Bayesian Identification of Seasonal Moving Average Models

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Abstract

This study approaches the Bayesian identification of seasonal moving average processes using an approximate likelihood function and a normal gamma prior density. The marginal posterior probability mass function of the model orders is developed in a convenient form. Then one may investigate the posterior probabilities over the grid of the orders and choose the orders combination with the highest probability to solve the identification problem. A comprehensive simulation study is carried out to demonstrate the performance of the proposed procedure and check its adequacy in handling the identification problem. In addition, the proposed Bayesian procedure is compared with the AIC automatic technique. The numerical results support the adequacy of using the proposed procedure in solving the identification problem of seasonal moving average processes.

Keywords: Identification, seasonal moving average processes, automatic techniques, normal gamma density, posterior probability mass function.

1. Introduction

The seasonal moving average models are very useful in modeling seasonal time series data that arise in many areas of scientific endeavor such as engineering, physics, business, marketing, and economics. In such cases, the time series repeats its behavior over a certain period of time, usually, a year. In practice, the seasonal moving average model orders q and Q are usually unknown and should be identified or estimated. Identifying the orders of seasonal moving average models is the first and one of the most important phases in time series analysis.

It is well known that the solution of the identification problem depends on subjective opinions as well as statistical argument. The most popular non Bayesian approach to identify the orders of ARMA(p,q) and seasonal ARMA(p,q)(P,Q) models is developed by Box and Jenkins (1970). Their methodology is based on matching the sample autocorrelation and partial autocorrelation functions with their theoretical counterparts. Their technique is explained in many references such as Chatfield (1980), Priestley (1981), Tong

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(1990), Harvey(1993),Wei (2005), Box et al (2008) and Liu (2009). Another non Bayesian approach, known as the automatic approach, is based on fitting all possible models and computing a certain criterion for each model and choosing the model which minimizes the proposed criterion. For more details about the automatic approach, the reader is referred to Akaike (1973, 1974), Hannan and Quinn (1979), Mills and Prasad (1992) and Beveridge and Oickle (1994).

On the other hand, the Bayesian identification of time series is being developed and the Bayesian literature devoted to the identification of ARMA and seasonal ARMA (SARMA) models is sparse. Diaz and Farah (1981) developed a direct Bayesian method to identify the autoregressive models. Monahan (1983) made an important contribution to the analysis of low-order ARMA models by developing a numerical technique which implements the identification, estimation, and forecasting phases of an ARMA process. Broemeling and Shaarawy (1988) have developed an approximate procedure to identify the orders of ARMA processes. Their technique is somewhat indirect and based on approximating the posterior distribution of the maximum number of coefficients by a multivariate t distribution, then the significance of coefficients is checked by a series of univariate t tests in a similar fashion to the backward elimination procedure used in linear regression analysis.

Recently, Daif et al. (2003) studied the efficiency of Diaz and Farah technique and compared it with Broemeling and Shaarawy technique for autoregressive models. Moreover, Shaarawy and Ali (2003) developed a direct Bayesian technique to identify the orders of seasonal autoregressive models. This last technique has been extended to non seasonal moving average models by Shaarawy et al. (2007). Furthermore, the direct technique was used to identify multivariate AR models by Shaarawy and Ali (2008).

For well-known reasons, the direct Bayesian procedure to identify the moving average models and consequently seasonal moving average models is difficult. The direct Bayesian procedure to identify the seasonal moving average models has not been explored yet. The current article extends the direct Bayesian technique, introduced by Shaarawy et al (2007), to the case of the seasonal moving average models. The main difficulty in dealing with the exact Bayesian analysis of SMA models is that the likelihood function is very complicated and analytically intractable. Therefore, the posterior distributions of the parameters are not standard. This problem arises because the errors of these models are nonlinear in their coefficients. Hence, the errors' sum of squares is not quadratic in the coefficients. Among different solutions for this problem, the use of analytical approximations appears to be a reasonable one (see Broemeling and Shaarawy (1988)). The use of analytical approximations simplifies the analysis since it approximates the errors' sum of squares as a quadratic function in the coefficients. Thus, the resulting approximate posterior distribution will be standard, namely, t distribution. The efficiency of the proposed identification procedure is assessed via a comprehensive simulation study.

