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Estimation of parameters in the Random walk distribution

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Abstract-:

The two parameters random walk distribution is defined and its characteristic are derived. Hazard function, mean, variance etc are worked out for random walk distribution. In this paper, moments and maximum likelihood estimators are obtained, we define various least square estimation method in general, and we then show how this method can be applied to the random walk. Numerical examples are given

Keywords: Inverse Gaussian distribution, Moments method, Brownian motion, least square method.

1) Introduction:

Statistical distributions such as Exponential, Weibull, Inverse Guassian, etc. have proved to be of considerable interest in terms of multivarious applications, in the field of reliability. These distributions are reasonable models for the life distribution of a device. Most of the distributions are motivated by considering either basic characteristics of fatigue process or a physical failure process or the wear out and aging properties of the devices.

In recent year many Inverse Guassian (IG) distribution we proposed notably by Johson and Kotz (1994) Akman and Gupta (1992) Abd-eEl- Hakim and Ahmad (1992) . .For some purposes, it is convenient to work with the reciprocal of the IG variable X, which will be denoted by Y, the probability density function of Y is given by

$$f(y) = \sqrt{\frac{\lambda}{2\pi y}} \exp\left(-\frac{\lambda}{2y}(y - 1/\mu)^2\right) \qquad y > 0 \qquad (1)$$

This distribution is called the Random Walk distribution see Wise (1966) and Wasan (1968). Random walk distribution their practical uses in Brownian motion. Knowledge of the properties of random walk provides a basis for understanding an extremely broad range of phenomena in physics, chemistry, biology and other areas. For this reason, random walk has been intensively investigated throughout this century. In recent years, considerable attention has been focused on the properties of random walk, fractals, and other non-Euclidean systems.

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The cdf corresponding to (1) is

$$F(y)=\phi[\sqrt{\lambda y}(1-1/(\mu y))]-e^{\frac{2\lambda}{\mu}}\phi[-\sqrt{\lambda y}(1+1/(\mu y))] \qquad (2)$$

Where

$$\phi(z)=1/\sqrt{2\pi}\int_{-\infty}^{z}e^{-x^2/2}dx$$

The reliability function and the hazard rate corresponding to the random walk distribution is

$$R(y)=1-\phi(\sqrt{\lambda y}(1-1/\mu y))+e^{\frac{2\lambda}{\mu}}\phi[-\sqrt{\lambda y}(1+1/\mu y)]$$

and

$$h(y)=\frac{\sqrt{\frac{\lambda}{2\pi y}}e^{-\frac{\lambda}{2y}(y-1/\mu)^2}}{1-\phi(\sqrt{\lambda y}(1-1/\mu y))+e^{\frac{2\lambda}{\mu}}\phi(-\sqrt{\lambda y}(1+1/\mu y))}$$

It can be shown that the hazard function for a random walk distribution is a monotonically decreasing function of y if $y\geq 2\lambda/\mu^2$.

$$h(y) = f(y)/(1-F(y))$$

then

$$h'(y) = (R(y)\, f'(y) + f^2(y))/R^2(y) \qquad (3)$$

The denominator is nonnegative for all y. Hence, it is sufficient to show that the numerator of equation (3) is < 0, i.e.

$$R(y)\, f'(y) + f^2(y) < 0 \qquad (4)$$

Now, the derivative of the density function (1) with respect to y terms is

$$df(y)/dy=-f(y)[\frac{y+\lambda(y^2-1/\mu^2)}{2y^2}]$$

So now the condition that must be satisfied is

$$f(y)[-\frac{(y+\lambda(y^2+1/\mu^2)}{2y^2}R(y)+f(y)]\leq 0$$

Since $f(y)\geq 0$ by definition and $R(y)\geq 0$, We may use the condition

$$\frac{y+\lambda y^2-\lambda/\mu^2}{2y^2}\int_y^\infty f(x)dx=[\frac{\mu^2+4\lambda^2}{8\lambda}-\frac{\lambda}{2\mu^2}(1/y-\frac{\mu^2}{2\lambda})^2]\int_y^\infty f(x)dx$$
$$\geq\int_y^\infty\frac{\mu^2+4\lambda^2}{8\lambda}-\frac{\lambda}{2\mu^2}(1/x-\frac{\mu^2}{2\lambda})^2 f(x)dx$$

