The Preservation of Some Classes of Discrete Distributions

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Abstract

In this paper, some new families of discrete life distributions are discussed. Definitions and Basic Results are introduced. Several properties of these classes are presented, including the preservation under convolution, closer under formation of parallel systems and mixing.

Key Words:formation of parallel systems, convolution, mixing andNBUE.

1. Introduction

Nonparametric aging classes of life distributions have been found to be useful in reliability analysis, engineering applications, maintenance policies, economics, biometry, queuing theory and many other fields. There are many situations where a continuous time is inappropriate for describing the lifetime of devices and other systems. For example, the life time of many devices in industry such as switches and mechanical tools, depends essentially on the number of times they are turned on and off or the number of shocks they receive. In such cases, the time to failure is often more appropriately represented by the number of times they are used before they fail, which is a discrete random variable.

Let X be a non-negative discrete random variable representing the lifetime of the unit. Without loss of generality, we will assume the N is support of X. The probability mass function (p.m.f) is given by $f(x) = Pr\{X = x\}, x = 0,1,2,\ldots$, the cumulative distribution function F of X satisfies $F(x) = \Pr\{X \le x\} = \sum_{i=0}^{x} f(i)$ for all $x \in N$, and the survival function $\bar{F}$ of X satisfies $\bar{F}(x) = 1 - F(x) = \sum_{i=x+1}^{\infty} f(i)$ for all $x \in N$ where $N = \{0,1,\ldots\}$. Furthermore, for all$x = -1,-2,\ldots,$, $F(x) = 0$and$\bar{F}(x) = 1$. The distribution of counting random variable is called a discrete life distribution. In particular, if $f(0) = \Pr\{X = 0\} = 0$, or a counting random variable X has a support on$N_+ = \{1,2,\ldots\}$, we say that the discrete distribution is zero-truncated. Moreover, $N_- = \{-1,0,1,\ldots\}$.

Similar to continuous distributions, discrete distribution can also be classified by the properties of failure rates, mean residual lifetimes, survival function. These classes of discrete distribution aging have been used extensively in different fields of statistics and probability such as insurance, finance, reliability, survival analysis, and others. See, for example, Barlow and Proshan(1981), Cai and Kalashnikov(2000), Cai and Willmot (2005), Hu et al (2003), Willmot and Lin (2000) and Willmot et al. (2005). Some commonly used classes of discrete distributions include the classes of discrete decreasing failure rate (D-DFR), discrete decreasing failure rate average (D-DFRA), discrete new worse than used (D-NWU), discrete increasing mean residual life (d-IMRL), discrete harmonic new worse than used in expectation (d-HNWUE), and their dual ones including the classes of discrete increasing failure rate(D-IFR), discrete increasing failure rate average (D-IFRA), discrete new better than used (D-NBU), discrete decreasing mean residual life (D-DMRL), and discrete harmonic new better than used in expectation (D-HNBUE).

Ahmad et al. (2006) defined the class of new better than used in increasing convex average order (NBUCA). This class is requiring the distribution function F of a random variable X to satisfy

$$\int_0^\infty \int_x^\infty \bar{F}(u+t)dudt \le \bar{F}(t)\int_0^\infty \int_x^\infty \bar{F}(u)dudx, for \;\; all\, t \ge 0.$$

And Ahmed (1990) defined a continuous random variable X (or its distribution function F) to be generalized harmonic new better than used in expectation if

$$\int_0^\infty \int_t^\infty \bar{F}(x)dx \le \mu^2, \qquad 0<\mu<\infty.$$

2. Basic Definitions.

Most of the nonparametric discrete classes of distributions that are commonly found in the reliability literature are based on some notion of aging. In this section we present the definitions of some classes of discrete distributions, which are used in the sequel.

Definition 2.1

A discrete distribution F is called discrete new better than used in convex ordering (D-NBUC) (or discrete new worse than used in convex ordering (D-NWUC)) if

$\sum_{i=x+y}^{\infty} \bar{F}(i) \le \bar{F}(x)\sum_{j=y}^{\infty} \bar{F}(j)$for all $x, y \in N$.

Definition 2.2

An integer valued random variable X (or its cdf F) is said to be discrete new better than used in convex average order (D-NBUCA) if

$\sum_{k=0}^{\infty}\sum_{i=k}^{\infty} \bar{F}(i) \le \bar{F}(j)\sum_{k=0}^{\infty}\sum_{i=0}^{\infty} \bar{F}(i)$.