In addition, a comparison between the proposed technique and some non-Bayesian automatic techniques is conducted.

2. Seasonal Moving Average Processes

Let $\underline{y} = [y(1) \quad y(2) \quad \dots \quad y(n)]'$ be a vector of n observations generated from the seasonal moving average process of orders q and Q, denoted by $SMA(q,Q)_s$, where s is the seasonal period, i.e. the number of seasons in the year. The model has the form (see Box and Jenkins (1970)),

$$y(t)=\theta(B)\Theta(B^s)\varepsilon(t) \tag{2.1}$$

Where,

B is the backward shift operator defined as $B^r y(t)=y(t-r)$, r = 1,2,...

y(t) denotes the time series observations, t=1,....,n.

ε(t) denotes the random errors assumed to be iid $N(0,\tau^{-1})$,$\tau>0$ is the precision parameter. Moreover,

$$\theta(B)=1-\theta_1 B-\theta_2 B^2-\dots-\theta_q B^q$$

and

$$\Theta(B^s)=1-\Theta_1 B^s-\Theta_2 B^{2s}-\dots-\Theta_Q B^{Qs}$$

The model is always stationary and is invertible if the roots of both $\theta(B)=0$ and $\Theta(B^s)=0$ lie outside the unit circle. The model can be written explicitly in the form,

$$y(t) = \varepsilon(t) - \sum_{i=1}^{q}\theta_i\varepsilon(t-i) - \sum_{j=1}^{Q}\Theta_j\varepsilon(t-js) + \sum_{i=1}^{q}\sum_{j=1}^{Q}\beta_{ij}\varepsilon(t-i-js) \tag{2.2}$$

Where, $\beta_{ij}=\theta_i\Theta_j$, $i=1,\dots,q$, $j=1,\dots,Q$.

The seasonal moving average class of models (2.1) is quite important in modeling seasonal time series data, see for instance Box and Jenkins (1970). In practice, the orders q and Q are unknown and one has to determine a value for each of them using n observations $\underline{y} = [y(1) \quad y(2) \quad \dots \quad y(n)]'$. Thus the statistical question is: "Given n observations generated from a seasonal moving average process, what are the values of q and Q?" The direct Bayesian answer of this question is to find the marginal posterior probability mass function of the orders and this has not been done yet for seasonal moving average process because of the complexity of the likelihood function. However, the indirect approach introduced by Broemeling and Shaarawy (1988) can be developed to choose initial adequate values for the orders q and Q. These values will be used later by our proposed

direct approach in order to develop an approximate posterior probability mass function for the model orders in a convenient form.

3. Direct Bayesian Identification of Seasonal Moving Average Processes

Except for some special cases, it is very hard to work with time series models including moving average parts. In order to avoid this difficulty, Broemeling and Shaarawy (1988) developed an indirect Bayesian procedure to identify the orders of mixed ARMA models. However, their procedure can be modified to identify initial values for the orders q and Q of seasonal moving average process. These values will be used later in this section to develop our proposed direct approach. Instead of working with the joint posterior probability mass function of q and Q, their indirect approach is proposed to focus on the posterior distribution of the coefficients vector

$$\underline{\gamma}^{(k+m+km)} = [\theta_1 \ \theta_2 \ldots \theta_k \ \Theta_1 \ \Theta_2 \ldots \Theta_m \ \beta_{11} \ \ldots \ \beta_{km}]' \qquad (3.1)$$

Where, k is the maximum value of q and m is the maximum value of Q.

