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Order Statistics from Non-identical Doubly-Truncated Generalized Power Function Random Variables and Applications

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Abstract: In this paper, we derive some recurrence relations for the single and product moments of order statistics from n independent and non-identically distributed generalized power function random variables. These recurrence relations are simple in nature and could be used systematically in order to compute all the single and product moments of all order staistics in a simple recursive manner. The results for order statistics from a multiple-outlier model (with a slippage of p observations) from generalized power function distributions are deduced as special cases. The results then generalized in the case of doubly truncated case. Numerical example is also presented.

1 Introduction

Let $X_1, X_2, ...X_n$ be independent random variables having probability density functions $f_1(x), f_2(x), ..., f_n(x)$ and cumulative distribution functions $F_1(x), F_2(x), ..., F_n(x)$, respectively. Let $X_{1:n} \leq X_{2:n} \leq ... \leq X_{n:n}$ denote the order statistics obtained by arranging the n $X_i's$ in increasing order of magnitude. Then, the density function of $X_{r:n}(1 \leq r \leq n)$ can be written as (David, 1981, p. 22).

$$f_{r:n}(x) = \frac{1}{(r-1)!(n-r)!} \sum_{p} \prod_{a=1}^{r-1} F_{i_a}(x) f_{i_r}(x) \prod_{b=r+1}^{n} \{1 - F_{i_b}(x)\}, \quad (1.1)$$

where $\sum_p$ denotes the summation over all $n!$ permutations $(i_1, i_2, ..., i_n)$ of $(1, 2, ..., n)$. Similarly, the joint density function of $X_{r:n}$ and $X_{s:n}$ $(1 \leq r < s \leq n)$ can be written as

$$f_{r,s:n}(x) = \frac{1}{(r-1)!(s-r-1)!(n-s)!}\sum_P \prod_{a=1}^{r-1} F_{i_a}(x) f_{i_r}(x) \prod_{b=r+1}^{s-1}\{F_{i_b}(y) - F_{i_b}(x)\}$$
$$f_{i_s}(y)\prod_{c=s+1}^{n}\{1 - F_{i_c}(y)\}, \quad x < y. \tag{1.2}$$

Vaughan and Venables (1972) gave alternative forms of the densities in (1.1) and (1.2) in terms of permanents of matrices.

Balakrishnan (1994a,b) have derived several recurrence relations for the single and product moments from non-identical right-truncated exponential random variables. Also Balakrishnan and Balasubramanian (1995) have studied the recurrence relations for the single and the product moments of order statistics arising from n independent and non-identically distributed power function random variables. Childs, Balakrishnan and Moshref (2000) have derived some recurrence relations for the single and product moments of order statistics for INID from right truncated Lomax distribution.

In this paper, we consider the case when X_i's are independent the having generalized power function distribution with density functions

$$f_i(x) = \frac{\nu_i}{\beta^{\nu_i}}\{x + \alpha\}^{\nu_i - 1}, \qquad -\alpha \le x \le \beta - \alpha, \ \ \nu_i > 0, \tag{1.3}$$

and cumulative distribution functions

$$F_i(x) = \left\{\frac{x+\alpha}{\beta}\right\}^{\nu_i}, \qquad -\alpha \le x \le \beta - \alpha, \ \ \nu_i > 0, \tag{1.4}$$

for $i = 1, 2, ..., n$. It is clear from (1.3) and (1.4) that the distributions satisfy the differential equations

$$x f_i(x) = \nu_i F_i(x) - \alpha f_i(x), \quad i = 1, 2, ..., n, \ \ \nu_i > 0. \tag{1.5}$$

Let us denote the single moments $E(X_{r:n}^k)$ by $\mu_{r:n}^{(k)}$, $1 \le r \le n$ and $k = 1, 2, ...$ and the product moments $E(X_{r:n}X_{s:n})$ by $\mu_{r,s:n}$ for $1 \le r < s \le n$. Let us also use $\mu_{r:n-1}^{[i](k)}$ and $\mu_{r,s:n-1}^{[i]}$ to denote the single and the product moments of order statistics arising from $n-1$ variables obtained by deleting X_i from the original n variables $X_1, X_2, ..., X_n$.

By making use the differential equations in (1.5), we establish several recurrence relations satisfied by the single and the product moments of order statistics. These relations will enable one to compute all the single and the product moments of all order statistics in a simple recursive manner. The results for the p-outlier model are deduced as special cases. Also, the results for the doubly truncated case are found. The mean, variance and covariances calculated up to sample size $n = 5$.

