THE EGYPTIAN STATISTICAL JOURNAL
ISSR, Cairo Univ., Vol.37, No.1, 1993

ON THE UNIFORM RATES OF CONVERGENCE IN THE CENTRAL LIMIT THEOREM FOR FUNCTIONS OF THE AVERAGE OF I.I.D.RANDOM VARIABLES

BY

PRASHANT KIRKIRE (BARODA)

Abstract : Let $\{ X_k, k \geq 1 \}$ be a sequence of i.i.d.r.v.s with common distribution function (d.f.) F. Suppose F belongs to the domain of normal attraction of a stable law G_α with index α, $1 < \alpha \leq 2$ and F satisfies some regularity conditions. Let $S_n = X_1 + \ldots + X_n$ and g be a real differentiable function such that $| g'(x) - g'(y) | \leq L | x - y |$, $L > 0$. We give uniform rate of convergance in the Central Limit Theorem(CLT) for the sequence :

$$\frac{n^{1-r}}{g'(0)} \{ g(\frac{S_n}{n}) - g(0) \}, \; n \geq 1, \; g'(0) \neq 0.$$

1. Introduction and Notation. Let $\{ X_k, k \geq 1 \}$ be a sequence of i.i.d.r.v.s. with a common d.f. F. Let $S_n = X_1 + X_2 + \ldots + X_n$. The asymptotic normality of the sequence $\{g(S_n/n), n \geq 1\}$, where g is real function such that g',the derivative of g,satisfies Lipschitz condition,has been considered by several authors. (See:[1] – [5]). Throughout these references, the d.f. F is assumed to have finite mean and variance i.e. d.f. F is assumed to be in the domain of normal attraction of normal law. The CLT type result and its rate of convergence for the sequence $g(S_n/n)$ for i.i.d.r.v.s. with d.f. F belonging to the domain of normal attraction of a stable law G_α, with index α, $1 < \alpha \leq 2$ has not been studied yet. It is therefore interesting to establish uniform rate of convergence in this set up. We wish to show that the techniques developed by Szynal D., Bartmanska B., and Morris

---

This research is supported by Senior Research Fellowships of University Grants Commission of India 1991.

K., ([1]-[5]), are also applicable in this set up.

Throughout this note we shall make use of the following notations.

We denote the d.f. of S_n/n^r, $r = 1/\alpha$, by $F_n(x)$. $G_\alpha(x)$ denotes the d.f. of a stable r.v. with index α, $1 < \alpha \le 2$.

We define two types of pseudomoments viz.:

$$\mu(k) = \int x^k \, d\{F(x) - G_\alpha(x)\}, \quad k = 0,1,2,\ldots$$

$$\pi(\delta) = \delta \int |x|^{\delta-1} \, | F(x) - G_\alpha(x) | \, dx, \quad \delta \ge 0$$

Write $s = 1 + [\alpha]$, where $[\alpha]$ is the greatest integer not exceeding α, the principal parameter of the stable law G_α.

Let D^* denote the class of all real, differentiable functions g such that g' satisfies the Lipschitz condition i.e.

(1) $$| g'(x) - g'(y) | \le L \, | x - y |$$

where L is a positive constant.

In what follows, $C, C_1, C_2, \ldots$ etc. stand for generic constants and they may change from one step to another.

2. Preliminary Results.

We shall use the following results.

LEMMA 2.1 ([6], p.16). Let X and Y be random variables, $F(x) = P[X \le x]$, $G(x) = P[X+Y \le x]$. Then, for any $\epsilon > 0$, $x \in R$ and any arbitrary function H,

(2) $$| G(x) - H(x) | \le \sup_x | F(x) - H(x) | + \max\{ |H(x-\epsilon) - H(x)|, | H(x+\epsilon) - H(x) | \} + P(|y| \ge \epsilon).$$

LEMMA 2.2 ([7], LEMMA 3). Let $\{ X_k, k \ge 1 \}$ be a sequence of i.i.d.r.v.s with common d.f. F. Let F belong to the domain of normal attraction of a stable law G_α, $1 < \alpha \le 2$. If $\mu(k) = 0$ for $k \le s-1$ and $\pi(\alpha) < \infty$. Then there exists a constant C_1 such that for all $n \ge 1$,

(3) $\sup_x \mid F_n(x) - G_\alpha(x) \mid \le C_1 n^{1-sr}$, $r = 1/\alpha$.