The indirect Bayesian technique is based on approximating the conditional likelihood function by a normal-gamma density on the parameter space. Then, one would expect a normal-gamma posterior analysis when the approximate likelihood function is combined, via Bayes theorem, with a normal-gamma prior density or with a Jeffreys' vague prior. When the approximate conditional likelihood function is combined with such prior, the marginal posterior of $\underline{\gamma}^{(k+m+km)}$ is a (k+m+km) dimensional multivariate t distribution. Thus, any single component of this vector has a univariate t distribution and the conditional distribution of any component given any other component has also a univariatc t distribution. For a discussion of these distributional results see DeGroot (1970, pp. 59–62). Then one can do a backward elimination procedure to identify initial values for the orders q and Q as follows:

1. Test $H_0 : \Theta_m = 0$ versus $H_1 : \Theta_m \neq 0$ using the marginal posterior distribution of Θ_m which is a univariate t distribution.
2. If the above H_0 is not rejected, test $H_0 : \Theta_{m-1} = 0$ versus $H_1 : \Theta_{m-1} \neq 0$ using the conditional distribution of Θ_{m-1} given $\Theta_m = 0$ which is also a univariate t distribution.
3. If the above H_0 is not rejected, test $H_0 : \Theta_{m-2} = 0$ versus $H_1 : \Theta_{m-2} \neq 0$ using the conditional distribution of Θ_{m-2} given $\Theta_m = \Theta_{m-1} = 0$.
4. The procedure is continued in this way until the hypothesis $\Theta_{Q_0} = 0$ is rejected for some Q_0 where $0 < Q_0 \leq m$. The value Q_0 is then the initial selected order for the seasonal part of the model.

5. The previous steps are repeated for the non seasonal part of the model starting with testing $H_0 : \theta_k = 0$ versus $H_1 : \theta_k \neq 0$ until one of the hypothesis is rejected for some q_0 where $0 < q_0 \leq k$. The value q_0 is then the initial selected order for the non seasonal part.

Given the initial values q_0 and Q_0, estimated above, this section is devoted to develop the proposed direct Bayesian identification for seasonal moving average processes. Unlike the indirect technique, the orders q and Q are assumed to be random variables and the problem is how to find the joint posterior probability mass function of q and Q in a convenient form. In order to do that, let $\underline{y} = [y(1) \quad y(2) \quad \ldots \quad y(n)]'$ be a vector of n observations generated from a seasonal moving average process of orders q and Q having the form (2.2) where the orders q and Q are non negative unknown integers. Conditioning on $\varepsilon_0 = \varepsilon_{-1} = \ldots = \varepsilon_{1-k-ms}$, the likelihood function is

$$L(\underline{\gamma}(q,Q),q,Q,\tau|\underline{y}) \propto (\tau)^{\frac{n}{2}}$$

$$\exp\frac{-\tau}{2}\sum_{t=1}^{n}\left[y(t)+\sum_{i=1}^{q}\theta_{qi}\varepsilon(t-i)+\sum_{j=1}^{Q}\Theta_{Qj}\varepsilon(t-js)-\sum_{i=1}^{q}\sum_{j=1}^{Q}\beta_{qQij}\varepsilon(t-i-js)\right]^2 \tag{3.2}$$

Where, $\underline{\gamma}^{(k+m+km)}$ is the coefficients vector defined in (3.1), $\tau > 0$, q = 1,2,..., k and Q = 1,2,..., m.

The likelihood function (3.2) is analytically intractable since the errors ε_{t-j}'s are nonlinear functions in the model coefficients $\theta_{k1},\ldots,\theta_{kk},\Theta_{m1},\ldots,\Theta_{mm},\beta_{kmk1},\ldots,\beta_{kmkm}$. Many suggestions have been done in order to simplify the likelihood function such as Shaarawy and Broemeling (1984), Newbold (1973), and Zellner and Reynolds (1978). All these simplifications and others are based on knowing the values of the model orders q and Q. Thus, one must use an adequate technique, preferably a Bayesian one, to determine or identify initial values for the unknown orders q and Q that can be used later to simplify the likelihood function. Here, following Shaarawy et al (2007), we propose to use the indirect procedure, explained above, to determine or identify such initial values, q_0 and Q_0. Then the values q_0 and Q_0 are used to estimate the errors ε_{t-j}'s recursively by the recurrence formula