2 Relations for single moments

In this section, we shall establish some recurrence relations satisfied by the single moments of order statistics by making use of the differential equations in (1.5).

Result 2.1 for $1 \leq r \leq n-1$ and $k = 1, 2, \ldots$.

$$\mu_{r:n}^{(k)} = \frac{1}{k + \sum_1^n \nu_i}\left\{ -k\alpha\mu_{r:n}^{(k-1)} + \sum_1^n \nu_i \mu_{r:n-1}^{[i](k)} \right\} \tag{2.1}$$

Result 2.2 For $n \geq 1$ and $k = 1, 2, \ldots$

$$\mu_{n:n}^{(k)} = \frac{1}{k + \sum_1^n \nu_i}\left\{ -k\alpha\mu_{n:n}^{(k-1)} + \beta^k \sum_1^n \nu_i \right\}. \tag{2.2}$$

Remark 1: Results (2.1)-(2.2) will enable one to compute all the single moments of all order statistics in a simple recursive way for any specific values of $\nu_i (i = 1, 2, .., n)$.

Remark 2: Result 2.2, along with a general relation established by Balakrishnan (1988) which expresses $\mu_{r:n}^{(k)}$ in terms of the k^{th} moment of the largest order statistics in samples of size up to n, will also enable one to compute all the single moments of all order statistics in a simple recursive way.

Remark 3: For the case when the $X_i's$ are independent and identically distributed as generalized power function random variables (that is, $\nu_1 = \nu_2 = \ldots = \nu_n = \nu$), Results 2.1 and 2.2 reduce to

$$\mu_{r:n}^{(k)} = \frac{1}{k + n\nu}\left\{ -k\alpha\mu_{r:n}^{(k-1)} + n\nu\mu_{r:n-1}^{(k)} \right\}, \tag{2.3}$$

and

$$\mu_{n:n}^{(k)} = \frac{1}{k + n\nu}\left\{ -k\alpha\mu_{n:n}^{(k-1)} + \beta^k n\nu \right\}. \tag{2.4}$$

Remark 4: At $\alpha = 0$ and $\beta = 1$, Results 2.1 and 2.2 reduce to

$$\mu_{r:n}^{(k)} = \frac{\sum_1^n \nu_i \mu_{r:n-1}^{[i](k)}}{k + \sum_1^n \nu_i} \tag{2.5}$$

and

$$\mu_{n:n}^{(k)} = \frac{\sum_1^n \nu_i}{k + \sum_1^n \nu_i} \tag{2.6}$$

These results were orginally drived by Balakrishnan and Balasubramanian (1995).

Remark 5:

For the cases when $\nu_i = 1$ and $\nu_i = 2$, we obtain the same results for rectangular and traingular distributions respectively (see Bhoj and Ahsanullah (1996)].

3 Relations for product moments

In this section, we shall establish some recurrence relations satisfied by the product moments of order statistics using the differential equations in (1.5).

Result 3.1 For $n \geq 3$ and $1 \leq r \leq n-2$

$$\mu_{r,r+1:n} = \frac{\sum_1^n \nu_i \mu_{r,r+1:n-1}^{[i]}}{2(1+\alpha) + \sum_1^n \nu_i}, \tag{3.1}$$

where $2(1+\alpha) + \sum_1^n \nu_i \neq 0$.

Result 3.2 For $1 \leq r \leq n-2$

$$\mu_{r,n:n} = \frac{(\beta-\alpha)\sum_1^n \nu_i \mu_{r:n-1}^{[i]}}{2(1+\alpha) + \sum_1^n \nu_i}, \tag{3.2}$$

where $2(1+\alpha) + \sum_1^n \nu_i \neq 0$,

Result 3.3 For $n \geq 2$

$$\mu_{n-1,n:n} = \frac{(\beta-\alpha)\sum_1^n \nu_i \mu_{n-1:n-1}^{[i]}}{2(1+\alpha) + \sum_1^n \nu_i}, \tag{3.3}$$

where $2(1+\alpha) + \sum_1^n \nu_i \neq 0$.

Result 3.4 For $1 \leq r < s \leq n-1$ and $s - r \geq 2$

$$\mu_{r,s:n} = \frac{\sum_1^n \nu_i \mu_{r,s:n-1}^{[i]}}{2(1+\alpha) + \sum_1^n \nu_i}, \tag{3.4}$$

where $2(1+\alpha) + \sum_1^n \nu_i \neq 0$.