Definition 2.3

An integer valued random variable X (or its cdf F) F is said to be discrete new better than used in second order (D-NBU (1)) if

$\sum_{j=0}^{k} \bar{F}(i+j) \leq \bar{F}(i) \sum_{j=0}^{k} \bar{F}(j)$ for all $i, k \in N$.

Definition 2.4

A discrete distribution F with finite mean μ is called discrete new better than used in expectation (D-NBUE) if for all $x \in N$.

$\sum_{i=x}^{\infty} \bar{F}(i) \leq \bar{F}(x) \sum_{j=0}^{\infty} \bar{F}(j)$ for all $x \in N$.

Definition 2.5

A discrete distribution F with expectation $\sum_{i=0}^{\infty} \bar{F}(i) = \mu$ is called discrete generalized harmonic new better than used in expectation (D-GHNBUE) if for all $x \in N$.

$$\sum_{k=0}^{\infty} \sum_{j=k}^{\infty} \bar{F}(j) \leq (\geq) m^2.$$

Theorem 2.1

If $X \in D - NBU(1)$ then $X \in D - NBUE$.

Proof

Since X is $D - NBU(1)$.

$\bar{F}(i) \sum_{j=0}^{k} \bar{F}(j) \geq \sum_{j=0}^{k} \bar{F}(i+j)$ for all $i, k \in N$.(2.1)

Letting k tends to infinity in equation (2.1), one gets

$$\bar{F}(i) \sum_{j=0}^{\infty} \bar{F}(j) \geq \sum_{j=0}^{\infty} \bar{F}(i+j),$$

Or equivalently

$$\mu \bar{F}(i) \geq \sum_{j=0}^{\infty} \bar{F}(i+j),$$

Which is $D - NBUE$.

3. Preservation Properties

3.1 Convolution

As an important reliability operation, convolutions of life distributions of a certain class are often paid much attention. The closure properties of IFR, NBU, NBUE, and IFRA can be found in Barlow and Proschan (1981). It has shown that the D-NBU and D-NBUC classes are closed under this operation see Pavlova, et al (2006). In this section we establish the closure property of the $D-NBU(2)$, D-*NBUCA* and D-*GHNWUE* classes under the convolution operation.

Theorem3.1

Suppose that F_1 and F_2 are two independent $D-NBUCA$ life distributions. Then their convolution is also $D-NBUCA$.

Proof

The survival functions of convolution of two life distribution F_1 and F_2 is

$$\bar{F}(u)=\sum_{t=0}^{u}\bar{F}_1(u-t)f_2(t), for\ all\ u\in N.$$

Let $x,y\in N.$ on one hand

$$\sum_{y=0}^{\infty}\sum_{i=0}^{\infty}\bar{F}(x+y+i)=\sum_{y=0}^{\infty}\sum_{i=0}^{\infty}\sum_{j=0}^{\infty}\bar{F}_1(x+y+i-j)f_2(j)$$

$$=\sum_{y=0}^{\infty}\sum_{j=0}^{x-1}f_2(j)\sum_{i=0}^{\infty}\bar{F}_1(x+y+i-j)$$
$$+\sum_{y=0}^{\infty}\sum_{i=0}^{\infty}\sum_{j=0}^{\infty}\bar{F}_1(y+i-j)f_2(x+j)$$
$$=I_1+I_2, say$$

Observe that

$I_1\le\sum_{y=0}^{\infty}\bar{F}_1(y)\sum_{j=0}^{x-1}f_2(j)\sum_{i=0}^{\infty}\bar{F}_1(x+i-j)$(3.1)

Also,

$$I_2=\sum_{y=0}^{\infty}\sum_{i=0}^{\infty}\bar{F}_2(x-1)\sum_{i=y}^{\infty}\bar{F}_1(i)+\sum_{y=0}^{\infty}\sum_{i=0}^{\infty}\sum_{j=0}^{\infty}\bar{F}_2(x+j)f_1(y+i-j).$$

$$=\sum_{y=0}^{\infty}\sum_{i=y}^{\infty}\bar{F}_1(i)+\sum_{y=0}^{\infty}\sum_{i=0}^{\infty}\sum_{j=0}^{y+i}\bar{F}_2(x+y+i-j)f_1(j).$$

$$=\bar{F}_2(x-1)\sum_{y=0}^{\infty}\sum_{i=y}^{\infty}\bar{F}_1(i)+\sum_{y=0}^{\infty}\sum_{j=y+1}^{\infty}\sum_{i=j-y}^{\infty}\bar{F}_2(x+y+i-j)f_1(j)$$

$$+\sum_{y=0}^{\infty}\sum_{i=0}^{\infty}\sum_{j=0}^{\infty}\bar{F}_2(x+y+i-j)f_1(j).$$

