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GENERAL RECURRENCE RELATIONS BASED ON UPPER RECORD VALUES

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Abstract:

In this paper, we establish general recurrence relations satisfied by the single and double expected values of any measurable function of upper record values. Then, we use these relations to obtain the corresponding recurrence relations for the moments, moment generating functions (MGFs), and factorial moment generating functions (FMGFs) of upper record values. We consider some examples including, exponenetial, Weibull, Pareto, generalized Pareto, Burr, logistic, half logistic, log-logistic, skewed and parabolic distributions. Finally, we present an application in life testing.

1 Introduction

Record values arise naturally in many real life applications involving data relating to weather, sport, economics and life testing studies. Many authors have studied record values and associated statistics; for example, see Chandler (1952), Galambos (1978), Resnick (1987), Nevzorov and Balakrishnan (1998), Nagaraja (1988), Ahsanullah (1980, 1988, 1990, 1993, 1995), Arnold and Balakrishnan (1989), and Arnold, Balakrishnan and Nagaraja (1992, 1998). Balakrishnan, Ahsanullah (1994a,b, 1995), Balakrishnan, Ahsanullah and Chan (1992, 1995), Balakrishnan and Chan (1993), and Balakrishnan, Chan and Ahsanullah (1993), have estabished some recurrence relations for moments of record values for the generalized Pareto, Lomax, exponential, Gumbel, logistic, Weibull and generalized extreme value distributions, respectively.

Let $X_{U(1)}, X_{U(2)}, \ldots, X_{U(n)}$ be the first n upper record values from a population whose density function (*pdf*) $f(x)$ and cumulative distribution function (*cdf*) $F(x)$.

Then, the *pdf* of $X_{U(n)}$ is given by

$$f_n(x) = \frac{1}{\Gamma(n)}\{-\log[1-F(x)]\}^{n-1}f(x), \quad -\infty < x < \infty, \ n = 1, 2, \ldots. \quad (1.1)$$

and the joint *pdf* of $X_{U(m)}$ and $X_{U(n)}$, $(m < n)$, is given by

$$\begin{aligned} f_{m,n}(x,y) &= \frac{1}{\Gamma(m)\Gamma(n-m)}\{-\log[1-F(x)]\}^{m-1} \\ &\times \frac{f(x)}{1-F(x)}\{-\log[1-F(y)]+\log[1-F(x)]\}^{n-m-1}f(y), \\ &\qquad -\infty < x < y < \infty, \ m, n = 1, 2, \ldots. \end{aligned} \quad (1.2)$$

In this paper, we extend the previous results of establishing recurrence relations between the moments of record values by deriving general forms for the expected values of any measurable functions of record values. Also, we establish recurrence relations between the moments, moment generating functions MGFs and factorial moment generating functions FMGFs of record values.

After presenting the main results in the following section, we use these results in Section 3 to establish some recurrence relations for the single and double moments of upper record values from some distributions including Weibull, Pareto, generalized Pareto, Burr, log-logistic, parabolic and skewed.

In Sections 4 and 5, we specialize the results given in Section 2 to establish some recurrence relations for the single and double MGFs and FMGFs of upper record values for exponential and logistic distributions, respectively. An application in life testing is finally presented in Section 6.

2 Main Results

Form (1.1), the expected value for any measurable function $g(x)$ is obtained to be

$$\begin{aligned} E\left(g(X_{U(n)})\right) &= E\left(g_n(X)\right) \\ &= \frac{1}{\Gamma(n)}\int_{-\infty}^{\infty} g(x)\{-\log[1-F(x)]\}^{n-1}f(x)dx, \ n = 1, 2, \ldots, \end{aligned} \quad (2.1)$$

and, the expected value for any measurable function $\tau(x,y)$ is obtained from (1.2) to be

$$\begin{aligned} E\left(\tau(X_{U(m)}, X_{U(n)})\right) &= E\left(\tau_{m,n}(X,Y)\right) \\ &= \frac{1}{\Gamma(m)\Gamma(n-m)}\int_{-\infty}^{\infty}\int_{x}^{\infty} \tau(x,y)\{-\log[1-F(x)]\}^{m-1} \\ &\times \frac{f(x)}{1-F(x)}\{-\log[1-F(y)]+\log[1-F(x)]\}^{n-m-1}f(y)dydx, \\ &\qquad m, n = 1, 2, 3, \ldots, \ m < n. \end{aligned} \quad (2.2)$$

The expected values given in (2.1) and (2.2), satisfy the following two theorems.

