpha}{2}+1)}{\Gamma(j-\frac{\alpha}{2}+1)}(1-x)^{j-\frac{\alpha}{2}}\\ &\qquad-\lambda_{n,\alpha}\sum_{k=0}^{n}\sum_{j=0}^{k}a_k\binom{k}{j}(-1)^j2^{k-j}(1-x)^{j-\frac{\alpha}{2}}=0.\end{aligned} \tag{9}$$

Expanding (9) we obtain an algebraic system to compute the coefficients a_k as follows

$$a_0\left(\frac{\Gamma(1+\frac{\alpha}{2})}{\Gamma(1-\frac{\alpha}{2})}\frac{\Gamma(1+\frac{\alpha}{2})}{\Gamma(1-\frac{\alpha}{2})}-\lambda_{n,\alpha}\right)+2a_1\binom{1}{0}\left(\frac{\Gamma(2+\frac{\alpha}{2})}{\Gamma(2-\frac{\alpha}{2})}\frac{\Gamma(1+\frac{\alpha}{2})}{\Gamma(1-\frac{\alpha}{2})}-\lambda_{n,\alpha}\right)$$

$$+a_1(-1)^1\binom{1}{1}\left(\frac{\Gamma(2+\frac{\alpha}{2})}{\Gamma(2-\frac{\alpha}{2})}\frac{\Gamma(2+\frac{\alpha}{2})}{\Gamma(2-\frac{\alpha}{2})}-\lambda_{n,\alpha}\right)(1-x)+2^2a_2\left(\frac{\Gamma(3+\frac{\alpha}{2})}{\Gamma(3-\frac{\alpha}{2})}\frac{\Gamma(1+\frac{\alpha}{2})}{\Gamma(1-\frac{\alpha}{2})}-\lambda_{n,\alpha}\right)$$

$$+2a_2(-1)^1\binom{2}{1}\left(\frac{\Gamma(3+\frac{\alpha}{2})}{\Gamma(3-\frac{\alpha}{2})}.\frac{\Gamma(2+\frac{\alpha}{2})}{\Gamma(2-\frac{\alpha}{2})}-\lambda_{n,\alpha}\right)(1-x)+a_2\binom{2}{2}\left(\frac{\Gamma(3+\frac{\alpha}{2})}{\Gamma(3-\frac{\alpha}{2})}.\frac{\Gamma(3+\frac{\alpha}{2})}{\Gamma(3-\frac{\alpha}{2})}-\lambda_{n,\alpha}\right)(1-x)^2$$

$$\vdots$$

$$+2a_n(-1)^{n-1}\binom{n}{n-1}\left(\frac{\Gamma(n+1+\frac{\alpha}{2})}{\Gamma(n+1-\frac{\alpha}{2})}.\frac{\Gamma(n+\frac{\alpha}{2})}{\Gamma(n-\frac{\alpha}{2})}-\lambda_{n,\alpha}\right)(1-x)^{n-1}$$

$$+a_n(-1)^n\binom{n}{n}\left(\frac{\Gamma(n+1+\frac{\alpha}{2})}{\Gamma(n+1-\frac{\alpha}{2})}\frac{\Gamma(n+1+\frac{\alpha}{2})}{\Gamma(n+1-\frac{\alpha}{2})}-\lambda_{n,\alpha}\right)(1-x)^n=0,$$

Equating the coefficients of $(1-x)^n$ to zero and choosing a_n an arbitrary number for example $a_n=(2\alpha)^n$ we find

$$\lambda_{n,\alpha}=\left[\frac{\Gamma(n+\frac{\alpha}{2}+1)}{\Gamma(n-\frac{\alpha}{2}+1)}\right]^2.$$

