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# Test for New Better than Used in Average Based on the Total Time on Test Transform

By
M. I. Hendi and A. S. Al-Ruzaiza
Department of Statistics, College of Science,
P. O. Box 2455, King Saud University,
Riyadh 11451, King Saud University

**Abstract**

*Let $F$ be a life distribution with survival function $\bar{F} = 1 - F$ and finite mean $\mu = \int_0^\infty \bar{F}(x)dx$. The scaled total time on test transform is defined by $\phi_F(t) = (1/\mu)\int^{F^{-1}(t)} \bar{F}(x)dx$. In this paper, the properties of $\phi_F(t)$ for new better than used in average (NBUA) or new worse than used in average (NWUA) are investigated. Test statistics for testing exponentiality against NBUA (NWUA) are proposed. Selected critical values are tabulated for sample size n =(1)20(5)60. Powers of the tests are estimated by simulation. An example of 40 patients of blood cancer disease demonstrate its practical application in medical Sciences.*

**Key Words:** NBUA, NWUA, Total time on test, Exponential distribution, Life distributions, Survival functions, Order statistics, TTT-test.

# 1 Introduction and Definitions

Statisticians and reliability analysts have shown interest in modeling survival data using classification of life distribution based on some aspects of ageing. Various classes of life distributions and their dual have been introduced in reliability to describe several types of deterioration (improvement).

These classes have been considered by different authors and mainly based on two concepts:

(i) The failure rate classes, e.g., IFR, IFRA, NBUFR, NBAFR, ..., etc.

(ii) The conditional survival classes, e.g., NBU, DMRL, NBUA, NBUE, ..., etc.

The implication among these classes of life distributions are

$$IFR \begin{cases} \Longrightarrow NBU & \Longrightarrow NBUA \quad \Longrightarrow NBUE \quad \Longrightarrow HNBUE \\ \Longrightarrow IFRA & \Longrightarrow NBU \quad \Longrightarrow NBUFR \quad \Longrightarrow NBAFR \end{cases}$$

Deshpande, et al (1986) introduce another set of classes in terms of stochastic dominance. One of their interesting classes are the new better than used of second order NBU(2), Abouammoh & Ahmed (1989) studied NBUAS (NBU(2)) for some reliability properties, Hendi & Rady (1994) studied NBUA (NBU(2)) for some extension reliability properties. In fact NBUA (NBU(2)) class is middle class between the NBU and NBUE classes, this prove useful in applications since it is less restrictive than NBU and easier to distinguish in practice than NBUE, and more study for this class on testing.

Moreover, the problem of testing exponentiality against various classes (IFR, IFRA, DMRL, NBUE ... etc.) of life distribution has seen a good deal of attention in literature. For examples Proschan and Pyke (1967), Ahmad (1975, 1994, 1995), Klefsjo (1983), Abouammoh et al (1987, 1989, 1993, 1994 and 1996), Hollander & Proschan (1972, 1975), Kanjo (1993) and Hendi et al (1996) among others.

**Definition 1.1.** A life distribution $F$ [i.e. $F(0-) = 0$] is called new better than used average (NBUA) or new worse than used average (NWUA), with finite mean $\mu = \int_0^\infty \bar{F}(u)du$ if

$$\int_0^x \bar{F}(u+t)du \leq (\geq)\bar{F}(t)\int_0^x \bar{F}(u)du; \qquad u \geq 0,\ x > 0,\ t \geq 0.$$

It is obvious that NBUA (NWUA) simultaneously if and only if (iff) $F$ is exponential. Note that NBUA is equivalents to NBUAS of Abouammoh & Ahmed (1989) and NBU(2) of Deshpande (1986).

The main theme of this paper is the problem of testing $H_0 : \bar{F} = \exp(-\lambda t), \lambda > 0, t > 0$ versus $H_1 : F =$ NBUA (NWUA). In section 2, we give a brief account of the TTT-transform and TTT-plot and the characterizations for NBUA (NWUA) are given. Test statistics for testing $H_0$ against $H_1$ are introduced by using the characterizations of NBUA (NWUA) in section 3. The power estimate of this test is investigated in section 4. An application in Medical Science is given in section 5.