If $y > 2\lambda/\mu^2$. Then

$$\frac{y+\lambda(y^2-1/\mu^2)}{2y^2}R(y)\geq\int_y^\infty\frac{x+\lambda(x^2-1/\mu^2)}{2x^2}f(x)dx=-\int_y^\infty df(x)=f(y)$$

to obtain

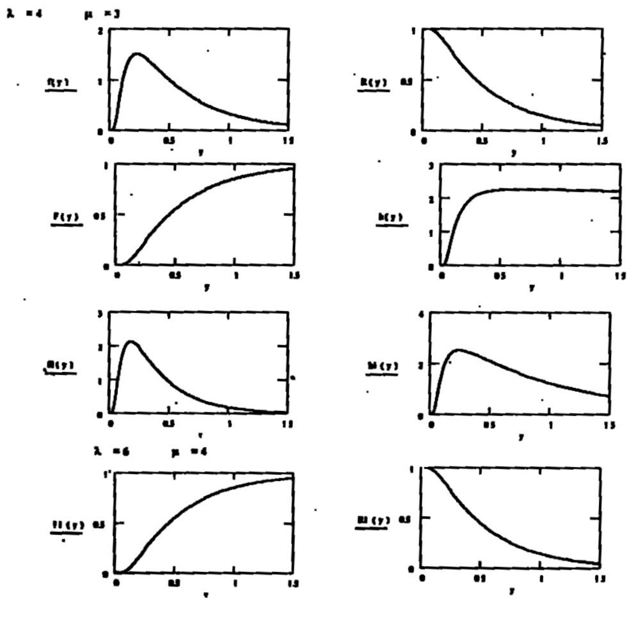
$$f(y)\leq\frac{y+\lambda y^2-\lambda/\mu^2}{2y^2}\int_y^\infty f(x)dx \qquad \text{if} \qquad y\geq\frac{2\lambda}{\mu^2}$$

so

$$f(y)[f(y)-\frac{y+\lambda y^2-\lambda/\mu^2}{2y^2}R(y)]\leq 0 \qquad \text{if} \qquad y\geq 2\lambda/\mu^2$$

and therefor the random walk hazard function is monotonically decreasing function of time y if $y \geq 2\lambda/\mu^2$. The plots of f(y), F(y), R(y) and h(y) for different values of λ and μ are shown in figure (1).

Figure (1)



The moment-generating function of the Random Walk distribution turns out to be.

$$M(t) = \frac{1}{\sqrt{1+2t/\lambda}} \exp(\frac{\lambda}{\mu}(1-\sqrt{1+2t/\lambda}))$$

and the corresponding cumulant – generating function

$$\Psi(t;\lambda,\mu) = \frac{\lambda}{\mu}(1-\frac{1}{\sqrt{1+2t/\lambda}}) - 1/2\log(1+2t/\lambda)$$

The mean and variance of RW distribution are now easily determined

$$E(y) = 1/\mu + 1/\lambda$$

And

$$v(y) = 2/\lambda^2 + 1/\lambda\mu$$

The distribution is unimodal and its shape depends only on the value of λ/μ. Hence (μ,λ) are not location / scale parameters in the usual sense. Also notice that,

$$\mathrm{mod}e(y) = (1/\mu)[\sqrt{1+\frac{\mu^2}{4\lambda^2}} - \mu/\lambda]$$

The coefficient of skewneesse is:

$$\gamma_1 = \frac{(4/\mu+8/\lambda)}{\sqrt{\lambda}(1/\mu+2/\lambda)^{3/2}}$$

And the coefficient of excess (or kurtosis) is:

$$\gamma_2 = \frac{3(16/\lambda+5/\mu)}{\lambda(2/\lambda+1/\mu)^2}$$

So, in general, the curve is leptokurtic, and is mesokurtic if $\lambda = 4\mu$ or $\lambda \Rightarrow \infty$ respectively, that is, RW approach normality $\gamma_2 \Rightarrow 3$. Wise (1966) has shown that the density function has two points of inflection, at values of y, satisfying the equation

$$u^4 + 2u^3 + 4 = (\lambda/\mu)^2 + 1/4$$

where

$$u = 1/2 + 1/2(\lambda/\mu)(y/\mu - \mu/y).$$

A measure of the reliability characteristic of a product, Component, or a system is the mean residual life function $L(y)>0$. It is defined as

$$L(y) = E(Y-y \mid Y \geq y) \qquad y\geq 0$$

In other words, the me   life, Y-y, given that the product, Component, or system has survival to time y Leemis(1995)