Remark 6: Results $3.1 - 3.4$ will enable one to compute all the product moments, and hence the covariance of all order statistics in a simple recursive manner for any specified values of $\nu_i (i = 1, 2, .., n)$

Remark 7: For the case when the variables are independent and identically distributed as generalized power function (that is, $\nu_1 = ... = \nu_n = \nu$). Results 3.1 – 3.4 reduce to

$$\mu_{r,r+1:n} = \frac{n\nu\mu_{r,r+1:n-1}}{2(1+\alpha) + n\nu}, \qquad 1 \leq r \leq n-2 \tag{3.5}$$

$$\mu_{r,n:n} = \frac{n(\beta-\alpha)\nu\mu_{r:n-1}}{2(1+\alpha) + n\nu}, \qquad n \geq 2 \tag{3.6}$$

$$\mu_{n-1,n:n} = \frac{n(\beta-\alpha)\nu\mu_{n-1:n-1}}{n\nu + 2(1+\alpha)}, \qquad 1 \leq r \leq n-2 \tag{3.7}$$

$$\mu_{r,s:n} = \frac{n\nu\mu_{r,s:n-1}}{2(1+\alpha)+n\nu}, \qquad 1 \leq r < s \leq n-1, s-r \geq 2, \tag{3.8}$$

where $2(1+\alpha)+n\nu \neq 0$.

Remark 8:

If $\alpha = 0$ and $\beta = 1$ the above results concide with those of Balakrishnan and Balasubramanian (1995).

4 Results for the p-outlier model

In this section, we shall present the results for the p-outlier model. Under this model, we assume that $X_1, X_2, ..., X_{n-p}$ are independent generalized power function random variables with parametr (ν), while $X_{n-p+1}, ..., X_n$ are independent generalized power function random variables with parameter (ν^*) (and independent of $X_1, X_2, ..., X_{n-p}$)(see Barnett and Lewis (1994), pp. 66-68). In this case, let us denote the single moments by $\mu_{r:n}^{(k)}[p]$ and the product moments by $\mu_{r,s:n}[p]$. Similarly, let us denote the single and product moments by $\mu_{r:n}^{(k)}[p-1]$ and $\mu_{r,s:n}[p-1]$, respectively, when a sample of size $n-1$ consists of $p-1$ outliers. Then, from the results established in Sections 2 and 3, we deduce the following:

(a) For $1 \leq r \leq n-1$ and $k = 1, 2, ...$

$$\mu_{r:n}^{(k)}[p] = \frac{-k\alpha\mu_{r:n}^{(k-1)}[p] + (n-p)\nu\mu_{r:n-1}^{(k)}[p] + p\nu^*\mu_{r:n-1}^{(k)}[p-1]}{k+(n-p)\nu+p\nu^*}, \tag{4.1}$$

(b) for $n \geq 1$ and $k = 1, 2, ...$

$$\mu_{n:n}^{(k)}[p] = \frac{-k\alpha\mu_{n:n}^{(k-1)}[p] + \beta^k\{(n-p)\nu+p\nu^*\}}{k+(n-p)\nu+p\nu^*}, \tag{4.2}$$

(c) for $1 \leq r \leq n-2$

$$\mu_{r,r+1:n}[p] = \frac{(n-p)\nu\mu_{r,r+1:n-1}[p] + p\nu^*\mu_{r,r+1:n-1}[p-1]}{2(1+\alpha)+(n-p)\nu+p\nu^*}, \tag{4.3}$$

(d) for $1 \leq r \leq n-2$

$$\mu_{r,n:n}[p] = \frac{(\beta-\alpha)\{(n-p)\nu\mu_{r:n-1}[p] + p\nu^*\mu_{r:n-1}[p-1]\}}{2(1+\alpha)+(n-p)\nu+p\nu^*}, \tag{4.4}$$

(e) for $n \geq 2$

$$\mu_{n-1,n:n}[p] = \frac{(\beta-\alpha)\{(n-p)\nu\mu_{n-1:n-1}[p] + p\nu^*\mu_{n-1:n-1}[p-1]\}}{2(1+\alpha)+(n-p)\nu+p\nu^*}, \tag{4.5}$$

(f) for $1 \leq r < s \leq n-1, s-r \geq 2$

$$\mu_{r,s:n}[p] = \frac{(n-p)\nu\mu_{r,s:n-1}[p] + p\nu^*\mu_{r,s:n-1}[p-1]}{2(1+\alpha)+(n-p)\nu+p\nu^*}, \tag{4.6}$$

where $2(1+\alpha)+(n-p)\nu+p\nu^* \neq 0$

Remark 9: The recurrence relations in $(4.1)-(4.6)$ will enable one to compute all the single and the product moments (in particular, the means, variance and covariance) of all order statistics from a p-outlier model in simple recursive manner.

