ON THE MEAN RANGE OF PARTIAL SUMS OF A FINITE NUMBER OF RANDOM VARIABLES

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SUMMARY

The problem of determining the mean of the range of partial sums of random variables is important in planning the storage capacity of reservoirs under the assumption of infinite storage capacity. In this paper, new formulae for the mean of the range (and adjusted range) of partial sums of a finite number of exchangeable random variables are given. These formulae are essentially based on a lemma given by Spitzer (1956) concerning the expected value of the maximum of the partial sums.

This lemma of Spitzer is estalbished by a simpler proof when the original random variables are indepenhently and symmetrically distributed.

INTRODUCTION

Let $X_1, X_2 \ldots, X_n$ be a sequence of exchangeble random variables, having the common probability density function (p.d.f.) f (x) with characteristic function (ch.f.) ψ (θ)

Define the partial sum s :

$$S_0 = 0 \quad , \quad S_r = \sum_{1}^{r} X_i \quad ; \qquad i=1,2, \ldots ,n$$

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Their maximum, minimum and range are defined respectively as :

$$M_n = \max (0, S_1, \ldots, S_n)$$

$$m_n = \min (0, S_1, \ldots, S_n)$$

$$R_n = M_n - m_n \tag{1.1}$$

When taking the partial sums of the deviations of Xi from their sample mean $X_n = \frac{1}{n} \sum_1^n x_i$, then the partial sums S_r^* will be called adjusted and denoted by S^*_r, and the corresponding adjusted : maximum, minimum and range are defined in anology with (1.1)

By using combinatorial analysis, Spitzer [5] derived a fundamental identity from which the expectation of the maximum M_n may be written in the form :

$$E(M_n) = \sum_1^n r^{-1} E\left[\max (0, S_r)\right] \tag{1.2}$$

In (1972), this lemma was used by Boes and Salus-La-Cruz [2] to show that the expected value of the range R_n may be presented as :

$$E(R_n) = \sum_1^n r^{-1} E(|S_r|) \tag{1.3}$$

whence, it was possible to state also that,

$$E(R_n^x) = \sum_1^n r^{-1} E(|S_r^x|) \tag{1.4}$$

The work done in this paper falls into two parts :

In the first part, we express equations (1.3) and (1.4) in a more practical form in which the expectations of R_n and R_n^x are given in terms of the ch.f. of t e original random variables X_i.

Because the expectation of the range is twice that of the maximum in the case of symmetric random variables, the second part is devoted to give a simple proof of Spitzer's lemma :

$$E(M_n) = \tfrac{1}{2} \sum_{1}^{n} r^{-1} E(|S_r|) \tag{1.5}$$

when X_i are independently and symmetrically distributed

The mean range and the mean adjusted range
in terms of the ch. f. of X_i.

The mean range :

In addition to the definitions and notations given before, we first introduce the following notations :

$\phi_r(.)$	the p.d.f. of $\lvert s_r \rvert$	(2.1)
$\eta_r(\theta)$	the ch.f. of $\phi_r(.)$	(2.2)
$\psi_r(\theta_1, \ldots, \theta_r)$	the joint ch.f. of the random vector $(X_1, \ldots, X_r)$	(2.3)

Next, since the p.d.f. of $Y_r = |s_r|$ is given by,

$\phi_r(y) = f_r(y) + f_r(-y)$

w ere $f_r(y)$ denotes the p.d.f. of S_r .

we have,

$$\eta_r(\theta) = \int_0^\infty e^{i\theta y} f_r(y)dy + \int_0^\infty e^{i\theta y} f_r(-y)dy$$

On the other hand, $\psi_r(1\theta)$ denotes, according to (2.3), the ch.f. of $f_r(.)$ where 1 is the unit raw-vector of order r. Thus, its complex conjugate $\psi_r(-1\theta)$ is given by :

$$\Psi_r(-1\theta) = \int_{-\infty}^{\infty} e^{-i\theta y} f_r(y)dy$$

$$= \int_{-\infty}^{0} e^{-i\theta y} f_r(y)dy + \int_0^\infty e^{-i\theta y} f_r(y)dy$$

$$= \int_0^\infty e^{i\theta y} f_r(y)dy + \int_0^\infty e^{-i\theta y} f_r(y)dy$$

Comparing (2.4) and (2.5), we find immediately that :

$$\eta_r(\theta) = \Psi_r(-i\theta) + 2i \int_0^{\infty} \sin(\theta y)\, f_r(y)\,dy$$

To find the last integral, we make use of the Fourier inversion formula.