Note that

$$L(y) = \frac{\int_0^\infty R(y+x)dx}{R(y)} = \frac{1}{R(y)} \int_y^\infty xf(x)dx - y$$

Where $R(y) \geq 0$ and that $L(0) = 1/\lambda + 1/\mu$ the mean life.

2:- Properties of the random walk distribution.

The random walk distribution has the following properties:

1 - If $y \neg RW(\lambda,\mu)$, then $z = 1/y \neg IG(\lambda,\mu)$ inverse Gaussian distribution with two parameters λ, μ.

2- If $y \neg RW(\lambda,\mu)$, then $Z \neg \chi^2(1)$ where $Z = (y-1/\mu)^2/y$

3 -For large values of μ, the RW distribution is very similar to a Gamma with parameters $(1/2, 2/\lambda)$.

We study the properties of the random walk distribution in section (2). Sometimes, maximum likelihood estimation breaks down for some models, In section (3) we desciss the maximum likelihood method, in this paper, we deal with some least square approaches for estimating the parameters of the RW model in section (4) and section (5). in section (6) we desciss the moment method, in section (7) we comparison between the least square and the maximum likelihood method, the comparison will base on the mean-square error.

3: Maximum Likelihood estimation

Ahmad (1993) has derived the maximum likelihood estimator, using the density function (1), let $y_1, y_2, \ldots, y_n$ be a random sample from the random walk distribution with parameters (λ, μ),

$$L(y_1, y_2, \ldots, y_n; \mu, \lambda) = [\frac{\lambda}{2\pi}]^{n/2} \prod_i^n y_i^{-1/2} \exp[-\lambda/2 \sum_i^n \frac{(y_i - 1/\mu)^2}{y_i}$$

$$\log L = -n/2 \log \lambda - n/2 \log 2\pi - \lambda/2 \sum \frac{(y_i - 1/\mu)^2}{y_i}$$

$$\frac{\partial \log L}{\partial \lambda} = \frac{-n}{2\lambda} - 1/2 \sum \frac{(y_i - 1/\mu)^2}{y_i}$$

$$\frac{\partial \log L}{\partial \mu} = -\frac{\lambda}{\mu^2} \sum \frac{(y_i - 1/\mu)}{y_i}$$

the MLE for the two unknown parameters will be

$$\hat{\mu} = \frac{\sum 1/y_i}{n}$$

and

$$1/\hat{\lambda} = \frac{\sum 1/y_i (y_i - 1/\hat{\mu})^2}{n} = \bar{y} - 1/\hat{\mu}$$

Let us study the properties of this estimator. Since the random varible y is Random Walk distribution with parameters λ and μ, 1/y is Inverse Gaussian distribution with parameters λ and μ, then $\mu^\wedge = 1/n\Sigma\, 1/y_i$ is Inverse Gaussian distribution with parameters nλ and μ. It happen to be unbiased – in fact, it is an minimum variance unbiased estimators, and the asymptotic variance-covariance matrix of the estimator λ and μ can be obtained by inverting the Fisher- information matrix I.

$$I = \begin{bmatrix} \frac{n}{2\hat{\lambda}^2} & 0 \\ 0 & \frac{n\hat{\lambda}}{\hat{\mu}^3} \end{bmatrix}$$

The MLE of the reliability at time y_0 is simply,

$$R(y_0)=1-\varphi(\sqrt{\hat{\lambda}}\, y_0(1-1/\hat{\mu}\, y_0))+e^{2\hat{\lambda}/\hat{\mu}}\phi(-\sqrt{\hat{\lambda}}\, y_0(1+1/\hat{\mu}\, y_0))$$

This estimator is not unbiased, but it is a very good estimator.