Similarly equating the coefficients of $(1-x)^{n-1}$ to zero we find

$$a_{n-1}(-1)^{n-1}\binom{n-1}{n-1}\left(\frac{\Gamma(n+\frac{\alpha}{2})}{\Gamma(n-\frac{\alpha}{2})}\frac{\Gamma(n+\frac{\alpha}{2})}{\Gamma(n-\frac{\alpha}{2})}-\lambda_{n,\alpha}\right)(1-x)^{n-1}$$

$$+2a_n(-1)^{n-1}\binom{n}{n-1}\left(\frac{\Gamma(n+1+\frac{\alpha}{2})}{\Gamma(n+1-\frac{\alpha}{2})}\frac{\Gamma(n+\frac{\alpha}{2})}{\Gamma(n-\frac{\alpha}{2})}-\lambda_{n,\alpha}\right)(1-x)^{n-1}=0,$$

$$\Rightarrow a_{n-1}=-\frac{2\binom{n}{n-1}}{\left[\frac{\Gamma(n+\frac{\alpha}{2})}{\Gamma(n-\frac{\alpha}{2})}\right]^2-\lambda_{n,\alpha}}\left[\frac{\Gamma(n+1+\frac{\alpha}{2})}{\Gamma(n+1-\frac{\alpha}{2})}.\frac{\Gamma(n+\frac{\alpha}{2})}{\Gamma(n-\frac{\alpha}{2})}-\lambda_{n,\alpha}\right]a_n.$$

Finally equating the coefficients of $(1-x)^{n-2}$ we compute a_{n-2} in terms of a_{n-1} and a_n as follows:

$$a_{n-2}(-1)^{n-2}\binom{n-2}{n-2}\left(\frac{\Gamma(n-1+\frac{\alpha}{2})}{\Gamma(n-1-\frac{\alpha}{2})}.\frac{\Gamma(n-1+\frac{\alpha}{2})}{\Gamma(n-1-\frac{\alpha}{2})}-\lambda_{n,\alpha}\right)(1-x)^{n-2}$$

$$+2a_{n-1}(-1)^{n-2}\binom{n-1}{n-2}\left(\frac{\Gamma(n+\frac{\alpha}{2})}{\Gamma(n-\frac{\alpha}{2})}.\frac{\Gamma(n-1+\frac{\alpha}{2})}{\Gamma(n-1-\frac{\alpha}{2})}-\lambda_{n,\alpha}\right)(1-x)^{n-2}$$

$$+2^2a_n(-1)^{n-2}\binom{n}{n-2}\left(\frac{\Gamma(n+1+\frac{\alpha}{2})}{\Gamma(n+1-\frac{\alpha}{2})}.\frac{\Gamma(n-1+\frac{\alpha}{2})}{\Gamma(n-1-\frac{\alpha}{2})}-\lambda_{n,\alpha}\right)(1-x)^{n-2}=0,$$

$$\Rightarrow a_{n-2}=-\frac{1}{\left[\frac{\Gamma(n-1+\frac{\alpha}{2})}{\Gamma(n-1-\frac{\alpha}{2})}\right]^2-\lambda_{n,\alpha}}\Bigg[2\binom{n-1}{n-2}\left(\frac{\Gamma(n+\frac{\alpha}{2})}{\Gamma(n-\frac{\alpha}{2})}.\frac{\Gamma(n-1+\frac{\alpha}{2})}{\Gamma(n-1-\frac{\alpha}{2})}-\lambda_{n,\alpha}\right)a_{n-1}$$

$$+2^2\binom{n}{n-2}\left(\frac{\Gamma(n+1+\frac{\alpha}{2})}{\Gamma(n+1-\frac{\alpha}{2})}.\frac{\Gamma(n-1+\frac{\alpha}{2})}{\Gamma(n-1-\frac{\alpha}{2})}-\lambda_{n,\alpha}\right)a_n.$$

Continuing the same procedure we find the recursive formula (7). □

Now we investigate orthogonality property of $T_{n,\alpha}$.