$$\hat{\varepsilon}(t) = y(t) + \sum_{i=1}^{q_0}\hat{\theta}_{q_0,i}\hat{\varepsilon}(t-i) + \sum_{j=1}^{Q_0}\hat{\Theta}_{Q_0,j}\hat{\varepsilon}(t-js) - \sum_{i=1}^{q}\sum_{j=1}^{Q}\hat{\beta}_{ij}\hat{\varepsilon}(t-i-js) \tag{3.3}$$

Where;

$\hat{\underline{\gamma}}(q_0,Q_0)=\left[\hat{\theta}_{q_0,1}\ldots\hat{\theta}_{q_0,q_0}\ \hat{\Theta}_{Q_0,1}\ \ldots\hat{\Theta}_{Q_0,Q_0}\ \hat{\beta}_{11}\ \ldots\ \hat{\beta}_{q_0,Q_0}\right]'$ is the nonlinear least squares estimate of the coefficients' vector $\underline{\gamma}(q_0,Q_0)$, found by minimizing

$$SSE=\sum_{t=1}^{n}\varepsilon_t^2$$

With respect to the coefficients vector, $\underline{\gamma}(q,Q)$, over the region of invertibility.

Once the estimated errors ε_{t-j}'s are obtained, they are substituted in the likelihood function (3.2) to get an approximate conditional likelihood function in the form

$$L^*(\underline{\gamma}(q,Q),q,Q,\tau|\underline{y})\propto(\tau)^{\frac{n}{2}}\exp-\frac{\tau}{2}\sum_{t=1}^{n}\left[y(t)+\sum_{i=1}^{q}\theta_{qi}\hat{\varepsilon}(t-i)+\sum_{j=1}^{Q}\Theta_{Qj}\hat{\varepsilon}(t-js)-\sum_{i=1}^{q}\sum_{j=1}^{Q}\beta_{ij}\hat{\varepsilon}(t-i-js)\right]^2 \quad (3.4)$$

$L^*(\underline{\gamma}(q,Q),q,Q,\tau|\underline{y})$ has a normal gamma form. If n is large, the approximate likelihood function L^* is expected to serve as a good approximation to the likelihood function.

The parameters vector $\underline{\gamma}(q,Q)$ is assigned, a priori, a conditional normal distribution given q, Q and τ with mean vector $\underline{\mu}(q,Q)$ and precision matrix $\tau R(q,Q)$. Assuming that τ and (q,Q) are independent where τ has a gamma density with parameters a and b and (q,Q) have a joint uniform prior function over the integers q=1,2,…,k, Q=1,2,…,m. Thus, one may write the joint prior distribution of the parameters $\underline{\gamma}(q,Q)$, q, Q and τ as follows:

$$g(\underline{\gamma}(q,Q),q,Q,\tau)\propto(2\pi)^{-\frac{1}{2}(q+Q+qQ)}|R(q,Q)|^{\frac{1}{2}}\tau^{\frac{1}{2}(q+Q+qQ+2a)-1}\exp-\frac{\tau}{2}\left\{2b+\left[\underline{\gamma}(q,Q)-\underline{\mu}(q,Q)\right]'R(q,Q)\left[\underline{\gamma}(q,Q)-\underline{\mu}(q,Q)\right]\right\} \quad (3.5)$$

If one is not quite confident about the hyperparameters $\underline{\mu}(q,Q)$, $R(q,Q)$, a, b of the prior density one might use Jeffreys' vague prior

$$g_1(\underline{\gamma},q,Q,\tau)\propto\tau^{-1} \qquad ,\tau>0 \quad (3.6)$$

The approximate conditional likelihood function (3.4) can be written in matrix notation as

$$L^*(\underline{\gamma}(q,Q),q,Q,\tau|\underline{y})\propto(\tau)^{\frac{n}{2}}\exp-\frac{\tau}{2}\sum_{t=1}^{n}\left[y(t)-\underline{\gamma}'(q,Q)\underline{\eta}_t(q,Q)\right]^2 \quad (3.7)$$

