ESTIMATION OF THE LOGNORMAL DISTRIBUTION FUNCTION

ESSAM KHALAF AL-HUSSAINI

Minimum variance unbiased estimates of the lognormal distribution with parameters μ and σ^2 are given for all possible cases: when σ^2 is known, when μ is known and when both μ and σ^2 are unknown. It is shown that the estimates of the lognormal distribution function can be estimated from the normal and Student's t-distribution tables.

1. INTRODUCTION

Aitcheson and Borwn [1] showed the domains of applications of the lognormal distribution in the areas of economics, biology, phsical and industrial processes, astronomy and other areas of special interest to

* *Departmen of Mathematics, University of Assiut, Assiut, A.R. Egypt.*

researchers in different fields. The methods (up to the year 1957) of estimating the parameters of two and three parameter lognormal distribution are summarized by Aitcheson and Brown as being essentially: the maximum likelihood method, the moments, quantiles, graphical and the mixed methods.

Chhikara and Folks [2] used a method due to Kolmogorov to obtain minimum variance unbiased estimates of the inverse Gaussian distribution function. The method is based on RaoBlackwell theorem which states that : given a complete sufficient statistic T for θ and an unbiased estimate $\tilde{g}(\theta)$ of a parametric function $g(\theta)$, the MVUE of $g(\theta)$ is given by $\bar{g}(\theta) = E[\tilde{g}(\theta)|T]$. We shall choose the initial unbiased estimate of F(x) for a random sample $X_1, \ldots, X_n$ as :

$$\tilde{g}(\theta) = \begin{cases} 1 , & X_1 < x \\ 0 , & \text{otherwise} \end{cases}$$

2. THE TWO-PARAMETER LOGNORMAL DISTRIBUTION FUNCTION

A random variable X is said to follow the lognormal distribution with parameters μ and σ^2, denoted by $X \sim \Lambda(\mu, \sigma^2)$ if and only if $Y = \log X \sim N(\mu, \sigma^2)$. The density function of a lognormal random variable X is given by:

$$f_X(X) = \begin{cases} \dfrac{1}{\sqrt{2\sigma^2\pi x}} \exp\{ -\dfrac{1}{2\sigma^2}(\log x-\mu)^2\}, & x > o \\ 0, & \text{otherwise} \end{cases} \qquad (2.1)$$

The lognormal distribution function is given by:

$$F_X(x) = \int_o^x f_X(z)dz = \Phi\left(\frac{\log x - \mu}{\sigma}\right) \qquad (2.2)$$

where Φ is the standard normal distribution at a. So that when both parameters μ, σ^2 are known, the lognormal distribution bution function can be evaluated using the normal table.

3. ESTIMATION OF $F(\alpha)$

3.1 M.V.U.E. OF $F(\alpha)$ WHEN σ^2 IS KNOWN

Given a r.s. $X_1,\ldots,X_n$ from a $\Lambda(\mu, \sigma^2)$ with known σ^2,

$$\bar{Y} = \frac{\sum_{i=1}^{n} \log X_i}{n} \sim N\left(\mu, \frac{\sigma^2}{n}\right)$$ is a complete sufficient for μ.

The MVUE of $F(\alpha)$ defined by $P(X_1 \leq \alpha \mid \bar{Y})$ is given by:

$$\hat{F}(\alpha) = P(X_1 \leq \alpha \mid \bar{Y}) = \int_0^{\alpha} f_{X_1 \mid \bar{Y}}(x_1 \mid \bar{y})\,dx_1, \quad \alpha > 0 \tag{3.1}$$

and $F(\alpha) = 0, \quad \alpha \leq 0$, where

$$f_{X_1 \mid \bar{Y}}(x_1 \mid \bar{y}) = \sqrt{\frac{n}{2\pi\,(n-1)}}\;\frac{1}{\sigma x_1} \exp\left[-\frac{n}{2(n-1)\sigma^2}(\log x_1 - \bar{y})^2\right], \quad x_1 > 0 \tag{3.2}$$

So that
$$\hat{F}(\alpha) = \begin{cases} 0 & \alpha \leq 0 \\ \Phi\left(\dfrac{\log\quad - \bar{y}}{\sqrt{\dfrac{n-1}{n}}\,\sigma}\right) & \alpha > 0 \end{cases} \tag{3.3}$$