Theorem 2.1

For any measurable function $g(x)$ of the n-th upper record value and $n = 1, 2, \ldots$

$$E\Big(g_{n+1}(X) - g_n(X)\Big) = \frac{1}{\Gamma(n+1)} \int_{-\infty}^{\infty} \{-\log[1-F(x)]\}^n \{1-F(x)\} dg(x). \qquad (2.3)$$

Theorem 2.2

For any measurable function $\tau(x, y)$ of record values and $n, m = 1, 2, \ldots, m < n$,

$$\begin{aligned} E\Big(\tau_{m,n+1}(X,Y) - \tau_{m,n}(X,Y)\Big) &= \frac{1}{\Gamma(m)\Gamma(n-m+1)} \int_{-\infty}^{\infty} \int_{x}^{\infty} \{-\log[1-F(x)]\}^{m-1} \\ &\times \frac{f(x)}{1-F(x)} \{-\log[1-F(y)] + \log[1-F(x)]\}^{n-m} \\ &\times \{1-F(y)\} d\tau_x(x,y), \end{aligned} \qquad (2.4)$$

where $d\tau_x(x, y)$ means that the differentiation is with respect to y.

3 Recurrence Relations for Moments of Record Values

In this section, we use Theorem 2.1 and Theorem 2.2 to establish general recurrence relations satisfied by the single and double moments of upper record values. Then, we present some examples.

3.1 General relations

By putting $g(x) = x^k$, $k = 0, 1, 2, \ldots$ in Theorem 2.1, we get

$$\mu_{n+1}^{(k)} - \mu_n^{(k)} = \frac{k}{\Gamma(n+1)} \int_{-\infty}^{\infty} x^{k-1} \{-\log[1-F(x)]\}^n \{1-F(x)\} dx, \qquad (3.1)$$

which represents general form for establishing recurrence relations for the single moments of upper record values.

Also, by putting $\tau(x, y) = x^j y^k$, $j, k = 0, 1, 2, \ldots$ in Theorem 2.2, we get

$$\begin{aligned} \mu_{m,n+1}^{(j,k)} - \mu_{m,n}^{(j,k)} &= \frac{k}{\Gamma(m)\Gamma(n-m+1)} \int_{-\infty}^{\infty} \int_{x}^{\infty} x^j y^{k-1} \{-\log[1-F(x)]\}^{m-1} \\ &\times \frac{f(x)}{1-F(x)} \{-\log[1-F(y)] + \log[1-F(x)]\}^{n-m} \\ &\times \{1-F(y)\} dy dx, \end{aligned} \qquad (3.2)$$

which represents general form for establishing recurrence relations for the double moments of upper record values.

3.2 Examples

In this subsection, we present some examples as special cases from the relations given in (3.1) and (3.2). These examples including Weibull, Pareto, generalized Pareto, Burr, log-logistic parabolic and skewed distributions. In all of these examples, we establish recurrence relations for the single and double moments of upper record values. Basically, this technique depends on writing $\{1 - F(.)\}$ in terms of $f(.)$ through the relationship between *pdf* and *cdf*.

Example 3.1. Weibull distribution

The *pdf* of Weibull distribution is given by

$$f(x) = px^{p-1}e^{-x^p}, \; p > 0, \; x \geq 0, \tag{3.3}$$

and

$$1 - F(x) = \frac{x^{1-p}}{p} f(x), \tag{3.4}$$

First, by using (3.12) in (3.1), we get

$$\mu_{n+1}^{(k)} - \mu_n^{(k)} = \frac{k}{p}\mu_{n+1}^{(k-p)}, \tag{3.5}$$

which represents recurrence relation for the single moments of upper record values from Weibull distribution.