Theorem 2 If m and n are nonnegative distinct integers then $T_{m,\alpha}(x)$ and $T_{n,\alpha}(x)$ are orthogonal in the following sense:

$$\int_{-1}^{1}\left(1-x^2\right)^{-\frac{\alpha}{2}}T_{m,\alpha}(x)T_{n,\alpha}(x)dx=0. \tag{10}$$

Moreover, the polynomials $P_{m,\alpha}(x)$ and $P_{n,\alpha}(x)$ are orthogonal in the following sense:

$$\int_{-1}^{1}\left(\frac{1+x}{1-x}\right)^{\frac{\alpha}{2}}P_{m,\alpha}(x)P_{n,\alpha}(x)dx=0. \tag{11}$$

Proof. We consider equation (1) for indices n and m as follows:

$$\left[{}^{c}D^{\alpha}_{1^-}(1-x^2)^{\frac{\alpha}{2}}D^{\alpha}_{-1^+}-\lambda_{n,\alpha}(1-x^2)^{\frac{-\alpha}{2}}\right]T_{n,\alpha}(x)=0,$$
$$\left[{}^{c}D^{\alpha}_{1^-}(1-x^2)^{\frac{\alpha}{2}}D^{\alpha}_{-1^+}-\lambda_{m,\alpha}(1-x^2)^{\frac{-\alpha}{2}}\right]T_{m,\alpha}(x)=0.$$

Multiplying the first equation by $T_{\alpha,m}$ and the second equation by $T_{\alpha,n}$, then subtracting the results we obtain:

$$\begin{aligned}T_{m,\alpha}(x){}^{c}D^{\alpha}_{1^-}(1-x^2)^{\frac{\alpha}{2}}D^{\alpha}_{-1^+}T_{n,\alpha}(x)-T_{n,\alpha}(x){}^{c}D^{\alpha}_{1^-}(1-x^2)^{\frac{\alpha}{2}}D^{\alpha}_{-1^+}T_{m,\alpha}(x)\\=\left[\lambda_{n,\alpha}-\lambda_{m,\alpha}\right](1-x^2)^{\frac{-\alpha}{2}}T_{n,\alpha}(x)T_{m,\alpha}(x),\end{aligned}$$

Now integrating over interval $[-1,1]$ and applying relation (4), we have

$$\left[\lambda_{n,\alpha}-\lambda_{m,\alpha}\right]\int_{-1}^{1}\left(1-x^2\right)^{-\frac{\alpha}{2}}T_{m,\alpha}(x)T_{n,\alpha}(x)dx=0,$$

Which leads to the orthogonality relation (10). By replacing $T_{n,\alpha}(x)=(1+x)^{\frac{\alpha}{2}}P_{n,\alpha}(x)$ and $T_{m,\alpha}(x)=(1+x)^{\frac{\alpha}{2}}P_{m,\alpha}(x)$ in the last equation , relation (11) is obtained. □

Definition 3 We define the Chebyshev norm of the function $f(x)$ as follows

$$\|f\|_C=\left(\int_{-1}^{1}(1-x^2)^{-\frac{\alpha}{2}}f(x)T_{n,\alpha}(x)dx\right)^{\frac{1}{2}}, \tag{12}$$

Corollary 1 Orthogonal properties in (10) and (11) imply that

$$\int_{-1}^{1}\left(1-x^2\right)^{-\frac{\alpha}{2}}T_{m,\alpha}(x)T_{n,\alpha}(x)dx=\delta_{mn}c_{n,\alpha}, \tag{13}$$

and

$$\int_{-1}^{1}\left(\frac{1+x}{1-x}\right)^{\frac{\alpha}{2}}P_{m,\alpha}(x)P_{n,\alpha}(x)dx=\delta_{mn}c_{n,\alpha}, \tag{14}$$

where $c_{n,\alpha}=\|T_{n,\alpha}\|^2_C$ is given by

$$c_{n,\alpha}=\sum_{j=0}^{n}\sum_{i=0}^{n}a_ia_j2^{i+j+1}\frac{\Gamma(i+j+\frac{\alpha}{2}+1)\Gamma(1-\frac{\alpha}{2})}{\Gamma(i+j+2)}. \tag{15}$$

TABLE 1. The first six polynomials $P_{n,\alpha}$ for $\alpha = 0.7$.