LEMMA 2.3 ([8], COROLLARY(2)). Suppose that $G_\alpha(x)$ is a symmetric stable d.f. with parameters C and α, i.e., a d.f. with a corresponding characteristic function $\exp\{-C|t|^\alpha\}$. Then for any positive x,

(4) $$1 - G_\alpha(x) \le \frac{C\pi^\alpha}{2^{\alpha+1} x^\alpha} .$$

3. Uniform estimates

Now we discuss the main results.

THEOREM 1. Let $\{ X_k, k \ge 1 \}$ be a sequence of i.i.d.r.v.s. with common d.f. F. Let F belong to the domain of normal attraction of a stable law G_α, $1 < \alpha \le 2$. If $\mu(k) = 0$ for $k \le 0,\dots,[\alpha]$ and $\pi(\alpha) < \infty$. Then , for every $g \in D^*$ with $g'(0) \neq 0$, for $n \ge 1$,

(5) $$\sup_{-\infty<x<\infty} \left| P\left[\frac{n^{1-r}}{g'(0)} \{g\left(\frac{S_n}{n}\right) - g(0)\} \le x \right] - G_\alpha(x) \right|$$

$$\le 5C_1 n^{1-sr} + (n^{1-r})^{(-\alpha/(2+\alpha))} \frac{|g'(0)|}{L}$$

$$+ 4\,(1-G_\alpha((n^{1-r})^{(2+\alpha)})$$

PROOF.

Put $$h(x) = \begin{cases} \dfrac{g(x) - g(0)}{x\, g'(0)} & \text{if } x \neq 0 \\ 1 & \text{if } x = 0 \end{cases}$$

Observe that

$$\frac{n^{1-r}}{g'(0)} \{ g(\frac{S_n}{n}) - g(0) \} = \frac{S_n}{n^r} \{ h(\frac{S_n}{n}) - 1 \} + \frac{S_n}{n^r}$$

Hence by Lemma 2.1 and 2.2, for any given $\epsilon_n > 0$, we have

$$(6)\qquad \left| P\left[\frac{n^{1-r}}{g'(0)} \{ g\left(\frac{S_n}{n}\right) - g(0) \} \le x \right] - G_\alpha(x) \right|$$

$$\le C_1 n^{1-sr} + \epsilon_n + P\left[\left| \frac{S_n}{n^r} \{ h\left(\frac{S_n}{n}\right) - 1 \} \right| \ge \epsilon_n \right]$$

Consider now ,for $\theta_n \to \infty$, θ_n to be chosen later appropriately,

$$(7)\ P\left[\left| \frac{S_n}{n^r} \{ h\left(\frac{S_n}{n}\right) - 1 \} \right| \ge \epsilon_n \right]$$

$$= P\left[\left| \frac{S_n}{n^r} \{ h\left(\frac{S_n}{n}\right) - 1 \} \right| \ge \epsilon_n ,\ \left| \frac{S_n}{n^r} \right| \ge \theta_n \right]$$

$$+ P\left[\left| \frac{S_n}{n^r} \{ h\left(\frac{S_n}{n}\right) - 1 \} \right| \ge \epsilon_n ,\ \left| \frac{S_n}{n^r} \right| < \theta_n \right]$$

$$\le P\left[\left| \frac{S_n}{n^r} \right| \ge \theta_n \right] + P\left[\left| h\left(\frac{S_n}{n}\right) - 1 \right| \ge \frac{\epsilon_n}{\theta_n} \right]$$

$$\le 2 \sup_x | F_n(x) - G_\alpha(x) | + 2(1 - G_\alpha(\theta_n))$$

$$+ P\left[\left| h\left(\frac{S_n}{n}\right) - 1 \right| \ge \frac{\epsilon_n}{\theta_n} \right]$$

$$\le 2C_1 n^{1-sr} + 2(1 - G_\alpha(\theta_n) + P\left[\left| h\left(\frac{S_n}{n}\right) - 1 \right| \ge \frac{\epsilon_n}{\theta_n} \right]$$

using Lemma 2.2.