3.2 MVUE OF $F(\alpha)$ WHEN μ IS KNOWN

The statistic $T = \dfrac{\sum_{i=1}^{n} (\log x_i - \mu)^2}{n-1} \sim (\dfrac{\sigma^2}{n})\chi^2(n)$, is a complete sufficient statistic for σ^2. It can be shown that

$$f_{X_1 | T}(x_1|t) = \frac{\Gamma(\frac{n}{2})}{\sqrt{\pi}\,\Gamma(\frac{n-1}{2})\sqrt{nt}\;\; x_1} \left[1 - \frac{(\log x_1 - \mu)^2}{nt}\right]^{(n-3)/2},$$

$$e^{\mu - \sqrt{nt}} < x_1 < e^{\mu + \sqrt{nt}} \qquad (3.4)$$

So that

$$\hat{F}(\alpha) = \begin{cases} 0 & , \quad \alpha \leq e^{\mu - \sqrt{nt}} \\ F_{n-1}(\beta) & , \quad e^{\mu - \sqrt{nt}} < \alpha < e^{\mu + \sqrt{nt}} \\ 1 & , \quad \alpha \geq e^{\mu + \sqrt{nt}} \end{cases} \qquad (3.5)$$

where $F_{n-1}(\beta)$ denotes the Student's t-distribution with (n-1) degrees of freedom evaluated at β with

$$\beta = \sqrt{n-1}\left(\frac{\log \alpha - \mu}{nt}\right)\left[1 - \frac{(\log \alpha - \mu)^2}{nt}\right] \qquad (3.6)$$

3.3 MVUE OF $F(\alpha)$ WHEN BOTH PARAMETERS ARE UNKNOWN

In this case the vector $T = (\bar{Y}, V)$, where $\bar{Y}$ is as defined in (3.1) and

$$V = \frac{\sum_{i=1}^{n} (\log x_i - \bar{Y})^2}{n-1},$$ forms a complete

sufficient statistic for (μ, σ^2). $\bar{Y}$ and V are independent, $\bar{Y} \sim N(\mu, \frac{\sigma^2}{n})$ and $V \sim (\frac{\sigma^2}{n-1})\, \chi^2(n-1)$. It can be shown that:

$$f_{X_1|\bar{Y},V}(x_1|\bar{y}, v) =$$

$$\frac{\Gamma(\frac{n-1}{2})\sqrt{n}}{\sqrt{\pi}\,\Gamma(\frac{n-2}{2})(n-1)\sqrt{v}\; x_1}\left[1 - \frac{n}{(n-1)^2 v}(\log x_1 - \bar{y})^2\right]^{\frac{n-4}{2}}, \quad L < x_1 < U \qquad (3.7)$$

where $L = e^{\bar{y}-(n-1)\sqrt{v/n}}$ and $U = e^{\bar{y}+(n-1)\sqrt{v/n}}$, (3.8)

So that

$$F(\alpha) \begin{cases} 0 & , \alpha \leqslant L \\ F_{n-1}(\gamma) & , L < \alpha < \\ 1 & , \alpha \geqslant U \end{cases} \qquad (3.9)$$

where $F_{n-1}(\gamma)$ denotes the Student's t-distribution with (n-1) degrees of freedom evaluated at γ, where

$$\gamma = \frac{\sqrt{n(n-2)}(\log \alpha - \bar{y})^2}{(n-1)\sqrt{v}} \left[1 - \frac{n(\log \alpha - \bar{y})^2}{(n-1)^2 v}\right]^{-\frac{1}{2}}$$

and L, U are as given by (3.8).

4. CONCLUDING REMARKS

It may be noted that the estimate obtained can be evaluated by using the normal and Student's t-tables.

Extension to the three-parameter case (say τ, μ, σ^2) is immediate if τ is known from prior information. In this case $F(\alpha)$ is the same as that in the two-parameter case except that a should be replaced by $\alpha-\tau$ throughout. If τ is unknown, however, the estimation of the three-parameter logormal distribution needs further investigation which will appear in forthcoming paper.

Estimation of the lognormal distribution function is of importance in the analysis of consumer's behaviour, the distribution of incomes, the problem of life testing and reliability.

[received september 1975, revised Febraury 1976]

REFERENCES

[1] Aitchison, J., and Brown, J.A.C., (1957), The *Lognormal Distribution*. Cambridge University Press, Cambridge, England.

[2] Chhikara, R.S., and Folks, J.L., "Estimation of the Inverse Gaussian Distribution Function", *Journal of the American Statistical Association*, *69(1974) 250-254*.