# 2 The concept of Total time on Test (TTT-test)

Let $F$ be the life distribution with survival function $\bar{F} = 1 - F$ and finite mean $\mu = \int_0^\infty \bar{F}(u)du$. We present the following definition of Barlow and Campo (1975).

**Definition 2.1.**

(i) The function

$$H^{-1}(t) = \int_0^{F^{-1}(t)} \bar{F}(u)du \qquad (2.1)$$

is called the TTT-transform. It is easy to verify that the mean $\mu$, of $F$ is given by

$$H^{-1}(1) = \int_0^{F^{-1}(1)} \bar{F}(u)du \qquad (2.2)$$

(ii) The function

$$\phi_F(t) = H^{-1}(t)/H^{-1}(1) \qquad (2.3)$$

is called the scaled TTT-transform.

Here we consider $F^{-1}(t)$ to be $\inf[x : F(x) \geq t]$. Note that if $F$ is the exponential distribution then the scaled TTT-transform is given by

$$\phi_F(t) = t, 0 \leq t \leq 1.$$

Now let $t_{(1)} < t_{(2)} < \ldots t_{(n)}$ to be an ordered sample from a life distribution $F$, where $t_{(0)} = 0$ and let

$$D_j = (n - j + 1)(t_{(j)} - t_{(j-1)}) \text{ for } j = 1, 2, \ldots, n \tag{2.4}$$

then

$$S_j = \sum_{k=1}^{j} D_k \text{ for } j = 1, 2, \ldots, n \tag{2.5}$$

denote TTT-transform at $t_{(j)}$, where $S_0 = 0$. The value $S_j/S_n$ is an estimate of the scaled TTT-transform. The TTT-plot is obtained by plotting

$$W_j = S_j/S_n \tag{2.6}$$

against $j/n$ for $j = 0, 1, \ldots, n$ and joining the points by straight lines.

It has been shown, Barlow et al (1972) (p. 237) that using Glivenko-Cantelli Lemma, that for strictly increasing $F$, $W_j$ converges to $\phi_F(t)$ with probability one and uniformly in [0,1] as $n$ tends to $\infty$ and $j/n$ converges to $t$.

Scaled TTT-transforms for some families of life distributions are given by Barlow and Compo (1975), Barlow (1979), Bergman (1979) and Klefsjo (1982, 1983). Here we present the following theorem where proof is found in these papers.

**Theorem 2.1.** Let $F$ be a life distribution and $\phi_F(t)$, for $0 \leq t \leq 1$, be the corresponding TTT-transform as defined in (2.3). Then we have

(i) $F$ is IFR (DFR) if and only if (iff) $\phi_F(t)$ is concave (convex) for $0 \leq t \leq 1$.

(ii) $F$ is IFRA (DFRA) iff $\phi_F(t)/t$ is decreasing (increasing) for $0 < t < 1$.

(iii) $F$ is HNBUE (HNWUE) iff $t^{-1}\log(1 - \phi_F(t)) \leq -\mu^{-1}, 0 < t \leq 1$.

(iv) $F$ is DMRL (IMRL) iff $[1-\phi_F(t))]/(1-t)$ is decreasing (increasing) for $0 \le t < 1$.

**Theorem 2.2.** Let $F$ and $\phi_F(t)$ (or simply $\phi(t)$) be as in Theorem 2.1) then we have the following:

$F$ is NBUA (NWUA) iff

$$\mu^{-1}\int_0^{F^{-1}(v)+F^{-1}(w)} \bar{F}(u)du - (1-w)\phi_F(v) - \phi_F(w) \le (\ge) 0 \tag{2.7}$$

**Proof.** The life distribution $F$ has NBUA (NWUA) property, if

$$\int_0^x \bar{F}(t+u)du \le (\ge) \bar{F}(t)\int_0^x \bar{F}(u)du \tag{2.8}$$

where $u > 0, t > 0$ and $x > 0$.