$= A_1 + A_2 + A_3$.(3.2)

Where

$$A_1 \le \sum_{y=0}^{\infty}\bar{F}_2(x-1)\mu_1\bar{F}_1(y) \le \mu_1\mu\bar{F}_2(x-1),$$

$$A_2 = \sum_{y=0}^{\infty}\sum_{j=y+1}^{\infty}f_1(j)\sum_{i=0}^{\infty}\bar{F}_2(x+i).$$

$$= \sum_{y=0}^{\infty}\bar{F}_1(y)\sum_{i=0}^{\infty}\bar{F}_2(x+i),$$

$$A_3 \le \sum_{y=0}^{\infty}\sum_{j=0}^{\infty}f_1(j)\left[\bar{F}_2(y-j)\sum_{i=0}^{\infty}\bar{F}_2(x+i)\right]$$

$$= \sum_{y=0}^{\infty}\left[\sum_{j=0}^{y}\bar{F}_2(y-j)\,f_1(j)\right]\sum_{i=0}^{\infty}\bar{F}_2(x+i).$$

$$= \sum_{y=0}^{\infty}[\bar{F}(y)-\bar{F}_1(y)]\sum_{i=0}^{\infty}\bar{F}_2(x+i).$$

$$= (\mu-\mu_1)\sum_{i=0}^{\infty}\bar{F}(x+i).$$

Hence

$I_2 \le \mu_1\mu\bar{F}_2(x-1) + \mu\sum_{i=0}^{\infty}\overline{F_2}(x+i)$ (3.3)

Combining equations (3.1), (3.2) and (3.3) we see that

$$\sum_{y=0}^{\infty}\sum_{i=0}^{\infty}\bar{F}(x+y+i) \le \mu\left\{\sum_{i=0}^{\infty}\sum_{i=0}^{x-1}\bar{F}_1(x+i-j)\,f_2(j)\right\}$$

$$\{\mu_1\bar{F}_2(x-1)+\overline{F_2}(x+i)\}.$$

Finally, implementing equation (3.3) with y=0, we get

$$\sum_{y=0}^{\infty}\bar{F}(y)\sum_{i=0}^{\infty}\bar{F}(x+i)$$

$$=\sum_{y=0}^{\infty}\bar{F}(y)\left\{\sum_{i=0}^{\infty}\sum_{i=0}^{x-1}\bar{F}_1(x+i-j)f_2(j)+\sum_{j=0}^{\infty}\sum_{i=j}^{\infty}\bar{F}_2(x+i-j)f_1(j)\right\}.$$

$$=\sum_{y=0}^{\infty}\bar{F}(y)\left\{\sum_{i=0}^{\infty}\sum_{i=0}^{x-1}\bar{F}_1(x+i-j)f_2(j)+\mu_1\bar{F}_2(x-1)\right.$$
$$\left.+\sum_{j=0}^{\infty}f_1(j)\sum_{i=0}^{\infty}\bar{F}_2(x+i)\right\}$$

$$\geq\sum_{y=0}^{\infty}\sum_{i=0}^{\infty}\bar{F}(x+y+i).$$

Which mean that F is D-NBUCA.

Theorem 3.2

Suppose that F_1 and F_2 are two independent D-*GHNBUE* lives

Proof

With F^*Gdenoting the convolution of the life distributions F and G, the result follows from the fact: If $\sigma_F \leq \mu_F, \sigma_G \leq \mu_G$and $\sigma_{F^*G} \leq \mu_{F^*G}$ then either μ_F orμ_G. See, Ahmed (1990).

3.2 Closure of the D − NBU(1) under formation of parallel system

In this subsection we show that the discrete new better than used in second order is closed under formation of parallel systems.

Theorem 3.3

Let $X_1, X_2, \dots, X_n$be independent and identically distributed integer valued random variables with distribution function F and F is$d-NBU(1)$. Then the random variable $Y = {}_{1\leq i\leq n}^{Max}X_i$ has distribution F_n is also $D-NBU(1)$.