Where,

$$\underline{\eta}_t'(q,Q)=[-\underline{\hat{\varepsilon}}'(t-i)\ -\underline{\hat{\varepsilon}}'(t-js)\ \underline{\hat{\varepsilon}}'(t-1-js)\ \dots\ \underline{\hat{\varepsilon}}'(t-q-js)]$$

$$\underline{\hat{\varepsilon}}'(t-i)=[\hat{\varepsilon}(t-1)\ \ \hat{\varepsilon}(t-2)\ \ \dots\ \ \hat{\varepsilon}(t-q)]$$

$$\underline{\hat{\varepsilon}}'(t-js)=[\hat{\varepsilon}(t-s)\ \ \hat{\varepsilon}(t-2s)\ \ \dots\ \ \hat{\varepsilon}(t-Qs)]$$

$$\underline{\hat{\varepsilon}}'(t-r-js)=[\hat{\varepsilon}(t-r-s)\ \ \hat{\varepsilon}(t-r-2s)\ \ \dots\ \ \hat{\varepsilon}(t-r-Qs)] \qquad ,r=1,2,\dots,q$$

Since the exponent is quadratic in $\underline{\gamma}(q,Q)$, L^* as a function on the parameter space is a normal gamma density. Thus, one would expect a normal gamma posterior density when L^* is combined, via Bayes theorem, with a normal gamma prior density or Jeffreys' vague prior. In order to formulate the posterior analysis, let us define the following quantities:

$$A^*(q,Q)=\begin{bmatrix} A_{11} & A_{12} & \dots & A_{1(q+2)} \\ A_{12}' & A_{22} & \dots & A_{2(q+2)} \\ \vdots & \vdots & \dots & \vdots \\ A_{1(q+2)}' & A_{2(q+2)}' & \dots & A_{(q+2)(q+2)} \end{bmatrix} \qquad (3.8)$$

Where the matrix A_{11} is of order $q\times q$, with ijth element $\sum_{t=1}^{n}\hat{\varepsilon}(t-i)\hat{\varepsilon}(t-j)$. The matrix A_{12} is of order $q\times Q$, with ijth element $\sum_{t=1}^{n}\hat{\varepsilon}(t-i)\hat{\varepsilon}(t-js)$. The matrix A_{1d}, d=3,4,...,q+2 is of order $q\times Q$, with ijth element $-\sum_{t=1}^{n}\hat{\varepsilon}(t-i)\hat{\varepsilon}(t-js-d+2)$. The matrix A_{22} is of order $Q\times Q$, with ijth element $\sum_{t=1}^{n}\hat{\varepsilon}(t-is)\hat{\varepsilon}(t-js)$. The matrix A_{2d}, d=3,4,..., q+2 is of order $Q\times Q$, with ijth element $-\sum_{t=1}^{n}\hat{\varepsilon}(t-is)\hat{\varepsilon}(t-js-d+2)$

The matrix A_{rd}, r=3,4,...,q+2, d=r,r+1,...,q+2 is of order $Q\times Q$, with ijth element

$\sum_{t=1}^{n}\hat{\varepsilon}(t-is-r+2)\hat{\varepsilon}(t-js-d+2)$.