$P_{n,\alpha}(x)$	Roots of $P_{n,\alpha}(x)$
$P_1 = 1.4x - 0.49$	0.35
$P_2 = 1.96x^2 - 0.686x - 0.5733$	$-0.3934,\ 0.7434$
$P_3 = 2.744x^3 - 0.9604x^2 - 1.5119x + 0.24826$	$-0.6714,\ 0.1557,$ 0.8657
$P_4 = 3.8416x^4 - 1.3446x^3 - 3.0911x^2 + 0.68861x + 0.28501$	$-0.7963, -0.2243$ $0.4526, 0.9179$
$P_5 = 5.3782x^5 - 1.8824x^4 - 5.683x^3 + 1.4385x^2 + 1.1459x - 0.12263$	$-0.8619, -0.4531$ $0.0996, 0.6206$ 0.9448
$P_6 = 7.529x^6 - 2.635x^5 - 9.848x^4 + 2.676x^3 + 3.127x^2 - 0.511x - 0.140$	$-0.9004, -0.5966$ $-0.1562, 0.3200$ $0.7229, 0.9603$

Proof. By definition of $T_{n,\alpha}$ and change of variable $1 + x = 2u$, we obtain:

$$\begin{aligned}\int_{-1}^{1} \left(1-x^2\right)^{-\frac{\alpha}{2}} T_{n,\alpha}(x)T_{n,\alpha}(x)dx &= \int_{-1}^{1} \left(\frac{1+x}{1-x}\right)^{\frac{\alpha}{2}} (\sum_{j=0}^{n} a_j(1+x)^j)(\sum_{i=0}^{n} a_i(1+x)^i)dx \\ &= \sum_{j=0}^{n}\sum_{i=0}^{n} a_j a_i \int_{-1}^{1} (1+x)^{i+j+\frac{\alpha}{2}}(1-x)^{-\frac{\alpha}{2}}dx \\ &= \sum_{j=0}^{n}\sum_{i=0}^{n} a_j a_i 2^{i+j+1} \int_{0}^{1} u^{i+j+\frac{\alpha}{2}}(1-u)^{-\frac{\alpha}{2}}du.\end{aligned}$$

Finally, using Beta function, relation (13) is obtained. Equation (14) is obtained by setting $T_{n,\alpha}(x) = (1+x)^{\frac{\alpha}{2}} P_{n,\alpha}(x)$ in relation (13). □

Remark 1 We introduced new family of polynomials $P_{n,\alpha}$ that are orthogonal with weight function $w_\alpha(x) = (\frac{1+x}{1-x})^{\frac{\alpha}{2}}$ on $[-1, 1]$.
We computed $P_{n,\alpha}$ and the corresponding roots for $\alpha = 0.7,\ 0.9$ and some values of n in Tables 1 and 2. Now we show that for $\alpha = 1$ the polynomials $P_{n,1}$ are identical to the classical Chebyshev polynomials of third kind.
Theorem 3 For $\alpha = 1$ we have $P_{n,1}(x) = v_n(x)$ where $v_n(x)$ is the classical Chebyshev polynomials of third kind.

Proof. Using $\alpha = 1$ in the FCDE (1) we have:

$$\left[D(1-x^2)^{\frac{1}{2}}D + (n+\frac{1}{2})^2(1-x^2)^{\frac{-1}{2}}\right]T_{n,1} = 0. \tag{16}$$

It is clear that $U_{n+\frac{1}{2}}(x) = \cos((n+\frac{1}{2})\arccos x)$ is also a solution of equation (16). Due to definition of third order Chebyshev polynomials [8, 18] we have:

$$U_{n+\frac{1}{2}}(x) = \sqrt{\frac{1+x}{2}} v_n(x). \tag{17}$$

Using induction we show that $\frac{1}{\sqrt{2}}T_{n,1} = U_{n+\frac{1}{2}}$. We have

$$\frac{1}{\sqrt{2}}T_{n,1} = \frac{1}{\sqrt{2}}(1+x)^{\frac{1}{2}}P_{n,1}(x)\ ,\ U_{n+\frac{1}{2}} = \frac{1}{\sqrt{2}}(1+x)^{\frac{1}{2}}v_n(x).$$

TABLE 2. The first six polynomials $P_{n,\alpha}$ for $\alpha = 0.9$.