Substituting by $t+u=w$ in L. H. S. of (2.8) we get

$$\int_t^{t+x} \bar{F}(w)dw \le (\ge) \bar{F}(t)\int_0^x \bar{F}(u)du$$

Therefore

$$\int_0^{t+x} \bar{F}(u)du - \int_0^t \bar{F}(u)du \le (\ge)[1-F(t)]\int_0^x \bar{F}(u)du$$

Using the transformation $x = F^{-1}(v), t = F^{-1}(w)$ we get

$$\int_0^{F^{-1}(v)+F^{-1}(w)} \bar{F}(u)du \le (\ge)(1-w)\int_0^{F^{-1}(v)} \bar{F}(u)du + \int_0^{F^{-1}(w)} \bar{F}(u)du$$

Therefore

$$\frac{1}{\mu}\int_0^{F^{-1}(v)+F^{-1}(w)} \bar{F}(u)du \le (\ge)(1-w)\frac{1}{\mu}\int_0^{F^{-1}(v)} \bar{F}(u)du + \frac{1}{\mu}\int_0^{F^{-1}(w)} \bar{F}(u)du$$

Therefore

$$\mu^{-1}\int_0^{F^{-1}(v)+F^{-1}(w)} \bar{F}(u)du \le (\ge)(1-w)\phi_F(v) + \phi_F(w) \tag{2.9}$$

where $\phi_F(v) = \frac{1}{\mu}\int_0^{F^{-1}(v)} \bar{F}(u)du$, $\phi_F(w) = \frac{1}{\mu}\int_0^{F^{-1}(w)} \bar{F}(u)du$ and $\mu = \int_0^\infty \bar{F}(u)du$ from (2.9) we get (2.7) and then the proof is complete. □

# 3 Test Statistics Based on the Scaled TTT-Transform

In this section a test statistic using the scaled TTT-transform for testing exponentiality or $H_0$ against $H_1$ or NBUA (NWUA) class (not exponential) based on a sample $T_1, \ldots, T_n$ from $F$. Since $F$ is NBUA (NWUA). We use the parameter $\Delta_F$ as a measure of departure from $H_0$ from (2.7), we write $\Delta_F$ as follows:

$$\Delta_F = \mu^{-1} \int_0^{F^{-1}(v)+F^{-1}(w)} \bar{F}(u)du - (1-w)\phi_F(v) - \phi_F(w) \leq (\geq) 0 \quad (3.1)$$

Note that under $H_0 : \Delta_F = 0$, and under $H_1 : \Delta_F \lessgtr 0$.

**Remark 3.1.** If $F(t) = 1 - \exp(-\lambda t)$, for $t \geq 0, \lambda > 0$, then the scaled TTT-transform is given by

$$\phi_F(t) = t \quad (3.2)$$

**Remark 3.2.**

An estimate of the scaled TTT-transform is known as empirical scaled TTT-transform and is obtained as

$$\phi_F\left(\frac{j-1}{n}\right) = W_{j-1}, \text{ for } j = 1, 2, \ldots, n \quad (3.3)$$

The TTT-plot is obtained by plotting $W_{j-1}$ against $\frac{(j-1)}{n}$, for $j = 1, 2, \ldots, n$, i.e. $(W_{j-1}, \frac{j-1}{n})$, for $j = 1, 2, \ldots, n$ and connecting the plotted points by straight lines.

Since the TTT-plot $W_{j-1}$ in (2.5) converges to the scaled TTT-transform $\phi_F(t)$ in (2.3) as $n \to \infty$ and $\frac{j-1}{n} \to t$. Thus, the TTT-plot based on an ordered sample

$$0 = t_{(0)} \leq t_{(1)} \leq \ldots \leq t_{(n)},$$

behave as $\phi_F(t)$ does. This suggests the following test statistic based on the scaled TTT-transform. We estimate $\Delta_F$ in (3.1) as follows. The left-hand side of (3.1) is estimated at a specified time $t$, by

$$\delta(i,j) = (\bar{t})^{-1}\sum_{k=1}^{l}\frac{(n-k+1)}{n}(t_{(k)} - t_{(k-1)}) - (1 - \frac{j-1}{n})W_{i-1} - W_{j-1}$$

where $i = 1,2,\ldots,n$, $j = 1,2,\ldots,n$, $t_{(1)}, t_{(2)},\ldots,t_{(n)}$ are the ordered statistics of the independent random sample $X_1, X_2, \ldots, X_n, t(0) = 0$ and $\bar{t}$ is the sample mean of $t_{(1)}, t_{(2)}, \ldots, t_{(n)}$ or $X_1, X_2, \ldots, X_n$. Note that

$$l = \begin{cases} \#t's \leq t_{(i)} + t_{(j)} & \text{if } t_{(i)} + t_{(j)} < t_{(n)} \\ n & \text{if } t_{(i)} + t_{(j)} \geq t_{(n)} \end{cases}$$