5 Truncated generalized power function

Let us consider the case when $X_i s$ are independent having doubly truncated generalized power function distributions with density functions

$$f_i(x) = \frac{\nu_i\{x+a\}^{\nu_i-1}}{B^{\nu_i}-A^{\nu_i}}, \qquad -\alpha \leq L \leq x \leq T \leq \beta-\alpha, \quad \nu_i > 0, \tag{5.1}$$

and cumulative distribution funcations

$$F_i(x) = \frac{\{x+a\}^{\nu_i}-A^{\nu_i}}{B^{\nu_i}-A^{\nu_i}}, \qquad -\alpha \leq L \leq x \leq T \leq \beta-\alpha, \quad \nu_i > 0, \tag{5.2}$$

where $A = L+\alpha$ and $B = T+\alpha$ for $i = 1,2,..,n$. It is clear from (5.1) and (5.2) that the distributions satisfy the differential equations

$$xf_i(x) = \nu_i\{F_i(x)+S_i\} - \alpha f_i(x), \tag{5.3}$$

where $S_i = \frac{A^{\nu_i}}{B^{\nu_i}-A^{\nu_i}}$. By proceeding on lines simliar to those in sections 2 and 3, we can establish the following recurrence relations for the single moments of order statistics:
(a) for $n \geq 2, k = 1,2,....$

$$\mu_{1:n}^{(k)} = \frac{-ka\mu_{1:n}^{(k-1)} + \sum_1^n \nu_i\{D_i\mu_{1:n-1}^{[i](k)} - S_iA^k\}}{k+\sum_1^n \nu_i}, \tag{5.4}$$

(b) for $2 \leq r \leq n-1, k = 1,2,...$

$$\mu_{r:n}^{(k)} = \frac{-k\alpha\mu_{r:n}^{(k-1)} + \sum_1^n \nu_i\{D_i\mu_{r:n-1}^{(k)} - S_i\mu_{r-1:n-1}^{[i](k)}\}}{k+\sum_1^n \nu_i}, \tag{5.5}$$

(c) for $n \geq 2, k = 1,2,...$

$$\mu_{n:n}^{(k)} = \frac{-k\alpha\mu_{n:n}^{(k-1)} + \sum_1^n \nu_i\{D_iB^k - S_i\mu_{n-1:n-1}^{[i](k)}\}}{k+\sum_1^n \nu_i}, \tag{5.6}$$

where $D_i = \frac{B^{\nu_i}}{B^{\nu_i}-A^{\nu_i}}$.

The recurrence relations for the product moments can be similarly derived for the doubly truncated case and are as given below

(d) for $n \geq 2$ and $1 \leq r \leq n-2$

$$\mu_{r,r+1:n} = \frac{\sum_1^n \nu_i\{D_i\mu^{[i]}_{r,r+1:n-1} - S_i\mu^{[i]}_{r-1,r:n-1}\}}{2(1+\alpha)+\sum_1^n \nu_i}. \tag{5.7}$$

(e) for $2 \leq r \leq n-1$

$$\mu_{r,n:n} = \frac{\sum_1^n \nu_i\{TD_i\mu^{[i]}_{r:n-1} - S_i\mu^{[i]}_{r-1,n-1:n-1}\}}{2(1+\alpha)+\sum_1^n \nu_i}, \tag{5.8}$$

(f) for $n \geq 2$

$$\mu_{1,n:n} = \frac{\sum_1^n \nu_i\{TD_i\mu^{[i]}_{1:n-1} - LS_i\mu^{[i]}_{n-1:n-1}\}}{2(1+\alpha)+\sum_1^n \nu_i}, \tag{5.9}$$

(g) for $2 \leq r < s \leq n-1$

$$\mu_{r,s:n} = \frac{\sum_1^n \nu_i\{D_i\mu^{[i]}_{r,s:n-1} - S_i\mu^{[i]}_{r-1,s-1:n-1}\}}{2(1+\alpha)+\sum_1^n \nu_i}, \tag{5.10}$$

where $D_i = \frac{B^{\nu_i}}{B^{\nu_i}-A^{\nu_i}}$ and $2(1+\alpha)+\sum_1^n \nu_i \neq 0$.