Next, by using (3.12) in (3.2), we get

$$\mu_{m,n+1}^{(j,k)} - \mu_{m,n}^{(j,k)} = \frac{k}{p}\mu_{m,n+1}^{(j,k-p)}, \; k > p, \; m < n, \; n = 1, 2, \ldots, \tag{3.6}$$

which represents recurrence relation for the double moments of upper record values from Weibull distribution.

Putting $p = 1$ in (3.3) and (3.4), we get the corresponding recurrence relations in the exponential case. Also, by putting $p = 2$ in (3.3) and (3.4), we get the corresponding recurrence relations from Rayleigh distribution.

Example 3.2. Pareto distribution

The *pdf* of Pareto distribution is given by

$$f(x) = \nu a^{\nu} x^{-\nu-1}, \; x \geq a, \; \nu > 0, \tag{3.7}$$

and

$$1 - F(x) = \frac{x}{\nu} f(x). \tag{3.8}$$

First, by using (3.8) in (3.1), we get

$$\left(1-\frac{k}{\nu}\right)\mu_{n+1}^{(k)}=\mu_n^{(k)},\ k\neq\nu, \tag{3.9}$$

which represents recurrence relation for the single moments of upper record values from Pareto distribution.

Next, by using (3.8) in (3.2), we get

$$\left(1-\frac{k}{\nu}\right)\mu_{m,n+1}^{(j,k)}=\mu_{m,n}^{(j,k)},\ k\neq\nu,\ m<n,\ n=1,2,\dots, \tag{3.10}$$

which represents recurrence relation for the double moments of upper record values from Pareto distribution.

Example 3.3. Generalized Pareto distribution

The *pdf* of generalized Pareto distribution is given by

$$f(x)=\begin{cases}(1+\beta x)^{-(1+1/\beta)}, & x\geq 0,\ \text{for}\ \beta>0,\\ & 0<x<-\frac{1}{\beta},\ \text{for}\ \beta<0,\\ e^{-x}, & x\geq 0\ \text{for}\ \beta=0,\\ 0 & \text{otherwise},\end{cases} \tag{3.11}$$

and

$$1-F(x)=(1+\beta x)f(x). \tag{3.12}$$

First, by using (3.12) in (3.1), we get

$$(1-k\beta)\mu_{n+1}^{(k)}=\mu_n^{(k)}+k\mu_{n+1}^{(k-1)}, \tag{3.13}$$

which represents recurrence relation for the single moments of upper record values from generalized Pareto distribution. For $\beta=0$, the above recurrence relation reduces to the corresponding one in the exponential distribution, see Balakrishnan and Ahsanullah (1994a).

Next, by using (3.12) in (3.2), we get

$$(1-k\beta)\mu_{m,n+1}^{(j,k)}=\mu_{m,n}^{(j,k)}+k\mu_{m,n+1}^{(j,k-1)},\ m<n,\ n=1,2,\dots, \tag{3.14}$$

which represents recurrence relation for the double moments of upper record values from generalized Pareto distribution.

Example 3.4. Burr distribution

The *pdf* of Burr distribution is given by

$$f(x)=\alpha p x^{p-1}(1+x^p)^{-(\alpha+1)},\ x\geq 0,\ p>1,\ x>0, \tag{3.15}$$

and

$$1 - F(x) = \frac{x^{1-p}}{\alpha p} f(x) + \frac{x}{\alpha p} f(x). \tag{3.16}$$

First, by using (3.16) in (3.1), we get

$$\left(1 - \frac{k}{\alpha p}\right) \mu_{n+1}^{(k)} = \mu_n^{(k)} + \frac{k}{\alpha p} \mu_{n+1}^{(k-p)}, \; k < p, \alpha p \neq k, \tag{3.17}$$

which represents recurrence relation for the single moments of upper record values from Burr distribution.