Proof

Since F is $D-NBU(1)$, we have

$$\sum_{j=0}^{k}\bar{F}(i+j)\leq\bar{F}(i)\sum_{j=0}^{k}\bar{F}(j).$$

This impliesthat

$$\sum_{j=i}^{i+k} \bar{F}(i+j) \leq \bar{F}(i) \sum_{j=0}^{i+k} \bar{F}(j).$$

Or

$$\sum_{j=i}^{i+k} [1-F(j)] \leq \bar{F}(i) \left[\sum_{j=0}^{i} \bar{F}(j) + \sum_{j=i+1}^{i+k} \bar{F}(j) \right].$$

Or equivalently

$$\sum_{j=i}^{i+k} \left[\frac{1-F(j)}{1-F(i)} - 1 - F(j) \right] \leq \sum_{j=0}^{k} [1-F(j)].$$

Since F is a distribution, we have the following

$$\sum_{j=0}^{k} [1-F(j)] \leq \sum_{j=0}^{k} [1-F^{n}(j)]. \qquad (3.4)$$

Hence

$$\sum_{j=i}^{i+k} \left[\frac{(1-F(j))F(i)}{1-F(i)} \right] \geq \sum_{j=i}^{i+k} \left[\frac{(1-F^{n}(j))F^{n}(i)}{1-F^{n}(i)} \right]. \qquad (3.5)$$

But

$$\sum_{j=i}^{i+k} \left[\frac{(1-F(j))F(i)}{1-F(i)} - \frac{(1-F^{n}(j))F^{n}(i)}{1-F^{n}(i)} \right] \geq 0$$

This implies that

$$\sum_{j=i}^{i+k} \frac{(1-F(j))F(i)}{1-F(i)} \left\{ 1 - F^{n-1}(j) \left[\frac{1-F^{n}(j)}{1-F^{n}(j)} \times \frac{1-F(i)}{1-F^{n}(i)} \right] \right\}$$

$$= \sum_{j=i}^{i+k} \frac{(1-F(j))F(i)}{1-F(i)} \left\{ 1 - F^{n-1}(j) \left[\frac{1+F(j)+\cdots+F^{n-1}(j)}{1+F(i)+\cdots+F^{n-1}(i)} \right] \right\}$$

$$\geq \sum_{j=i}^{i+k} \frac{(1-F(j))F(i)}{1-F(i)} \left\{ 1 - F^{n-1}(j) \left[\frac{1+F^{-1}(j)+\cdots+F^{-(n-1)}(j)}{1+F(i)+\cdots+F^{n-1}(i)} \right] \right\} \geq 0.$$

Since

$$F(j) \leq F^{-1}(j) \leq F^{-1}(i) \quad for\ all\ j \leq i$$

From (3.4) and (3.5) we get

$$\sum_{j=i}^{i+k} \frac{(1-F^n(j))F^n(i)}{1-F^n(i)} \le \sum_{j=0}^{k} [1-F^n(j)],$$

equivalently

$$\sum_{j=i}^{i+k} \frac{F_n(i)\bar{F}_n(j)}{\bar{F}_n(i)} \le \sum_{j=0}^{k} \bar{F}_n(j),$$

the above inequality may be written as follows

$$\sum_{j=i}^{k} \frac{\bar{F}_n(j)}{\bar{F}_n(i)} \le \sum_{j=0}^{i} \bar{F}_n(j) + \sum_{j=i}^{k} \bar{F}_n(j).$$

Hence

$$\sum_{j=0}^{k} \frac{\bar{F}_n(i+j)}{\bar{F}_n(i)} \le \sum_{j=0}^{k} \bar{F}_n(j),$$

or

$$\sum_{j=0}^{k} \bar{F}_n(i+j) \le \bar{F}_n(i) \sum_{j=0}^{k} \bar{F}_n(j).$$

show that F_n is also $D-NBU(1)$

3.3 Mixtures

In this subsection we discuss preservation the $D-NWU(1)$,*D-NWUCA* and D-*GHNWUE* under mixing.

Theorem 3.4

The $D-NWU(1)$ Class is preserved under mixing.

Proof

Let g be the mixing p.f. Applying Chebyshev's inequality we obtain

$$\bar{F}(i)\sum_{j=0}^{k} \bar{F}(j) = \sum_{l=0}^{\infty} \bar{F}_l(i)g(l)\left[\sum_{j=0}^{k}\sum_{l=0}^{\infty} \bar{F}_l(j)g(l)\right].$$

$$\leq \sum_{l=0}^{\infty}\left[\bar{F}_l(i)\sum_{j=0}^{k}\bar{F}_l(j)\right]g(l)$$