$P_{n,\alpha}(x)$	Roots of $P_{n,\alpha}(x)$
$P_1 = 1.8x - 0.81$	0.45
$P_2 = 3.24x^2 - 1.458x - 0.8613$	$-0.3375, 0.7875$
$P_3 = 5.832x^3 - 2.6244x^2 - 3.0268x + 0.66441$	$-0.6397, 0.2002$ 0.8895
$P_4 = 10.498x^4 - 4.7239x^3 - 8.0869x^2 + 2.3833x + 0.70144$	$-0.7763, -0.1906$ $0.4843, 0.9326$
$P_5 = 18.90x^5 - 8.503x^4 - 19.30x^3 + 6.422x^2 + 3.722x - 0.540$	$-0.8484, -0.4280$ $0.1281, 0.6435$ 0.8483
$P_6 = 34.01x^6 - 15.31x^5 - 43.25x^4 + 15.39x^3 + 13.26x^2 - 2.906x - 0.569$	$-0.8905, -0.5776$ $-0.1324, 0.3431$ $0.7400, 0.9675$

Indeed we have to show that for $\alpha = 1$ the polynomials $P_{n,1}(x)$ and $v_n(x)$ are identical. For $n = 0$ we have $P_{0,1} = v_0 = 1$. Now suppose for $j < n$ we have $P_{j,1} = v_j$. We can write $P_{n,1}$ in the following form

$$P_{n,1} = \sum_{k=0}^{n} B_k^n v_k = B_n^n v_n + \sum_{k=0}^{n-1} B_k^n P_{k,1}. \tag{18}$$

Multiplying both sides of (18) by $\left(\frac{1+x}{1-x}\right)^{\frac{1}{2}} P_{j,1}$ and integrating over $[-1,1]$ and using orthogonality of $P_{k,1}$ we find

$$0 = \int_{-1}^{1} \left(\frac{1+x}{1-x}\right)^{\frac{1}{2}} P_{n,1}(x) P_{j,1}(x) dx = B_j^n c_{j,1}.$$

Since $c_{j,1} \neq 0$ we conclude $B_j^n = 0$ for $j < n$. Using (18) we find $P_{n,1} = B_n^n v_n$, the leading coefficient in $P_{n,1}$ and v_n is 2^n. Thus, $B_n^n = 1$ and $P_{n,1} = v_n$. This completes the proof. □

A new class of orthogonal functions may be defined by fractional derivative of $T_{n,\alpha}$ as follows:

Corollary 2 The functions $P_{n,\alpha}^{+}$ defined by

$$P_{n,\alpha}^{+}(x) = D_{-1^+}^{\alpha} T_{n,\alpha}(x)$$

solve the fractional differential equation:

$$\left[D_{-1^+}^{\alpha} (1-x^2)^{\frac{\alpha}{2}} {}^{c}D_{1^-}^{\alpha} (1-x^2)^{\frac{\alpha}{2}} - \lambda_{n,\alpha}\right] P_{n,\alpha}^{+}(x) = 0. \tag{19}$$

Moreover, $P_{n,\alpha}^{+}$ are orthogonal in the following sense

$$\int_{-1}^{1} (1-x)^{\frac{\alpha}{2}} P_{n,\alpha}^{+}(x) P_{m,\alpha}^{+}(x) dx = \delta_{mn} \lambda_{n,\alpha} c_{n,\alpha} \tag{20}$$

Proof. Using relation (1) we have

$$(1-x^2)^{\frac{\alpha}{2}} {}^{c}D_{1^-}^{\alpha} (1-x^2)^{\frac{\alpha}{2}} D_{-1^+}^{\alpha} T_{n,\alpha}(x) - \lambda_{n,\alpha} T_{n,\alpha}(x) = 0.$$