Next, by using (3.16) in (3.2), we get

$$\left(1 - \frac{k}{\alpha p}\right) \mu_{m,n+1}^{(j,k)} = \mu_{m,n}^{(j,k)} + \frac{k}{\alpha p} \mu_{m,n+1}^{(j,k-p)}, \; k < p, \; \alpha p \neq k, \tag{3.18}$$

which represents recurrence relation for the double moments of upper record values from Burr distribution.

Example 3.5. Log-logistic distribution

The *pdf* of log-logistic distribution is given by

$$f(x) = \frac{\beta x^{\beta-1}}{(1 + x^{\beta})^2}, \; x \geq 0, \beta > 0, \tag{3.19}$$

and

$$1 - F(x) = \frac{1}{\beta} \left(x^{1-\beta} + x\right) f(x). \tag{3.20}$$

First, by using (3.20) in (3.1), we have

$$\left(1 - \frac{k}{\beta}\right) \mu_{n+1}^{(k)} = \mu_n^{(k)} + \frac{k}{\beta} \mu_{n+1}^{(k-\beta)}, \; \beta \neq k, \tag{3.21}$$

which represents recurrence relation for the single moments of upper record values from log-logistic distribution.

Next, by using (3.20) in (3.2), we get

$$\left(1 - \frac{k}{\beta}\right) \mu_{m,n+1}^{(j,k)} = \mu_{m,n}^{(j,k)} + \frac{k}{\beta} \mu_{m,n+1}^{(j,k-p)}, \; \beta \neq k, \tag{3.22}$$

which represents recurrence relation for the double moments of upper record values of log-logistic distribution.

Example 3.8. Parabolic distribution

The *pdf* of parabolic distribution is given by

$$f(x) = 6x(1-x),\ 0 < x < 1, \tag{3.23}$$

and

$$1 - F(x) = \frac{1}{6}\left(1 + \frac{1}{x} - 2x\right) f(x). \tag{3.24}$$

First, by using (3.24) in (3.1), we get

$$\left(1 + \frac{k}{3}\right)\mu_{n+1}^{(k)} = \mu_n^{(k)} + \frac{k}{6}\left(\mu_{n+1}^{(k-1)} + \mu_{n+1}^{(n-2)}\right), \tag{3.25}$$

which represents recurrence relation for the single moments of upper record values from parabolic distribution.

Next, by using (3.24) in (3.2), we get

$$\left(1 + \frac{k}{3}\right)\mu_{m,n+1}^{(j,k)} = \mu_{m,n}^{(k)} + \frac{k}{6}\left(\mu_{m,n+1}^{(j,k-1)} + \mu_{j,n+1}^{(j,k-2)}\right), \tag{3.26}$$

which represents recurrence relation for the double moments of upper record values from parabolic distribution.

Example 3.9. Skewed distribution

The *pdf* of skewed distribution is given by

$$f(x) = 12x^2(1-x),\ 0 < x < 1, \tag{3.27}$$

and

$$1 - F(x) = \frac{1}{12}\left(1 + \frac{1}{x} + \frac{1}{x^2} - 3x\right) f(x). \tag{3.28}$$

First, by using (3.28) in (3.1), we get

$$\left(1 + \frac{k}{4}\right)\mu_{n+1}^{(k)} = \mu_n^{(k)} + \frac{k}{12}\left(\mu_{n+1}^{(k-1)} + \mu_{n+1}^{(n-2)} + \mu_{n+1}^{(k-3)}\right), \tag{3.29}$$

which represents recurrence relation for the single moments of upper record values from skewed distribution.

Next, by using (3.28) in (3.2), we get

$$\left(1 + \frac{k}{4}\right)\mu_{m,n+1}^{(j,k)} = \mu_{m,n}^{(k)} + \frac{k}{12}\left(\mu_{m,n+1}^{(j,k-1)} + \mu_{m,n+1}^{(j,k-2)} + \mu_{m,n+1}^{(j,k-3)}\right), \tag{3.30}$$

which represents recurrence relation for the double moments of upper record values from skewed distribution.