Taking into account the definition of h, and (1), we get, for some $\theta, 0 < \theta < 1$,

$$(8)\qquad P\left[\left| h\left(\frac{S_n}{n}\right) - 1 \right| \ge \frac{\epsilon_n}{\theta_n} \right]$$

$$= P\left[\left| \frac{g\left(\frac{S_n}{n}\right) - g(0)}{\frac{S_n}{n} g'(0)} - 1 \right| \ge \frac{\epsilon_n}{\theta_n} \right]$$

$$= P\left[\left| \frac{g'(\theta \frac{S_n}{n}) - g'(0)}{g'(0)} \right| \geq \frac{\epsilon_n}{\theta_n} \right]$$

$$\leq P\left[\left| \frac{S_n}{n^r} \right| \geq \frac{\epsilon_n}{\theta_n} \frac{n^{1-r} |g'(0)|}{L} \right]$$

$$\leq 2 \sup_x |F_n(x) - G_\alpha(x)| + 2 \left(1 - G_\alpha \left(\frac{\epsilon_n}{\theta_n} \frac{n^{1-r} |g'(0)|}{L} \right)\right)$$

$$\leq 2\, C_1\, n^{1-sr} + 2\left(1 - G_\alpha \left(\frac{\epsilon_n}{\theta_n} \frac{n^{1-r} |g'(0)|}{L} \right)\right)$$

Putting in (8), $\epsilon_n = \frac{(\theta_n)^2}{n^{1-r}} \frac{L}{|g'(0)|}$, we get

$$\text{(9)} \qquad P\left[\left| h\left(\frac{S_n}{n}\right) - 1 \right| \geq \frac{\theta_n}{n^{1-r}} \frac{L}{|g'(0)|} \right]$$

$$\leq 2\, C_1\, n^{1-sr} + 2\,(1 - G_\alpha(\theta_n))$$

Combing (6) - (9) we get,

$$\sup \left| P\left[\frac{n^{1-r}}{g'(0)} \left\{ g\left(\frac{S_n}{n}\right) - g(0) \right\} \leq x \right] - G_\alpha(x) \right|$$

$$\leq 5\, C_1 n^{1-sr} + \frac{(\theta_n)^2}{n^{1-r}} \frac{L}{|g'(0)|} + 4(1-G_\alpha(\theta_n))$$

Notice that G_α is a stable d.f.and, therefore ,if we select $\theta_n = (n^{1-r})^{(2+\alpha)}$ then the required result follows.

THEOREM 2. Let $\{ X_k, k \geq 1 \}$ be a sequence of i.i.d.r.v.s. with common d.f. F. Let F belong to the domain of normal attraction of a symmetric stable law G_α, $1 < \alpha \leq 2$. If $\mu(k) = 0$ for $k \leq 0,..,[\alpha]$ and $\pi(\alpha) < \infty$, then ,there exists a constant C_2 such that,for every $g \in D^*$ with $g'(0) \neq 0$, for all $n \geq 1$,

$$(10)\quad \sup_{-\infty<x<\infty} \left| P\left[\frac{n^{1-r}}{g'(0)} \{ g\left(\frac{S_n}{n}\right) - g(0) \} \le x \right] - G_\alpha(x) \right|$$

$$\le 5C_1 n^{1-sr} + C_2 (n^{1-r})^{(-\alpha/(2+\alpha))}$$

PROOF: In light of Theorem 1 and Lemma 2.3 ,the Theorem follows immediately.

References

[1] Morris K.W.,Szynal D.; On a Functional CLT of a Function of the average of Independent R.V.S. Annales Universitatis Marie Curie-skodowska,Lublin-Polonia, 1981,Vol.25(12), Sect.A, 101-106.

[2] Morris K.W.,Szynal D.; On the covergence rate in the CLT of some functions of the average of independent r.v.s.; PAMS Vol.3, Fasc.1,1982,p.85-95.

[3] Szynal D.; On the rate of convergence in the CLT for functions of the average of independent r.v.s.; PAMS, Vol.7,Fasc.2,1986,p.115-123.

[4] Bartmanska B.; Szynal D.; On non uniform estimates of the rates of convergence in the CLT for functions of the average of independent r.v.c. MSPT; (M.L.Puri et al.(eds)) Vol.A,23-31, 1987.

[5] Bartmanska B.; Szynal D.; On Levy's and Dudley's type estimates of the rate of convergence in the CLT for functions of the average of independent r.v.s.; PAMS, Vol.9, Fasc 2, 1988, p.67-75.

[6] Petrov V.V.; Sums of Independent random variables. Springer.Verlag, 1975.

[7] Satybaldina K.I.; Absolute Estimates of the rate of convergence to stable laws; Theory of Prob. and its application 1972,726-728, Vol.17,4.

[8] N. G. Ushakov; Some inequalities for characteristic functions of unimodal distributions, 1981, Theo. Prob. Appl. 595-599.

Department of Statistics,
Faculty of Science,
M.S.University of Baroda,
Baroda,Gujarat,
INDIA - PIN:390002.