Taking left Riemann-Liouville derivative from both sides of the last equation implies (19). To prove the second part, we use definition of $P^{+}_{n,\alpha}(x)$, relations (3) and (1) as follows

$$\begin{aligned}\int_{-1}^{1}(1-x^2)^{\frac{\alpha}{2}}P^{+}_{n,\alpha}(x)P^{+}_{m,\alpha}(x)dx &= \int_{-1}^{1}(1-x^2)^{\frac{\alpha}{2}}D^{\alpha}_{-1^+}T_{n,\alpha}(x).D^{\alpha}_{-1^+}T_{m,\alpha}(x)dx \\ &= \int_{-1}^{1}T_{m,\alpha}(x)^{c}D^{\alpha}_{1^-}(1-x^2)^{\frac{\alpha}{2}}D^{\alpha}_{-1^+}T_{n,\alpha}(x)dx \\ &= \lambda_{n,\alpha}\int_{-1}^{1}(1-x^2)^{\frac{-\alpha}{2}}T_{m,\alpha}(x)T_{n,\alpha(x)dx}.\end{aligned}$$

Finally, by using relation(13), we obtain (20). □

Corollary 3 Since some functions can be represented as a series of Chebyshev polynomials we have for

$$f(x)=\sum_{n=0}^{\infty}a_nT_{n,\alpha}(x),\ g(x)=\sum_{n=0}^{\infty}b_nT_{n,\alpha}(x).$$

the following relations

$$\int_{-1}^{1}(1-x^2)^{\frac{\alpha}{2}}D^{\alpha}_{-1^+}f.D^{\alpha}_{-1^+}gdx=\sum_{n=0}^{\infty}a_nb_n\lambda_{n,\alpha}c_{n,\alpha}.$$

Proof. The proof is immediate result of integration by parts, relations (13) and (1). □

4. Integral transform and their applications

In this section, first we define a sequence of integral transforms corresponding to $T_{n,\alpha}$ and the sequence of the corresponding inverse transforms. Then we find the solution of some nonhomogeneous fractional differential equations as an infinite series in terms of the sequence of inverse transforms.

Definition 4 The integral transform of a function $f\in L_2[-1,1]$ in terms of $T_{n,\alpha}$ is a sequence $F(n)$ defined by

$$F(n)=T[f](n)=\int_{-1}^{1}(1-x^2)^{\frac{-\alpha}{2}}f(x)T_{n,\alpha}(x)dx. \tag{21}$$

The corresponding inverse transform is a sequence defined by

$$T^{-1}[F(n)](x)=\sum_{n=0}^{\infty}\frac{1}{c_{n,\alpha}}F(n)T_{n,\alpha}(x). \tag{22}$$

Lemma 2 Define the fractional operator J_α as follows

$$J_\alpha={}^{c}D^{\alpha}_{1^-}(1-x^2)^{\frac{\alpha}{2}}D^{\alpha}_{-1^+}. \tag{23}$$

Then we have

$$T\left[(1-x^2)^{\frac{\alpha}{2}}J_\alpha f\right]=\lambda_{n,\alpha}F(n). \tag{24}$$

Proof. Using (21), (23) and (4) we conclude

$$\begin{aligned} T\left[(1-x^2)^{\frac{\alpha}{2}} J_\alpha f(x)\right] &= \int_{-1}^{1} J_\alpha f(x) T_{n,\alpha}(x)dx = \int_{-1}^{1} f(x) J_\alpha T_{n,\alpha}(x)dx \\ &= \lambda_{n,\alpha} \int_{-1}^{1} (1-x^2)^{\frac{-\alpha}{2}} f(x) T_{n,\alpha}(x)dx = \lambda_{n,\alpha} F(n). \end{aligned}$$

□

Now we give some applications in finding the solutions of some special fractional differential equations. We use $T_{n,\alpha}$ to solve typical fractional differential equations. Indeed, we generate particular solution in the form of series in terms of $T_{n,\alpha}$ using the integral transform (21).