Sometimes, it is not easy to establish recurrence relations for the moments of record values. Alternatively, in the next two sections, we will consider the problem of establishing recurrence relations based on MGFs and FMGFs of record values. This will enable us to establish the recurrence relations of moments of record values through direct differentiations.

4 Recurrence Relations for MGFs of Record Values

In this section, we use Theorem 2.1 and Theorem 2.2 to establish general recurrence relations satisfy by the single and double MGFs of upper record values. Then, we present some examples including exponential and logistic distributions.

4.1 General relations

First, by putting $g(x) = e^{tx}$, in Theorem 2.1, we get

$$M_{n+1}(t) - M_n(t) = \frac{t}{\Gamma(n+1)} \int_{-\infty}^{\infty} e^{tx}\{-\log[1-F(x)]\}^n\{1-F(x)\}dx. \tag{4.1}$$

which represents general form for establishing recurrence relations for the single MGFs of upper record values.

Next, by putting $\tau(x,y) = e^{t_1x+t_2y}$, in Theorem 2.2, we get

$$\begin{aligned} M_{m,n+1}(t_1,t_2) - M_{m,n}(t_1,t_2) &= \frac{t_2}{\Gamma(m)\Gamma(n-m+1)} \int_{-\infty}^{\infty}\int_{x}^{\infty} e^{t_1x+t_2y}\{-\log[1-F(x)]\}^{m-1} \\ &\times \frac{f(x)}{1-F(x)}\{-\log[1-F(y)]+\log[1-F(x)]\}^{n-m} \\ &\times \{1-F(y)\}dydx, \end{aligned} \tag{4.2}$$

which represents general form for establishing recurrence relations for the double MGFs of upper record values.

4.2 Examples

In this subsection, we present some examples as special cases from the relations given in (4.1) and (4.2). These examples including exponential and logistic distributions.

Example 4.1. Exponential distribution

First, by using (3.4) in (4.1), we have

$$(1-t)M_{n+1}(t) = M_n(t), \ t \neq 1, \tag{4.3}$$

which represents recurrence relation of the single MGFs of upper record values from exponential distribution. The above recuurence relation gives the corresponding recurrence relation in (3.5) through direct differentiation with respect to t at $t = 0$.

Next, by using (3.4) in (4.2), we get

$$(1-t_2)M_{m,n+1}(t_1,t_2) = M_{m,n}(t_1,t_2), \ t_2 \neq 1 \tag{4.4}$$

which represents recurrence relation of the double MGFs of upper record values from exponential distribution. It it easy to note that, the above recurrence relation gives the corresponding recurrence relation in (3.6) through direct differentiation with respect to t_2 at $t_2 = 0$.

Example 4.2. Logistic distribution

The *pdf* of logistic distribution is given by

$$f(x) = \frac{e^{-x}}{(1+e^{-x})^2}, \quad -\infty < x < \infty, \tag{4.5}$$

and

$$1 - F(x) = (1 + e^{-x})\, f(x), -\infty < x < infty. \tag{4.6}$$

First, by using (4.6) in (4.1), we have

$$(1-t)M_{n+1}(t) = M_n(t) + tM_{n+1}(t-1), \tag{4.7}$$

which represents recurrence relation of the single MGFs of upper record values from logistic distribution.

Next, by using (4.6) in (4.2), we have

$$(1-t_2)M_{m,n+1}(t_1,t_2) = M_n(t_1,t_2) + t_2M_{n+1}(t_1,t_2-1), \tag{4.8}$$

which represents recurrence relation for the double MGFs of upper record values from logistic distribution.

5 Recurrence Relations for FMGFs of Record Values

In this section, we use Theorem 2.1 and Theorem 2.2 to establish general recurrence relations satisfy by the single and double FMGFs of upper record values. Then, we present some examples including exponential logistic and half logistic distributions.

5.1 General relations

First, by putting $g(x) = t^x$, and $t > 0$ in Theorem 2.1, we get

$$\Psi_{n+1}(t) - \Psi_n^(t) = \frac{\log t}{\Gamma(n+1)} \int_{-\infty}^{\infty} t^x \{-\log[1-F(x)]\}^n \{1-F(x)\} dx, \tag{5.1}$$

which represents general form for establishing recurrence relation for the single FMGF of record values.