Maximum likelihood method can choose a distribution which makes the obtained sample values (locally) most probable. A weaker alternative is to suggest that the sample values be as near as possible, in some sense, to their expected values. The usual sense (classically) has been based on the ordinary Euclidean distance.

4: Least squares method

The method of least squares provides an efficient and unbiased estimator of the distribution parameters. The method defines the best fit as one that minimizes the sum of squared error between the observed data and the fitted distribution (Elsayed and Boucher (1994)).

The following method goes back to Bain and Antle(1976):

Let $x_1, x_2, \ldots, x_n$ be independent random variables that are P_θ-distributed, $\theta \in \Theta$ is an unknown parameter. Suppose we observe only some order statistics $X_{(i)}$, $i \in I$, where for example in the case of type II censoring i= (r+1, r+2,.......n).

The general idea of the method is to choose θ in that way that for some suitable function U_θ, the values U_θ (X) are close to their expected values, i.e. we want to minimize over $\theta \in \Theta$ or some reasonable- chosen regarding the observations- subset θ_x of θ

$d[U_\theta(X), E_\theta(U_\theta(X))]$ where $X = \{x_1, x_2, \ldots, x_n\}$ and d denotes a suitable metric.

The choice $U_\theta = F_\theta$, the cumulative function of P_θ , has the advantage that $E_\theta[U_\theta(X_{(i)})] = i/(n+1)$, i=1,2,.....,n independent from θ, hence we want to solve

$$\mathrm{Min}_\theta\ d[F_\theta(X_{(i)}),\ i/(n+1),\ i \in I] \qquad (5)$$

An alternative least square method is the following . Let F^- denotes the quintal function of a cumulative distribution function F , we may try to fit the observations $X_{(i)}$, $i \in I$ to $F^-[i/(n+1)]$ $i \in I$ i.e. we may want to solve the minimizing problem

$$\mathrm{Min}_\theta\ d[X_{(i)}, F^-_\theta(i/(n+1)), i \in I] \qquad (6)$$

For the special choice of $d = d_2$, the Euclidean metric, we have to minimize the statistics

$$\sum_{i\in I}[X_{(i)} - F^{-}(\frac{i}{n+1})]^2$$

Therefore, in section 5 we deal with some least squares approach for estimating of the two-parameters Random walk distribution.

5- Least Square estimators for the random walk distribution.

Let $y_1, y_2, \ldots, y_n$ represent the ordered failure times random walk distribution with pdf given by (1). As we have seen, the survival distribution of y can depend on unknown parameters. Now we will demonstrate a parameter estimation on basis of observations. A reasonable restriction of the admissible region of the minimization problem in $\Theta_0 = \{\lambda, \mu, \lambda > 0, \mu > 0\}$, after squaring and using the Euclidean metric $d = d_2$, we get the following problem with $\xi_i = i/(n+1)$, $i \in I$,

$$\min_{\lambda,\mu}\sum[\phi(\sqrt{\lambda y_i}(1-1/\mu y_i) - e^{2\lambda/\mu}\phi(-\sqrt{\lambda y_i}(1+1/\mu y_i) - \zeta_i]^2 \ldots\ldots\ldots\ldots(7)$$

At least two different observation values the minimization problem has a solution by differentiating (7) with respect to λ and μ and equating to zero, we have

and

$$\sum(A_i - \zeta_i)\sqrt{\frac{y_i}{2\pi\lambda}}\exp(-\frac{\hat{\lambda}}{2}y_i(1-1/\mu\hat{y}_i)^2 - \frac{2}{\mu}e^{\frac{2\lambda}{\mu}}\phi(-\sqrt{\lambda y_i}(1+\frac{1}{\mu y_i})) = 0\ldots\ldots\ldots(9)$$

where

$$\phi(z) = \int_{-\infty}^{z}\frac{1}{\sqrt{2\pi}}e^{-x^2/2}dx$$

and

$$A_i = \phi_i^{\wedge}(\sqrt{\hat{\lambda}}y_i(1-1/\hat{\mu}y_i)) - e^{\frac{2\lambda}{\hat{\mu}}}\phi_i^{\wedge}(-\sqrt{\hat{\lambda}}y_i(1+/\hat{\mu}y_i))$$