Lemma 3 Assume $\lambda \neq \lambda_{n,\alpha}$ for $n \in \mathbb{N}_0$ and $g \in L_2[-1,1]$. If the integral transform of function g satisfies the condition $|G(n)| \leqslant M\sqrt{c_{n,\alpha}} n^\beta$ for $n = n_0, n_0+1, \ldots$ and $\alpha > \frac{1+\beta}{2}$. Then the solution of fractional differential equation

$$\left[(1-x^2)^{\frac{\alpha}{2}} J_\alpha - \lambda\right] f = g, \tag{25}$$

is given by the following infinite series

$$f(x) = \sum_{n=0}^{\infty} \frac{1}{c_{n,\alpha}} \frac{G(n)}{\lambda_{n,\alpha} - \lambda} T_{n,\alpha}. \tag{26}$$

Proof. The $T_{n,\alpha}$ integral transform of equation (25) implies

$$[\lambda_{n,\alpha} - \lambda] F(n) = G(n),$$

from which we get the Chebyshev transform of solution f(x) as

$$F(n) = \frac{G(n)}{\lambda_{n,\alpha} - \lambda},$$

that yields the solution in the form of a series of Chebyshev polynomials

$$f(x) = \sum_{n=0}^{\infty} \frac{1}{c_{n,\alpha}} \frac{G(n)}{\lambda_{\alpha,n} - \lambda} T_{n,\alpha}.$$

The convergence of this series is immediate consequence of boundedness assumption of $G(n)$. For $n > n_0$ using (12) we have

$$\| \frac{1}{c_{n,\alpha}} \frac{G(n)}{\lambda_{n,\alpha} - \lambda} T_{n,\alpha} \|_C \leq \frac{1}{|\lambda_{n,\alpha} - \lambda|} M n^\beta.$$

Using asymptotic property of the eigenvalues [13] we find

$$\lambda_{n,\alpha} \cong (n+1)^{2\alpha}, \quad n \longrightarrow \infty. \tag{27}$$

If $\alpha > \frac{1+\beta}{2}$, there exists $p > 1$ such that $\alpha \geq \frac{p+\beta}{2}$. Comparing the resulting series with the Dirichlet series $\sum_{n=1}^{\infty} \frac{1}{n^p}$ and using (27) we find

$$\frac{M n^\beta \frac{1}{|\lambda_{n,\alpha} - \lambda|}}{\frac{1}{n^p}} \cong \frac{M}{n^{2\alpha - p - \beta}} \longrightarrow 0.$$

Convergence of Dirichlet series for $p > 1$ implies uniform convergence of resulting series in $[-1,1]$ which indicates that series (26) is continuous in the interval $[-1,1]$.

□

TABLE 3. Results for Example 4 with $\lambda = 2$ in $[-1, 1]$.

m	$\|e_{m,\alpha}\|_\infty$ for $\alpha = 0.7$	$\|e_{m,\alpha}\|_\infty$ for $\alpha = 0.8$	$\|e_{m,\alpha}\|_\infty$ for $\alpha = 0.9$
2	7.5×10^{-16}	3.8×10^{-16}	2.4×10^{-15}
4	6.1×10^{-14}	3.0×10^{-14}	1.4×10^{-13}
6	2.0×10^{-13}	6.7×10^{-12}	7.9×10^{-12}

Example 1 For fix $m \in \mathbb{N}$ we consider the following nonhomogeneous equation in $[-1, 1]$

$$\left[(1 - x^2)^{\frac{\alpha}{2}} J_\alpha - \lambda\right] f = T_{m,\alpha}(x). \tag{28}$$

The $T_{n,\alpha}$ integral transform of above equation gives,

$$F(m) = c_{m,\alpha} \frac{1}{\lambda_{m,\alpha} - \lambda},$$

and $F(n) = 0$ for $n \neq m$. Using relation (22), the particular solution of nonhomogeneous fractional equation (28) is obtained as follows:

$$f(x) = \frac{1}{\lambda_{m,\alpha} - \lambda} T_{m,\alpha}(x).$$

We substitute $f(x)$ in (28) and denote the difference of both sides by $e_{m,\alpha}$. The infinite norm of $e_{m,\alpha}$ computed in Table 3 for different values of m and α.