From the above results $(a-g)$, we deduce the following recurrence relations in p-outlier case,

$$\begin{aligned}\mu^{(k)}_{1:n}[p] &= \frac{1}{k+(n-p)\nu+p\nu^*}\Big\{-k\alpha\mu^{(k-1)}_{1:n}[p] + (n-p)\nu\left[D_\nu\mu^{(k)}_{1:n-1}[p] - S_\nu A^k\right] \\ &+ p\nu^*\left[D_{\nu^*}\mu^{(k)}_{1:n-1}[p-1] - S_{\nu^*}A^k\right]\Big\}, \\ & \qquad n \geq 2,\ \ k=1,2,..\end{aligned} \tag{5.11}$$

$$\begin{aligned}\mu^{(k)}_{r:n}[p] &= \frac{1}{k+(n-p)\nu+p\nu^*}\Big\{-k\alpha\mu^{(k-1)}_{r:n}[p] + (n-p)\nu\left[D_\nu\mu^{(k)}_{r:n-1}[p] - S_\nu\mu^{[i](k)}_{r-1:n-1}[p]\right] \\ &+ p\nu^*\left[D_{\nu^*}\mu^{(k)}_{r:n-1}[p-1] - S_{\nu^*}\mu^{[i](k)}_{r-1:n-1}[p-1]\right]\Big\}, \\ & \qquad 2 \leq r \leq n-1,\ \ k=1,2,...\end{aligned} \tag{5.12}$$

$$\begin{aligned}\mu^{(k)}_{n:n}[p] &= \frac{1}{k+(n-p)\nu+p\nu^*}\Big\{-k\alpha\mu^{(k-1)}_{n:n}[p] + (n-p)\nu\left[B^kD_\nu - S_\nu\mu^{(k)}_{n-1:n-1}[p]\right] \\ &+ p\nu^*\left[B^kD_{\nu^*} - S_{\nu^*}\mu^{(k)}_{n-1:n-1}[p-1]\right]\Big\}, \\ & \qquad n \geq 2, k=1,2,..\end{aligned} \tag{5.13}$$

$$\mu_{r,r+1:n}[p] = \frac{1}{2(1+\alpha)+(n-p)\nu+p\nu^*}\Big\{(n-p)\nu\left[D_\nu\mu_{r,r+1:n-1}[p]-S_\nu\mu_{r-1,r:n-1}[p]\right]$$
$$+\; p\nu^*\left[D_{\nu^*}\mu_{r,r+1:n-1}[p-1]-S_{\nu^*}\mu_{r-1,r:n-1}[p-1]\right]\Big\},$$
$$n\geq 2, 1\leq r\leq n-2, \qquad (5.14)$$

$$\mu_{r,s:n}[p] = \frac{1}{2(1+\alpha)+(n-p)\nu+p\nu^*}\Big\{(n-p)\nu\left[D_\nu\mu_{r,s:n-1}[p]-S_\nu\mu_{r-1,s-1:n-1}[p]\right]$$
$$+\; p\nu^*\left[D_{\nu^*}\mu_{r,s:n-1}[p-1]-S_{\nu^*}\mu_{r-1,s-1:n-1}[p-1]\right]\Big\},$$
$$1\leq r< s\leq n-1, s-r\geq 2, \qquad (5.15)$$

$$\mu_{r,n:n}[p] = \frac{1}{2(1+\alpha)+(n-p)\nu+p\nu^*}\Big\{(n-p)\nu\left[TD_\nu\mu_{r:n-1}[p]-S_\nu\mu_{r-1,n-1:n-1}[p]\right]$$
$$+\; p\nu^*\left[TD_{\nu^*}\mu_{r:n-1}[p-1]-S_{\nu^*}\mu_{r-1,n-1:n-1}[p-1]\right]\Big\},$$
$$2\leq r\leq n-1, \qquad (5.16)$$

$$\mu_{1,n:n}[p] = \frac{1}{2(1+\alpha)+(n-p)\nu+p\nu^*}\Big\{(n-p)\nu\left[TD_\nu\mu_{1:n-1}[p]-LS_\nu\mu_{r-1:n-1}[p]\right]$$
$$+\; p\nu^*\left[TD_{\nu^*}\mu_{1:n-1}[p-1]-LS_{\nu^*}\mu_{n-1:n-1}[p-1]\right]\Big\},\quad n\geq 2, \qquad (5.17)$$

where $2(1+\alpha)+(n-p)\nu+p\nu^*\neq 0$. Note that these recurrence relations reduce (by setting $p=0$, $L=-\alpha=0$ and $T=\beta-\alpha$) to those derived in Sections 2 and 3 for independent and identically distributed generalized power function random variables. Thus by starting with the above equations for $p=0$, all of the i.i.d. single and product moments can be determined. These same relations could then be used again, this time with $p=1$, to determine all of the single and product moments of all order statistics from a sample containing a single outlier. Continuing in this manner, the above relations could be used to compute all the single and product moments (and hence covariances) of all order statistics from a p-outlier model in a simple recursive manner.