The estimation of the parameters can be obtained by solving equations (8) and (9) simultaneously, we obtain

$$\hat{\mu} = \frac{2\sqrt{2\pi\hat{\lambda}}\sum C_j\phi(-\sqrt{\hat{\lambda}}y_j(1+1/\mu\hat{y}_j))}{\sum S_j v_j\sqrt{y_j}}$$

and

$$\hat{\lambda} = 2\pi [\frac{\sum \zeta_i \phi(-\sqrt{\hat{\lambda}} y_i (1+1/\hat{\mu} y_i))}{\sum B_i \phi(-\sqrt{\hat{\lambda}} y_i (1+1/\hat{\mu} y_i)}]^2$$

where,

$$C_j = F(y_j) - \zeta_j$$

$$v_j = e^{-\frac{2\hat{\lambda}}{\hat{\mu}}(1+1/\hat{\mu} y_j)}$$

and

$$B_j = \int_0^{y_j} x^{-1/2} e^{-\frac{\lambda}{2x}(x-1/\mu)^2} dx$$

The solution of these equation given a vector θ = (λ , μ), the system of the two nonlinear equations in $\theta \in (\lambda,\mu)$ is can be solved numerically by using the Newton-Raphson method or trial and error.

6: Moments estimations

The main idea of the method of moments is to equate certain sample characteristics such as mean and variance to the corresponding population expected values and then solve the resulting equations to obtain the estimates of the unknown parameters values.

Let $y_1, y_2, \ldots, y_n$ represent a random sample from the RW distribution with parameters λ and μ. The mean and variance are given by

$$E(y) = 1/\lambda + 1/\mu \quad \text{and } v(y) = 2/\lambda^2 + 1/\lambda\mu$$

We replace E(y) and $E(y^2)$ by their estimator M_1 and M_2 respectively

$$M_1 = 1/\mu + 1/\lambda \qquad (10)$$

$$M_2 - M_1^2 = 2/\lambda^2 + 1/\mu \qquad (11)$$

Where

$$M_k = 1/n \; \Sigma \, y_i^k \quad \text{for} \quad k=1,2,\ldots \qquad (12)$$

Solving the equations (10), (11) and (12) simultaneously yields.

$$\bar{\lambda} = \frac{2}{\sqrt{4M_2 - 3M_1^2} - M_1}$$

and

$$\bar{\mu} = \frac{2}{3M_1 - \sqrt{4M_2 - 3M_1^2}}$$

7 :- NUMIRICAL EXAMPLE:

The values of {EMBED Equation.3} and {EMBED Equation.3} are too hard to derived mathematically. Instead, in the present section, we use three methods of estimation, least square LSE, moment (ME) and maximum likelihood method (MLE). An iteration procedure will be developed for finding the MLE's using the Newton-Raphson method. For some values of μ and λ randome , sample of size 500 is generated. Table 1 summarizes the results we obtained; ME, MLE and LSE. It is clear from the results that the ME method is better than MLE method with respect to {EMBED Equation.3} , where MLE is better than ME with respect to {EMBED Equation.3}.

Table 1

Exact values	ME	MSE	MLE	MSE	LSE	MSE
$\mu = 8$	7.843	.0031	7.877	.0019	7.292	.063
$\lambda = 20$	19.780	.00242	19.567	.0094	19.359	.021
$\mu = 4$	3.890	.0033	3.972	.00019	3.589	.041
$\lambda = 6$	6.056	.00052	5.867	.0029	5.392	.062
$\mu = 3$	2.981	.00012	2.995	.00001	2.485	.078
$\lambda = 10$	9.927	.00053	9.778	.0049	9.126	.067
$\mu = 1$	1.005	.00025	1.006	.00002	0.921	.0062
$\lambda = 5$	4.904	.0018	4.889	.0025	4.562	.038
$\mu = 4$	3.957	.00046	3.962	.00036	4.162	.0039
$\lambda = 12$	11.784	.0039	11.736	.0059	11.445	.025
$\mu = 10$	10.050	.00025	10.056	.0003	9.621	.014
$\lambda = 50$	49.035	.0185	48.896	.024	48.392	.0517

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