Lemma 4 Assume $k_\alpha \in \mathbb{R}^-$, $\alpha > \frac{1+\beta}{2}$, $\lambda \neq \lambda_{\alpha,n}$ and $g \in L_2[-1, 1]$. Let the $T_{n,\alpha}$ integral transform of function g satisfy: $|G(n)| \leqslant M\sqrt{c_{n,\alpha}} n^\beta$ for $n > n_0$. Then, the partial fractional differential equation

$$k_\alpha (1 - x^2)^{\frac{\alpha}{2}} J_{\alpha,x} u(x, t) = \frac{\partial u(x, t)}{\partial t}, \ x \in [-1, 1], \ t \geqslant 0, \tag{29}$$

with the initial condition $u(x, 0) = g(x)$ for $t_0 > 0$ has a continuous solution in $[-1, 1] \times [t_0, \infty]$ given by

$$u(x, t) = \sum_{n=0}^{\infty} \frac{1}{c_{n,\alpha}} G(n) e^{k_\alpha \lambda_{n,\alpha} t} T_{n,\alpha}(x).$$

Proof. The $T_{n,\alpha}$ integral transform of equation (29) with respect to x gives,

$$k_\alpha \lambda_{n,\alpha} U(n, t) = \frac{\partial U(n, t)}{\partial t}.$$

Solving the last equation and applying the initial condition we find

$$U(n, t) = G(n) e^{k_\alpha \lambda_{n,\alpha} t}.$$

Thus, the solution of problem (29) is obtained by taking inverse $T_{n,\alpha}$ transform:

$$u(x, t) = \sum_{n=0}^{\infty} \frac{1}{c_{n,\alpha}} G(n) e^{k_\alpha \lambda_{n,\alpha} t} T_{n,\alpha}(x).$$

We prove the convergence of the series for $\geqslant t_0 > 0$. Taking Chebyshev norm with respect to x we have

$$\| \frac{1}{c_{n,\alpha}} G(n) e^{k_\alpha \lambda_{n,\alpha} t} T_{n,\alpha}(x) \|_C \leq M n^\beta e^{k_\alpha \lambda_{n,\alpha} t_0}.$$

If $\alpha > \frac{1+\beta}{2}$, there exists $p > 1$ such that $\alpha \geq \frac{p+\beta}{2}$. Using asymptotic form (27), the dominant series is convergent due to comparison test comparing with convergent Dirichlet series $\sum_{n=1}^{\infty} \frac{1}{n^p}$:

$$\frac{Mn^{\beta} e^{k_\alpha \lambda_{n,\alpha} t_0}}{\dfrac{1}{n^p}} \cong Mn^{\beta+p} e^{k_\alpha \lambda_{n,\alpha} t_0} \longrightarrow 0,$$

Convergence of Dirichlet series for $p > 1$ implies that our series is uniformly convergent in $[-1,1] \times [t_0, \infty]$ which indicates that series (30) is continuous in the interval $[-1,1] \times [t_0, \infty]$. $\square$

5. Conclusions

In this paper we introduce a typical fractional Chebyshev differential equation which leads to a family of orthogonal polynomials $P_{n,\alpha}(x)$, where α is a positive real number. For $\alpha = 1$ we prove that $P_{n,1}(x)$ coincide with the classical Chebyshev polynomials of third kind. Moreover, by defining an integral transform corresponding to $T_{n,\alpha}(x) = (1+x)^{\frac{\alpha}{2}} P_{n,\alpha}(x)$, we find the series solutions of some fractional differential equations. According to our knowledge, the orthogonality and the relevant properties obtained in this paper are different from the others used in the literature.

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