Taking the summation of $\delta(i,j)$ over $i$ and $j$ gives the corresponding test statistics $\hat{\Delta}_n$ as

$$\hat{\Delta}_n = \sum_{i=1}^{n}\sum_{j=1}^{n}\left[(\bar{t}^{-1})\sum_{k=1}^{l}\frac{(n-k+1)}{n}(t_{(k)} - t_{(k-1)}) - \left(\frac{n-j+1}{n}\right)W_{i-1} - W_{j-1}\right].$$

To reduce the size of the test statistics we consider the version

$$\hat{\Delta}_n^* = \hat{\Delta}_n / n \qquad (3.4)$$

Note that

$H_0 : \Delta_F = 0$ if $F$ is exponential

$H_1 : \Delta_F < (>) 0$ if $F$ is NBUA (NWUA)

## 3.1 Simulation of small samples

It is difficult to find the exact distribution for $\hat{\Delta}_n^*$ statistics and hence simulation percentiles for small samples are commonly used by applied statisticians and reliability analysts. We have simulated the lower and upper percentile points for $\alpha = 0.01, 0.05$ and $0.10$. Table (3.1) gives these percentile points of the statistic $\hat{\Delta}_n^*$ and the calculations are based on 1,000 simulated samples of sizes $n = 5(1)20(5)50$.

**Table (3.1)** Critical Values for the $\hat{\Delta}_n^*$-statistic

| sample size $n$ | Significance level $\alpha$ | | | | | |
|---|---|---|---|---|---|---|
| | Lower Percentile | | | Upper Percentile | | |
| | 0.01 | 0.05 | 0.10 | 0.10 | 0.05 | 0.01 |
| 5 | -0.4116 | -0.0456 | 0.1550 | 1.2412 | 1.3553 | 1.5258 |
| 6 | -0.3827 | -0.0033 | 0.1537 | 1.2064 | 1.3398 | 1.5290 |
| 7 | -0.4142 | -0.0511 | 0.1474 | 1.1978 | 1.2946 | 1.5077 |
| 8 | -0.4980 | -0.1064 | 0.0174 | 1.1521 | 1.2758 | 1.4797 |
| 9 | -0.6217 | -0.2549 | -0.0623 | 1.1296 | 1.2538 | 1.4844 |
| 10 | -0.6481 | -0.2881 | -0.0647 | 1.1165 | 1.2337 | 1.4214 |
| 11 | -0.7633 | -0.3395 | -0.1017 | 1.1291 | 1.2571 | 1.4189 |
| 12 | -0.8583 | -0.3919 | -0.1491 | 1.0552 | 1.1873 | 1.4008 |
| 13 | -0.8978 | -0.3755 | -0.1409 | 1.0478 | 1.1777 | 1.4096 |
| 14 | -0.9329 | -0.5038 | -0.2156 | 1.0615 | 1.1960 | 1.4409 |
| 15 | -1.0342 | -0.4967 | -0.2648 | 1.0555 | 1.1684 | 1.3790 |
| 16 | -0.9050 | -0.3876 | -0.1592 | 1.0642 | 1.2134 | 1.3699 |
| 17 | -1.0536 | -0.4898 | -0.2267 | 1.0239 | 1.1485 | 1.4039 |
| 18 | -1.0456 | -0.5330 | -0.2873 | 1.0563 | 1.1999 | 1.4902 |
| 19 | -1.1631 | -0.6025 | -0.3607 | 1.0544 | 1.2333 | 1.4735 |
| 20 | -1.0811 | -0.6508 | -0.3176 | 1.0661 | 1.2205 | 1.4765 |
| 25 | -1.1042 | -0.6356 | -0.4501 | 1.0313 | 1.2142 | 1.4648 |
| 30 | -1.3094 | -0.7974 | -0.4792 | 1.0767 | 1.2303 | 1.5664 |
| 35 | -1.4600 | -0.8344 | -0.5938 | 1.1079 | 1.2957 | 1.6102 |
| 40 | -1.6761 | -1.0555 | -0.7264 | 1.0921 | 1.3255 | 1.5796 |
| 45 | -1.7295 | -1.1479 | -0.7700 | 1.1335 | 1.3088 | 1.6520 |
| 45 | -1.9654 | -1.1205 | -0.8143 | 1.1602 | 1.3665 | 1.7160 |