$$\leq \sum_{l=0}^{\infty}\sum_{j=0}^{k}\bar{F}_l(j)\,\bar{F}_l(i)g(l)$$

$$\leq \sum_{l=0}^{\infty}\sum_{j=0}^{k}\bar{F}_l(i+j)\,g(l)$$

$$= \sum_{j=0}^{k}\sum_{l=0}^{\infty}\bar{F}_l(i+j)\,g(l)$$

$$= \sum_{j=0}^{k}\bar{F}(i+j).$$

As required

Theorem 3.5

The D-NWUCA *Class is preserved under mixing.*

Proof: let g be the mixing p.f. Applying Chebyshev's inequality we obtain

$$\bar{F}(j)\sum_{k=0}^{\infty}\sum_{i=0}^{\infty}\bar{F}(i) = \left[\sum_{l=0}^{\infty}\bar{F}_l(j)g(l)\right]\left[\sum_{k=0}^{\infty}\sum_{i=0}^{\infty}\sum_{l=0}^{\infty}\bar{F}_l(i)g(l)\right]$$

$$= \left[\sum_{l=0}^{\infty}\bar{F}_l(j)g(l)\right]\left\{\sum_{l=0}^{\infty}\left[\sum_{k=0}^{\infty}\sum_{i=0}^{\infty}\bar{F}_l(i)\right]g(l)\right\}$$

$$\leq \sum_{m=0}^{\infty}\left[\bar{F}_m(j)\sum_{k=0}^{\infty}\sum_{i=0}^{\infty}\bar{F}_m(i)\right]g(m)$$

$$\leq \sum_{m=0}^{\infty}\sum_{k=0}^{\infty}\sum_{i=k}^{\infty}\bar{F}_m(i)\,g(m)$$

$$= \sum_{k=0}^{\infty}\sum_{i=k}^{\infty}\sum_{m=0}^{\infty}\bar{F}_m(i)\,g(m)$$

$$= \sum_{k=0}^{\infty}\sum_{i=k}^{\infty}\bar{F}(i).$$

As required.

Theorem 3.6

The D-GHNWUE class is preserved under mixing.

Proof

Let $F_i(j), i \in N$, be D-GHNWUE and $F(j) = \sum_{i=0}^{\infty} F_i(j) g(i)$where $g(i), i \in N$ is a probability function, then $\mu_j = \sum_{i=0}^{\infty} \bar{F}_j(i) > 1, i \in N$ and $= \sum_{j=0}^{\infty} \bar{F}(j) > 1$ .Utilizing Taylor's expansion for two-dimensional functions one may show that$\varphi(s,t) = S^{1-j}(t-1)^j, s > 0, t > 1, j \in N$, is a convex real function. Then the two-dimensional Jensen's inequality (see Billingsley, 1995, p. 82) leads to

$\varphi(E(S), E(T)) \leq E\varphi(S,T)$ (3.6)

where$S$ and T are identically distributed random variables with joint distribution defined by

$$F_r(S = \mu_i, T = \mu_l) = \begin{cases} g(i) & i = l \\ 0 & otherwise \end{cases}$$

and with marginal distributions defined by$F_r\{S = \mu_i\} = g(i), i \in N$Therefore, (7) reduces to

$$\{E(S)\}^{1-j}\{E(T) - 1\}^j = \{E(S)\}^{1-j}\{E(T-1)\}^j \leq E\{S^{1-j}(T-1)\}$$

Which is equivalent to

$$\left\{\sum_{i=0}^{\infty} \mu_i \; g(i)\right\}^{1-j} \left\{\sum_{l=0}^{\infty} (\mu_l - 1) \; g(l)\right\}^j \leq \sum_{i=0}^{\infty} \mu_i^{1-j}(\mu_i - 1)^j \; g(i).$$

Or

$$\left\{\sum_{i=0}^{\infty} \mu_i \; g(i)\right\}\left\{1 - \frac{1}{\sum_{l=0}^{\infty} \mu_l \; g(l)}\right\} \leq \sum_{i=0}^{\infty} \mu_i^{1-j}\left(1 - \frac{1}{\mu_i}\right)^j g(i).$$

The desired result is now established since

$$\sum_{k=0}^{\infty}\sum_{j=k}^{\infty} \bar{F}(j) = \sum_{i=0}^{\infty}\left\{\sum_{k=0}^{\infty}\sum_{j=k}^{\infty} \bar{F}_i(j)\right\} g(i) \geq \sum_{k=0}^{\infty}\sum_{j=k}^{\infty} \mu_i \left(1 - \frac{1}{\mu_i}\right)^k g(i)$$

$$\geq \sum_{i=0}^{\infty} \mu_i^2 \; g(i) = \mu^2$$

As required.

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