Next, by putting $\tau(x,y) = t_1^x t_2^y$, and $t_1, t_2 > 0$ in Theorem 2.2, we get

$$\begin{aligned}\Psi_{m,n+1}(t_1,t_2) - \Psi_{m,n}(t_1,t_2) &= \frac{\log t_2}{\Gamma(m)\Gamma(n-m+1)} \int_{-\infty}^{\infty}\int_{x}^{\infty} t_1^x t_2^y \{-\log[1-F(x)]\}^{m-1} \\ &\times \frac{f(x)}{1-F(x)} \{-\log[1-F(y)] + \log[1-F(x)]\}^{n-m} \\ &\times \{1-F(y)\} dy dx,\end{aligned} \quad (5.2)$$

which represents general form for establishing recurrence relations for the double FMGF of record values.

5.2 Examples

In this subsection, we present some examples as special cases from the relations in (5.1) and (5.2). These examples including exponential, logistic and half logistic distributions.

Example 5.1. Exponential distribution

First, by using (3.4) in (5.1), we have

$$(1 - \log t)\Psi_{n+1}(t) = \Psi_n(t), \; t \neq e, \quad (5.3)$$

which represents recurrence relation of the single FMGFs of upper record values from exponential distribution. Again, the above recuurence relation gives the corresponding recurrence relation in (3.5) through direct differentiation with respect to t at $t = 1$.

Next, by using (3.4) in (5.2), we get

$$(1 - \log t_2)\Psi_{m,n+1}(t_1,t_2) = \Psi_{m,n}(t_1,t_2), \quad (5.4)$$

which represents recurrence relation of the double FMGFs of upper record values from expopnential distribution. It it easy to note that the above recuurence relation gives the corresponding recurrence relation in (3.6) through direct differentiation with respect to t_2 at $t_2 = 1$.

Example 4.2. Logistic and half logistic distributions

First, by using (4.6) in (5.1), we have

$$(1 - \log t)\Psi_{n+1}(t) = \Psi_n(t) + \log t \Psi_{n+1}(t/e), \quad (5.5)$$

which represents recurrence relation of the single FMGFs of upper record values from logistic and half logistic distributions.

Next, by using (4.6) in (5.2), we have

$$(1 - \log t_2)\Psi_{m,n+1}(t_1,t_2) = \Psi_n(t_1,t_2) + \log t_2 \Psi_{n+1}(t_1, t_2/e), \quad (5.6)$$

which represents recurrence relation of the double FMGFs of upper record values from logistic and half logistic distribution.

6 Application

In this section, we use Theorem 2.1 to state the following result, that gives the expected value for the differece between the reliability function at n-th and $(n+1)$-th upper record values.

Theorem 3
For $n = 1, 2, \ldots$.

$$E\Big(R_{n+1}(t) - R_n(t)\Big) = \frac{-1}{2^{n+1}}, \tag{6.1}$$

where $R_n(t) = 1 - F(t_{U(n)})$ represents the reliability function at n-th record value.

Proof
By putting $g(t) = 1 - F(t)$ in Theorem 2.1, we get

$$\begin{aligned} E\Big(R_{n+1}(t) - R_n(t)\Big) &= \frac{-1}{\Gamma(n+1)} \int_{-\infty}^{\infty} [-\log(1-F(t))]^n \{1 - F(t)\} f(t) dt, \\ &= \frac{-1}{n!} \int_0^1 [-\log y]^n y dy, \end{aligned}$$

substituting $y = e^{-z}$ into the above integral, we get

$$\begin{aligned} E\Big(R_{n+1}(t) - R_n(t)\Big) &= \frac{-1}{\Gamma(n+1)} \int_0^{\infty} z^n e^{-2z} dz, \\ &= \frac{-1}{2^{n+1}}. \end{aligned} \tag{6.2}$$

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