Furthermore,

$$\underline{B}^*(q,Q)=\begin{bmatrix} \underline{B}_1 \\ \underline{B}_2 \\ \vdots \\ \underline{B}_{(q+2)} \end{bmatrix} \qquad (3.9)$$

Where, $\underline{B}_1$ is a column vector of order q, with i^{th} element $-\sum_{t=1}^{n} y(t)\hat{\varepsilon}(t-i)$. $\underline{B}_2$ is a column vector of order Q, with i^{th} element $-\sum_{t=1}^{n} y(t)\hat{\varepsilon}(t-is)$. $\underline{B}_r$, r=3,4,...,q+2 is a column vector of order Q, with i^{th} element $\sum_{t=1}^{n} y(t)\hat{\varepsilon}(t-is-r+2)$. Moreover, let

$$C^* = \sum_{t=1}^{n} y^2(t) \tag{3.10}$$

Using the quantities $A^*(q,Q)$, $\underline{B}^*(q,Q)$ and C^* in (3.8), (3.9) and (3.10) respectively, the approximate conditional likelihood function (3.7) can be written in matrix notation in the form

$$L^*\left(\underline{\gamma}(q,Q),q,Q,\tau|\underline{y}\right) \propto (\tau)^{\frac{n}{2}} \exp -\frac{\tau}{2}\left[\underline{\gamma}'(q,Q)A^*(q,Q)\underline{\gamma}(q,Q) - 2\underline{\gamma}'(q,Q)\underline{B}^*(q,Q) + C^*\right] \tag{3.11}$$

In addition, define

$$\begin{aligned} A(q,Q) &= A^*(q,Q) + R(q,Q) \\ \underline{B}(q,Q) &= \underline{B}^*(q,Q) + R(q,Q)\underline{\mu}(q,Q) \\ C(q,Q) &= 2b + C^* + \underline{\mu}'(q,Q)R(q,Q)\underline{\mu}(q,Q) \end{aligned} \tag{3.12}$$

Where $R(q,Q)$, $\underline{\mu}(q,Q)$ and b are hyperparameters in the joint prior (3.5).

Using the above information, one may assert the following theorem and corollary:

Theorem (3.1)

Using the approximate conditional likelihood function in (3.11) and the prior density function in (3.5), the joint posterior mass function of the orders q and Q is

$$h\left(q,Q|\underline{y}\right) \propto |R(q,Q)|^{\frac{1}{2}} |A(q,Q)|^{-\frac{1}{2}} \Gamma(\nu_1)\left[C(q,Q) - \underline{B}'(q,Q)A^{-1}(q,Q)\underline{B}(q,Q)\right]^{-\nu_1} \tag{3.13}$$

Where, $q = 1,2,...,k$, $Q = 1,2,...,m$ *and* $2\nu_1 = n + 2a$. *$A(q,Q)$, $\underline{B}(q,Q)$ and $C(q,Q)$ are defined in (3.12).*

If one knows very little, a priori, about the parameters, one might use the following corollary:

Corollary (3.1)

Using the approximate conditional likelihood function in (3.11) and the Jeffreys' prior in (3.6), the joint posterior mass function of the orders q and Q is

$$h(q,Q|\underline{y}) \propto (\pi)^{-\nu_1} |A^*(q,Q)|^{-\frac{1}{2}} \Gamma(\nu_2(q,Q)) \left[C^* - \underline{B}^{*'}(q,Q) A^{*-1}(q,Q) \underline{B}^*(q,Q) \right]^{-\nu_2(q,Q)} \quad (3.14)$$

Where, $2\nu_2(q,Q) = n - q - Q - qQ$, $q=1,2,...,k$ *and* $Q=1,2,...,m$. *The matrix* $A^*(q,Q)$, *the column vector* $B^*(q,Q)$ *and* C^* *are as defined in (3.8), (3.9) and (3.10) respectively.*

After deriving the posterior mass function, then one may calculate all possible posterior probabilities to study the behavior of the marginal posterior mass function for all possible orders. Then one may choose the orders with the maximum posterior probability as point estimates for the orders (q,Q).

4. A Numerical Study

This section aims to assess the performance and efficiency of the proposed direct technique in identifying the order of seasonal moving average processes. In order to achieve this goal, four simulation studies have been conducted. The proposed technique is employed, with three different prior distributions, to identify the orders of $SMA(1,1)_4$, $SMA(1,2)_4$ and $SMA(1,3)_4$ models with various parameter values. The parameters in some cases are chosen to be well inside the invertibility domain while in some other cases they are chosen to be near the boundaries. All computations are performed using Matlab 7.