Thus, if $\psi_r(i\theta)$ is absolutly integrable function, f (y) can be substituted by its inversion transform,

$$f_r(y) = \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{-ity}\, \Psi_r(it)\,dt$$

Hence,

$$\eta_r(\theta) = \Psi_r(-i\theta) + \frac{i}{\pi} \int_0^{\infty} \int_{-\infty}^{\infty} \sin(\theta y) e^{-ity}\, \Psi_r(it)\,dt\,dy \qquad (2.6)$$

Since we are interested in the expectation of $|S_r|$, and according to the fact that,

$$E(|S_r|) = i^{-1} \frac{d}{d\theta} \eta_r(\theta) \Big|_{\theta=0}$$

we get

$$E(|S_r|) = -r\,E(X_i) + \frac{1}{\pi} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} y\, e^{-ity}\, \Psi_r(it)\,dt\,dy.$$

Inverting the order of integrations, and integrating by parts, first with respect to y, then with respect to t, and making use of the fact that :

$$\lim_{t \to \pm\infty} \Psi_r(it) = 0,$$

we obtain

$$E\,|S_r| = -r\,E(X_1) - \frac{1}{\pi} \int_{-\infty}^{\infty} t^{-1}\, d\,\Psi_r(it) \qquad (2.7)$$

going back to equation (1.3) of the mean range, and applying (2.7) we find that,

$$E(R_n) = -n\,E(X_1) - \frac{1}{\pi}\sum_{1}^{n} r^{-1} \int_{-\infty}^{\infty} t^{-1}\, d\,\Psi_r(1t) \tag{2.8}$$

This formula is valid for any set of exchangeable random variables for which ψ_r (1t) is absolutly integrable function.

We notice that, in the special case of independece, the function ψ_r (1t) reduces to ψ^r (t), hence,

$$E(R_n) = -n\,E(X_1) - \frac{1}{\pi}\sum_{1}^{n} \int_{-\infty}^{\infty} t^{-1}\, \Psi^{r-1}(t)\, d\,\Psi(t). \tag{2.9}$$

The mean adjusted range :

Using the same notations above but with the superscript (x) to denote the adjusted case, we can find, following the same method, that :

if ψ_r^x (1t) is absolutly integrable function, then

$$E(R_n^{*}) = -\frac{1}{\pi}\sum_{1}^{n} r^{-1} \int_{-\infty}^{\infty} t^{-1}\, d\,\Psi_r^{*}(1t), \tag{2.10}$$

$$\text{Since } E(X_i^{*}) = 0$$

where ψ_r^x (1t) reduces, in the independence case, to :

$\psi^{r}\,[\,(1-r/n)\,t]\,\psi^{n-r}\,(-r\,t/n)$

Example :

Suppose X_i, i=1, ... , n are symmetrically distributed random variables, having the multivariate normal distribution with,

$E(x_i) = \mu$, var $(X_i) = \sigma^2$ and $E(X_i, X_j) = \rho\sigma^2$ $i \neq j$

$i.j = 1, \dots, n.$

According to equation (2.8) we have,

$$\Psi_r(1t) = \exp [it\, E(S_r) - 1/2\, t^2 \operatorname{var}(S_r)]$$

$$\text{where} \quad E(S_r) = r\mu \quad ,$$

$$\operatorname{var}(S_r) = r\,\sigma^2 + r(r-1)\rho\sigma^2$$

Thus,

$$E(R_n) = -n\mu - \frac{1}{\pi} \sum_1^n \int_{-\infty}^{\infty} (i\mu t^{-1} - C_r) \exp(ir\mu t - 1/2\, rC_r t^2)\, dt$$

$$\text{where} \quad C_r = \sigma^2 [\, 1+r(r-1)\rho].$$

$$= -n\mu + 2 \sum_1^n [\mu Z(r/C_r)^{1/2} \mu) + (C_r/r)^{1/2} z((C_r/r)^{1/2} \mu)], \qquad (2.11)$$

where Z (.) is the standard normal distribution and z(.) is the standard normal density function.