Table 1. Expected Values in the Presence of Multiple Outlier

i	n	p	$\mu_{i:n}[p]$	i	n	p	$\mu_{i:n}[p]$	i	n	p	$\mu_{i:n}[p]$	i	n	p	$\mu_{i:n}[p]$
1	5	0	0.3608	1	5	1	0.3306	1	5	2	0.3016	1	5	3	0.2737
2	5	0	0.7512	2	5	1	0.6031	2	5	2	0.4791	2	5	3	0.3756
3	5	0	1.3651	3	5	1	0.9215	3	5	2	0.6173	3	5	3	0.4294
4	5	0	2.4254	4	5	1	1.2974	4	5	2	0.7143	4	5	3	0.5072
5	5	0	4.5859	5	5	1	1.3513	5	5	2	0.7360	5	5	3	0.5734

Table 2. Variances and Covariances in the Presence of Multiple Outlier

i	j	n	p	$\sigma_{i,j:n}[p]$	i	j	n	p	$\sigma_{i,j:n}[p]$
1	1	5	0	0.6572	1	1	5	2	0.7245
1	2	5	0	0.7282	1	2	5	2	0.8277
1	3	5	0	0.4479	1	3	5	2	0.4997
1	4	5	0	0.2587	1	4	5	2	0.2857
1	5	5	0	0.1155	1	5	5	2	0.1267
2	2	5	0	0.8299	2	2	5	2	0.9845
2	3	5	0	0.5091	2	3	5	2	0.5913
2	4	5	0	0.2936	2	4	5	2	0.3370
2	5	5	0	0.1309	2	5	5	2	0.1491
3	3	5	0	0.5459	3	3	5	2	0.6460
3	4	5	0	0.3145	3	4	5	2	0.3675
3	5	5	0	0.1402	3	5	5	2	0.1624
4	4	5	0	0.3299	4	4	5	2	0.3897
4	5	5	0	0.1469	4	5	5	2	0.1721
5	5	5	0	0.1524	5	5	5	2	0.1798
1	1	5	1	0.6873	1	1	5	3	0.7650
1	2	5	1	0.7738	1	2	5	3	0.8847
1	3	5	1	0.4708	1	3	5	3	0.5345
1	4	5	1	0.2704	1	4	5	3	0.3052
1	5	5	1	0.1203	1	5	5	3	0.1351
2	2	5	1	0.9042	2	2	5	3	1.0656
2	3	5	1	0.5479	2	3	5	3	0.6388
2	4	5	1	0:3139	2	4	5	3	0.3639
2	5	5	1	0.1394	2	5	5	3	0.1607
3	3	5	1	0.5928	3	3	5	3	0.7035
3	4	5	1	0.3392	3	4	5	3	0.4004
3	5	5	1	0.1505	3	5	5	3	0.1766
4	4	5	1	0.3575	4	4	5	3	0.4272
4	5	5	1	0.1585	4	5	5	3	0.1882
5	5	5	1	0.1649	5	5	5	3	0.1975

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Appendix

Proof of Result 2.1: From (1.1), let us consider for $1 \leq r \leq n-1$, $k = 1, 2, ...$

$$\begin{aligned}(r-1)!(n-r)!\mu_{r:n}^{(k)} &= \sum_{P} \int_{-\alpha}^{\beta-\alpha} x^k \prod_{a=1}^{r-1} F_{i_a}(x) f_{i_r}(x) \prod_{b=r+1}^{n} \{1 - F_{i_b}(x)\} dx \\ &= \sum_{P} \nu_{i_r} \int_{-\alpha}^{\beta-\alpha} x^{k-1} \prod_{a=1}^{r} F_{i_a}(x) \prod_{b=r+1}^{n} \{1 - F_{i_b}(x)\} dx \\ &- \alpha \sum_{P} \int_{-\alpha}^{\beta-\alpha} x^{k-1} \prod_{a=1}^{r-1} F_{i_a}(x) f_{i_r}(x) \prod_{b=r+1}^{n} \{1 - F_{i_b}(x)\} dx.\end{aligned}$$