Here, our main concern is to study the effectiveness of the proposed direct technique by calculating the percentage of correct identification. Such effectiveness will be examined with respect to the time series length as well as the parameters of the selected models. For all models, sample size and parameter sets the precision of the noise term is fixed at two.

Simulation 1, for illustration, begins by generating 200 data sets of normal variates, each of size 2500, to represent the noise ε_t. These data sets are then used to generate 200 realizations, each of size 2000, from $SMA(1,1)_4$ process with coefficients $\theta_1 = 0.5$ and $\Theta_1 = 0.5$. Note that, the first 500 observations are ignored to remove the initialization effect. For a specific prior, the second step of simulation 1 is to carry out all computations, assuming certain maximum orders k and m, required to identify each of the 200 realizations and to find the percentage of correct identification. Note that the computations include the application of the indirect technique to get initial values q_0 and Q_0 and use them in the application of the proposed direct technique. Such computations are done for a specific time series length n using the first n observations of each generated realization. The second step is repeated for each chosen time series length, maximum order, and prior combinations. The time series length n is taken to be 200, 400, 600, 800, 1500 and 2000, while the maximums k and m are taken to be (2,2) and (3,3).

With respect to the prior probability mass function of the orders q and Q, which is combined with the vague prior of $\underline{\gamma}(q,Q)$ and τ, the following three priors are used

- Prior 1: $\zeta(q,Q) \propto \frac{1}{k} \times \frac{1}{m}$ q=1,2,...,k ; Q=1,2,...,m.
- Prior 2: $\zeta(1,1)=0.35, \zeta(1,2)=\zeta(2,1)=0.25, \zeta(2,2)=0.15$ **for** $k=m=2$.

  $\zeta(1,1)=0.28, \zeta(1,2)=\zeta(2,1)=0.18, \zeta(1,3)=\zeta(2,2)=\zeta(3,1)=0.08$
  $\zeta(2,3)=\zeta(3,2)=0.045$ **and** $\zeta(3,3)=0.03$ **for** $k=m=3$.
- Prior 3: $\zeta(q,Q) \propto (0.5)^{q+Q}$ q=1,2,...,k ; Q=1,2,...,m .

The first prior assigns equal probabilities to all possible combinations of the orders q and Q. The second prior is chosen in such a way to give probabilities that decreases arbitrarily with the order, while the third prior is chosen in such a way to give probabilities that decline exponentially with the order.

Simulation 2 is done in a similar manner but using $SMA(1,1)_4$ with coefficients $\theta_1 = 0.8$ and $\Theta_1 = 0.8$. While simulation 3 is done similarly but using $SMA(1,2)_4$ model with coefficients $\theta_1 = -0.5$, $\Theta_1 = 0.1$ and $\Theta_2 = 0.9$. And finally, simulation 4 is done similarly but using $SMA(1,3)_4$ model with coefficients $\theta_1 = 0.5$, $\Theta_1 = 0.2$, $\Theta_2 = -0.2$ and $\Theta_3 = 0.5$. The results of the four simulation studies are presented in Table 1. For the four simulation studies, the percentages of correct identification using the indirect procedure, with Jeffreys' prior density, are also reported in the same table.

For the matter of comparison, the well-known Akaike's information criteria, denoted by AIC has been calculated for all 200 generated series in each of the above mentioned four simulation studies. The selected model is the one that achieves the smallest value of the criterion. Then, the percentages of correct identification are computed for AIC for various time series lengths and the results are also given in Table 1 for the same two assumed maximums of the orders q and Q.