when taking $\mu = 0$, $E(R_n)$ reduces to :

$$E(R_n) = (2/\pi)^{1/2} \sum_1^n r^{-1} (\operatorname{var}(S_r))^{1/2}$$

$$= (2/\pi)^{1/2}\, \sigma \sum_1^n r^{-1/2} (1+(r-1)\rho)^{1/2} \qquad (2.12)$$

Moreover, when $\rho = 0$, we get,

$$E(R_n) = (2/\pi)^{1/2}\, \sigma \sum_1^n r^{-1/2}$$

Which is the same form as the result given by Anis and Lloyd [1]. Also, according to equation (2.10) of the mean adjusted range, we have,

$$\Psi^*_r(1t) = \exp -1/2\, [\, t^2 \operatorname{var}(S^*_r)\,],$$

$$\text{where} \operatorname{var}(S^*_r) = r/n\, (n-r)(1-\rho)\sigma^2$$

Thus

$$E(R^*_n) = \frac{1}{\pi} \sum_1^n r^{-1} \int_{-\infty}^{\infty} \operatorname{var}(S^*_r) . \exp - 1/2\, [t^2 \operatorname{var}(S^*_r)]\, dt$$

$$\doteq (2/\pi)^{1/2} \sum_{1}^{n} r^{-1} (\text{var}(S_r^*))^{1/2}$$
$$= (2/\pi(1-\rho)\cdot)^{1/2} \sigma \sum_{1}^{n} \frac{(n-r)^{1/2}}{(nr)^{1/2}} \tag{2.14}$$

when $\rho = 0$, $\sigma = 1$ we get,

$$E\left(R_n^* \right) = (2 / \pi)^{1/2} \sum_{1}^{n} \frac{(n-r)^{1/2}}{nr^{1/2}} \tag{2.15}$$

which is equivalent to the result given by Solari and Anis (4)

3. A new proof of Spitzer's lemma

We now give a new proof to the formula

$$E(M_n) = \tfrac{1}{2} \sum_{1}^{n} r^{-1} E(|S_r|)$$

Consider the probability,

$$h_n = \Pr(S_1 < 0, S_2 < 0, \ldots, S^{n-1} < 0, S_n > 0), \tag{3.1}$$

and let q_n be defined as,

$$q_n = \Pr(S_n = M_n)$$
$$= \Pr(S_n > S_i, i = 0,1, \ldots, n\ldots 1) \tag{3.2}$$

By taking the variables $X_1, \ldots, X_n$ in reverse order the partial sums of the new variables $X'_1, \ldots, X'_n$, will be $S'_r = S_n — S_{n-r}$ and the correspondence X_i ------→ X'_i maps any event defined by S_r into a similar event defined by S'_r of equal probability. Thus,

$$q_n = \Pr(S'_1 > 0, \ldots\ldots, S'_n > 0)$$
$$= \Pr(S_1 > 0, \ldots\ldots, S_n > 0) \tag{3.3}$$

we state now following lemmas :

Lemma (1) : The generating function of h_n is given by.

$h(\theta) = 1—(1—\theta)^{1/2}$ (See Feller [3] p. 396)

Lemma (2) : The generating function of q_n is given by

$q(\theta) = 1/(1-\theta)^{1/2}$ (Feller [3] p. 379, theorem 4)

Lemma (3) : $Pr(S_r = M_n) = q_r q_{n-r}$ for all $r \leq n$.

(Feller (3) p. 398).

corolary (1) : from lemma (1) and (2), it is deduced that :

$$h_n = \frac{q_n}{2n-1}$$

Lemma (4) : For any symmetric random variable X,

$E(|X|) = E(X/X > 0)$.

Now, the expectation of the maximum M_n may be written in terms of the conditional expectations as follows :

$$E(M_n) = \sum_1^n E(S_r/S_r = M_n) \cdot Pr(S_r = M_n)$$

$$= \sum_1^n E(S_r/S_r = M_n) \cdot q_r q_{n-r}$$

according to lemma (3)

But, $E(S_r/S_r = M_n) = E(S_r/S_r > S_i, r \neq i, i = 0, 1, \ldots, n)$

$= E(S_r/S_r > S_i, i = 0, 1, \ldots\ldots, r\ldots1)$,

for reason of independence

$= E(S_r/S_r = M_r)$

Thus,

$$E(M_n) = \sum_1^n E(S_r/S_r = M_r)\, q_r q_{n-r} \tag{3.4}$$

we shall now prove the following identity

$$E(S_r/S_r = M_r)\, q_r = \sum_1^r (2k)^{-1} q_{r-k} E|S_k| \tag{3.3}$$

Proof. The proof is given using induction

The identity (3.5) is ture for $r = 1$ in virtue of lemma (4).