Upon using (1.5). Integrating now by parts (in the first term) treating x^{k-1} for integration and the rest of integrand for differentiation, we get

$$k(r-1)!(n-r)!\left[\mu_{r:n}^{(k)}+a\mu_{r:n}^{(k-1)}\right]$$
$$=\sum_p \nu_{i_r}\Big[\ -\sum_{j=1}^{r}\int_{-\alpha}^{\beta-\alpha} x^k f_{i_j}(x)\prod_{\substack{a=1\\a\neq j}}^{r} F_{i_a}(x)\prod_{b=r+1}^{n}\{1-F_{i_b}(x)\}dx$$
$$+\sum_{j=r+1}^{n}\int_{-\alpha}^{\beta-\alpha} x^k \prod_{a=1}^{r} F_{i_a}(x) f_{i_j}(x)\prod_{\substack{b=r+1\\b\neq j}}^{n}\{1-F_{i_b}(x)\}dx\Big]$$

$$(A.1)$$

Upon splitting the second set of integrals (ones with postive sign) on the RHS of (A.1) into two each through the term $F_{i_r}(x)=1-(1-F_{i_r}(x))$, we obtain

$$k(r-1)!(n-r)!\left[\mu_{r:n}^{(k)}+a\mu_{r:n}^{(k-1)}\right]$$
$$=\sum_p \nu_{i_r}\Big[\ -\sum_{j=1}^{r}\int_{-\alpha}^{\beta-\alpha} x^k f_{i_j}(x)\prod_{\substack{a=1\\a\neq j}}^{r} F_{i_a}(x)\prod_{b=r+1}^{n}\{1-F_{i_b}(x)\}dx$$
$$-\sum_{j=r+1}^{n}\int_{-\alpha}^{\beta-\alpha} x^k \prod_{a=1}^{r-1} F_{i_a}(x) f_{i_j}(x)\prod_{\substack{b=r+1\\b\neq j}}^{n}\{1-F_{i_b}(x)\}dx\Big]$$
$$+\sum_{j=r+1}^{n}\int_{-\alpha}^{\beta-\alpha} x^k \prod_{a=1}^{r-1} F_{i_a}(x) f_{i_j}(x)\prod_{\substack{b=r\\b\neq j}}^{n}\{1-F_{i_b}(x)\}dx\Big].$$

$$(A.2)$$

Result (2.1) is derived simply by rewritng Eq. (A.2)

Proof of Result 2.1: From (1.2), let us consider for $1\leq r\leq n-2$

$$\begin{aligned}(r-1)!(n-r-1)!\mu_{r,r+1:n} &= (r-1)!(n-r-1)!E((X_{r:n}X_{r+1:n})\\ &= \sum_p\int_{-\alpha}^{y} t_{-\alpha}^{\beta-\alpha}\int_{-\alpha}^{y} xy\prod_{a=1}^{r-1}F_{i_a}(x)f_{i_r}(x)\\ &\quad f_{i_{r+1}}(y)\prod_{b=r+2}^{n}\{1-F_{i_b}(y)\}dxdy\\ &= \sum_p\int_{-\alpha}^{\beta-\alpha} yJ(y)f_{i_{r+1}}(y)\prod_{b=r+2}^{n}\{1-F_{i_b}(y)\}dy\end{aligned}$$

$$(A.3)$$

where

$$J(y) = \int_{-\alpha}^{y} x \prod_{a=1}^{r-1} F_{i_a}(x) f_{i_r}(x) dx$$
$$= \nu_{i_r} \int_{-\alpha}^{y} \prod_{a=1}^{r} F_{i_a}(x) dx - \alpha \int_{-\alpha}^{y} \prod_{a=1}^{r-1} F_{i_a}(x) f_{i_r}(x) dx.$$

Upon using (1.5). Integrating by parts (the first term) now yields

$$J(y) = \nu_{i_r}\left[y \prod_{a=1}^{r} F_{i_a}(y) - \sum_{j=1}^{r} \int_{-\alpha}^{y} x f_{i_j}(x) \prod_{\substack{a=1 \\ a\neq j}}^{r} F_{i_a}(x) dx \right] - \alpha \int_{-\alpha}^{y} \prod_{a=1}^{r-1} F_{i_a}(x) f_{i_r}(x) dx.$$

Which, when substituted in (A.3), gives

$$(r-1)!(n-r-1)!(1+\alpha)!\mu_{r,r+1:n}$$
$$= \sum_{p} \nu_{i_r} \left[\int_{-\alpha}^{\beta-\alpha} y^2 \prod_{a=1}^{r} F_{i_a}(y) f_{i_{r+1}(y)} \prod_{b=r+2}^{n} \{1 - F_{i_b}(y)\} dy \right.$$
$$- \sum_{j=1}^{r} \int_{-\alpha}^{\beta-\alpha} \int_{a}^{y} xy f_{i_j}(x) \prod_{\substack{a=1 \\ a\neq j}}^{r} F_{i_a}(x)$$
$$\left. f_{i_{r+1}}(y) \prod_{b=r+2}^{n} \{1 - F_{i_b}(y)\} dx dy \right]. \qquad (A.4)$$