Table 1

The Percentages of Correct Identification for SMA(q,Q)$_4$ Models Using Bayesian and Automatic Techniques

		SMA(1,1)				SMA(1,2)		SMA(1,3)
		θ_1=0.5,Θ_1=0.5		θ_1=0.8,Θ_1=-0.8		θ_1=-0.5,Θ_1=0.1, Θ_2=-0.9		θ_1=0.5,Θ_1=0.2, Θ_1=-.2,Θ_3=0.5
Series length	Technique	Max(2,2)	Max(3,3)	Max(2,2)	Max(3,3)	Max(2,2)	Max(3,3)	Max(3,3)
200	Prior 1	87	84	80.5	72	97	81	91
	Prior 2	90	89	82.5	78.5	98	84.5	91.5
	Prior 3	91.5	90	85	79.5	98.5	83.5	91.5
	Indirect	79	56.5	66.5	37	90.5	55.5	78
	AIC	70.5	62	68.5	57	81	60	76.5
400	Prior 1	95.5	92.5	91.5	88	98	91	94.5
	Prior 2	96.5	94	92.5	89	98.5	94	95
	Prior 3	97.5	95	94	89.5	98.5	94	95.5
	Indirect	82	66.5	76	46.5	94.5	65	83
	AIC	70.5	61	73	64.5	84.5	67.5	80.5
600	Prior 1	94.5	92.5	90	84.5	97.5	91	95
	Prior 2	96	95.5	90	86.5	97.5	94	98.5
	Prior 3	97	97.5	90.5	87	98.5	93.5	98.5
	Indirect	85	63	76.5	50	94	70	88.5
	AIC	75.5	62.5	70.5	63	87.5	71	84.5
800	Prior 1	95	93	91.5	86.5	99	93	95
	Prior 2	95.5	94	92	87.5	99	93.5	95.5
	Prior 3	96	94.5	93.5	88.5	99.5	93.5	95.5
	Indirect	81	64.5	75.5	53.5	92.5	71	87
	AIC	67.5	59.5	70.5	60.5	83.5	67.5	81.5
1500	Prior 1	95.5	95	95.5	90.5	99	93	94
	Prior 2	97	96	97	93.5	99	94.5	94
	Prior 3	97.5	96.5	97	94	99	94.5	94
	Indirect	83.5	71	79.5	56	94.5	71	84
	AIC	71	61.5	70	59.5	83	67.5	80
2000	Prior 1	96	93.5	92	90	98.5	96	94
	Prior 2	96	94.4	92.5	91	98.5	97	94.5
	Prior 3	96.5	95	94	92.5	98.5	96.5	94.5
	Indirect	82	72.5	74.5	54	92.5	75.5	85.5
	AIC	66.5	59.5	72.5	65.5	85	66	81.5

Inspection of the numerical results in table 1 shows the following conclusions:

1. The numerical efficiency increases as the time series length increases for all models, all techniques and all maximum orders.
2. The numerical efficiency of the proposed direct technique is high, being greater than 71%, for all models, all time series lengths and all maximum orders.
3. The numerical efficiency of the proposed direct technique is better than those of both the indirect and the AIC techniques for all models, all time series lengths and all maximum orders.

4. The numerical efficiency of the indirect technique is fairly high and better than that of the AIC technique almost everywhere.
5. The numerical efficiency of the proposed direct technique using prior 3 is better than that of prior 2, which in turn is better than that of prior 1.
6. The results corresponding to maximum orders (2,2) are better than those of maximum order (3,3), for all models, all time series lengths and all techniques. This is reasonable since searching for the appropriate model among a smaller number of alternatives is easier.

5. Conclusion

This article has proposed a direct Bayesian technique to identify the orders of seasonal moving average processes. The joint posterior probability mass function of the model orders has been developed in a convenient form using an approximate conditional likelihood function and a normal-gamma prior density. Then one may easily inspect the behavior of the posterior probabilities and choose the order combination with highest posterior probability to solve the identification problem. The efficiency of the proposed direct technique has been checked and compared with a well known non Bayesian automatic technique through simulation studies. The analysis of the numerical results shows that the proposed direct technique can efficiently identify the orders of seasonal moving average processes.

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