# 4 The Power Estimates

The power of the test statistic $\hat{\Delta}_n^*$ is considered for the significance level $\alpha = 0.05$ and for commonly used distributions in reliability modeling. These distributions are

| | | | |
|---|---|---|---|
| (i) | Linear failure-rate | : $\bar{F}_1(t) = \exp(-t - \frac{1}{2}\theta t^2)$ | $\theta \geq 0, t \geq 0$ |
| (ii) | Makeham | : $\bar{F}_2(t) = \exp[-\{t + \theta(t + e^{-t} - 1)\}]$ | $\theta > 0, t \geq 0$ |
| (iii) | Weibull | : $\bar{F}_3(t) = \exp(-t^\theta)$ | $\theta \geq 0, t \geq 0$ |
| (iv) | Gamma | : $\bar{F}_4(t) = \frac{1}{\Gamma(\theta)} \int_t^\infty x^{\theta-1} \exp(-x)$ | $\theta > 0, t \geq 0$ |

All these distributions IFR (for an appropriate restriction on $\theta$), hence they all belong to a wider class. Moreover, all these reduce to the exponential distribution for an appropriate value of $\theta$. Table 4.1 contains the power estimates for the $\hat{\Delta}_n^*$ test statistic with respect to these distributions. The estimates are based on 1000 simulated samples of sizes $n =$ 10, 20, 30 and significance level $\alpha = 0.05$.

The power estimates in Table (4.1) shows clearly the departure from exponentiality towards (NBUA) properties as $\theta$ increases. In fact Table (4.1) shows how reliable our proposed test can be based life distribution with tractable ageing criteria.

Table (4.1) Power estimate for $\hat{\Delta}^*_n$-statistics

| Distribution | Parameter | Sample size | | |
|---|---|---|---|---|
| | $\theta$ | n=10 | n=20 | n=30 |
| $F_1$ | 2 | 0.199 | 0.327 | 0.510 |
| (Linear failure) | 3 | 0.216 | 0.428 | 0.607 |
| rate | 4 | 0.267 | 0.459 | 0.679 |
| $F_2$ | 2 | 0.081 | 0.098 | 0.135 |
| (Makeham) | 3 | 0.086 | 0.112 | 0.172 |
| | 4 | 0.102 | 0.144 | 0.176 |
| $F_3$ | 2 | 0.666 | 0.960 | 0.999 |
| (Weibull) | 3 | 0.984 | 1.000 | 1.000 |
| | 4 | 0.999 | 1.000 | 1.000 |
| $F_4$ | 2 | 0.324 | 0.603 | 0.828 |
| (Gamma) | 3 | 0.618 | 0.941 | 0.998 |
| | 4 | 0.831 | 0.994 | 1.000 |

# 5 Applications

In this section we calculate the $\hat{\Delta}^*_n$ test statistic for the data represent 40 patients suffering from blood cancer from one of ministry of Health Hospitals in Saudi Arabia see Abouammoh et al (1994). The ordered life times (in days) are

115, 181, 255, 418, 441, 461, 516, 739, 743, 789, 807, 865, 924, 983, 1024, 1062, 1063, 1165, 1191, 1222, 1222, 1251, 1277, 1290, 1357, 1369, 1408, 1455, 1478, 1549, 1578, 1578, 1599, 1603, 1605, 1696, 1735, 1799, 1815, 1852.

It was found that the test statistics for the set of data, by using equation (3.4) is $\hat{\Delta}^*_n = -5.939$ .

It is clear from the computed value of the test statistics that we accept

$H_1$ which states that the set of data have NBUA property under significant level $\alpha = 0.01$.

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M. I. Hendi and A. S. Al-Ruzaiza
Department of Statistics, College of Science,
P. O. Box 2455, King Saud University,
Riyadh 11451, King Saud University.