Next, from Eq.(1.2) let us write for $1 \le r \le n-2$

$$(r-1)!(n-r-1)!\mu_{r,r+1:n} = (r-1)!(n-r-1)!E(X_{r:n}X_{r+1:n})$$
$$= \sum_{p} \int_{-\alpha}^{\beta-\alpha} \int_{x}^{\beta-\alpha} xy \prod_{a=1}^{r-1} F_{i_a}(x) f_{i_r}(x)$$
$$f_{i_{r+1}}(y) \prod_{b=r+2}^{n} \{1 - F_{i_b}(y)\} dy dx$$
$$= \sum_{p} \int_{-\alpha}^{\beta-\alpha} x I(x) \prod_{a=1}^{r-1} F_{i_a}(x) f_{i_r}(x) dx, \qquad (A.5)$$

where

$$\begin{aligned} I(x) &= \int_x^{\beta-\alpha} y f_{i_{r+1}}(y) \prod_{b=r+2}^{n} \{1-F_{i_b}(y)\} dy \\ &= \nu_{i_{r+1}} \int_x^{\beta-\alpha} F_{i_{r+1}}(y) \prod_{b=r+2}^{n} \{1-F_{i_b}(y)\} dy \\ &- \alpha \int_x^{\beta-\alpha} f_{i_{r+1}}(y) \prod_{b=r+2}^{n} \{1-F_{i_b}(y)\} dy. \end{aligned}$$

Upon using (1.5). Integrating by parts (the first term) now yields

$$\begin{aligned} I(x) &= \nu_{i_{r+1}} \Big[-xF_{i_{r+1}}(x) \prod_{b=r+2}^{n} \{1-F_{i_b}(x)\} - \int_x^{\beta-\alpha} y f_{i_{r+1}}(y) \prod_{b=r+2}^{n} \{1-F_{i_b}(y)\} dy \\ &+ \sum_{j=r+2}^{n} \int_x^{\beta-\alpha} y F_{i_{r+1}}(y) f_{i_j}(y) \prod_{\substack{b=r+2 \\ b \neq j}}^{n} \{1-F_{i_b}(y)\} dy \Big] \\ &- \alpha \int_x^{\beta-\alpha} f_{i_{r+1}}(y) \prod_{b=r+2}^{n} \{1-F_{i_b}(y)\} dy. \end{aligned}$$

Upon substituting this expression of $I(x)$ in (A.5), we get

$$\begin{aligned} (r-1)!(n-r-1)!(1+\alpha)\mu_{r,r+1:n} &= \sum_P \nu_{i_{r+1}} \Big[-\int_{-\alpha}^{\beta-\alpha} x^2 \prod_{a=1}^{r-1} F_{i_a}(x) f_{i_r}(x) F_{i_{r+1}}(x) \prod_{b=r+2}^{n} \{1-F_{i_b}(x)\} dx \\ &- \int_{-\alpha}^{\beta-\alpha} \int_x^{\beta-\alpha} xy \prod_{a=1}^{r-1} F_{i_a}(x) f_{i_r}(x) f_{i_{r+1}}(y) \prod_{b=r+2}^{n} \{1-F_{i_b}(y)\} dy dx \\ &+ \sum_{j=r+2}^{n} \int_{-\alpha}^{\beta-\alpha} \int_x^{\beta-\alpha} xy \prod_{a=1}^{r-1} F_{i_a}(x) F_{i_r}(x) f_{i_r}(x) f_{i_j}(y) \\ &\prod_{\substack{b=r+2 \\ b \neq j}}^{n} \{1-F_{i_b}(y)\} dy dx \Big]. \end{aligned} \tag{A.6}$$

On adding Eqs. (A.4) and (A.6) and simplifying the resulting expression, we obtain,

$$\begin{aligned} (r-1)!(n-r-1)!(2+2\alpha)\mu_{r,r+1:n} &= -(r-1)!(n-r-1)!(\sum_1^n \nu_i)\mu_{r,r+1:n} \\ &+ (r-1)!(n-r-2)!(n-r-1) \sum_1^n \nu_i \mu_{r,r+1:n-1}^{[i]}. \end{aligned}$$

Result (3.1) is derived simply by rewriting the above equation.