Assume the identity for all $j < r$.

$$E(S_r/S_r = M_r)\, q_r = E(S_r/S_r > S_i, i=0,1, \ldots, r\ldots 1).$$

$$\Pr(S_i > 0, i = 1,2, \ldots\ldots, r) \qquad (3.6)$$

We take the variables X_j, $j = 1, \ldots\ldots, r$ in reverse order, so we define the disjoint events :

$$\beta_1 = \{S'_i > 0\}$$

$$\beta_j = \{S'_1 < 0, S'_2 < 0, \ldots, S'_{j-1} < 0, S'_j > 0\}, j=2,3, \ldots, r \qquad (3.7)$$

In addition, we denote by C_j the event $(S_i > 0, i = 1, \ldots, j), j = 1, \ldots, r.$

It is thus evident that :

$\beta_j \cap C_r$ are disjoint events, and,

$$C_r = \bigcup_{j=1}^{r} \cap \beta_j C_r.$$

But,

$$\beta_j \cap C_r = \{S'_1 < 0, \ldots, S'_{j-1} < 0, S'_j > 0; S_r > 0, \ldots, S_1 > 0\}$$

$$= \{S'_1 < 0, \ldots, S'_{j-1} < 0, S'_j > 0, S_{r-j} > 0, \ldots, S_1 > 0\}$$

$= \beta_j \cap C_{r-j}$, the intersection of independent events

Thus, equation (3.6) can be rewritten in the form :

$$E(S_r/S_r = M_r) q_r = E(S_r/\bigcup_{j=1}^{r} \cap \beta_j C_{r-j}) \Pr(\bigcup_{1}^{r} \cap \beta_j C_{r-j})$$

$$= \sum_{j=1}^{r} E(S_r/\cap \beta_j C_{r-j}) \Pr(\cap \beta_j C_{r-j}),$$

since β_j are disjoint.

$$= \sum_{1}^{r} [E(S_{r-j}/C_{r-j}) + E(S_j/\beta_j)] \Pr(\beta_j). \Pr(C_{r-j})$$

$$= \sum_{1}^{r-1} E(S_j/S_i > 0, i=1,\ldots,j)\, q_j\, h_{r-j}$$

$$+ \sum_{1}^{r} E(S_j/S_j > 0, S_i < 0, i=1,\ldots,j-1) h_j\, q_{r-j} \qquad (3.8)$$

For reason of symmetry, $E(S_j/S_j > 0, S_i < 0, i=1, \ldots, j-1)\, h_j$ is similarly obtinable as $E(S_j/S_i > 0, i = 1, \ldots, j)\, q_j$, since the corresponding integrals are of the same type which means that, if this latter is assumed as given by (3.5) for all $j < r$ then the former must be equal to

$$- \sum_{1}^{j} (2k)^{-1}\, h_{j-k}\, E\,|S_k| \tag{3.9}$$

Thus, applying (3.5) and (3.9) to (3.8) for all $j < r$, and making use of corrolary (1), we get :

$$E(S_r/S_r = M_r)\, q_r$$

$$= \sum_{j=1}^{r-1} \sum_{k=1}^{j} (2k)^{-1} \left[2(r-k)-1\right]^{-1} q_{j-k}\, q_{r-j} \cdot E\,|S_k|$$

$$- \sum_{j=1}^{r-1} \sum_{k=1}^{j} (2k)^{-1} \left[2(j-k)-1\right]^{-1} q_{j-k}\, q_{r-j} \cdot E\,|S_k|$$

$$+ E(S_r/ S_r > 0,\ S_i < 0,\ i=1,\ldots, j-1)\, h_r.$$

Inverting the summation signs in the first and second double sums above, it is found that most of the terms cancel, leaving :

$$E(S_r/S_r = M_r)q_r - E(S_r/S_r > 0, S_i < 0,\ i=1,\ldots,r-1)\, h_r$$

$$= \sum_{k=1}^{r-1} (2k)^{-1} q_{r-k} \cdot E\,|S_k| - (-1) \sum_{k=1}^{r-1} (2k)^{-1} \left[2(r-k)-1\right]^{-1} \cdot q_{r-k}\, E\,|S_k|$$

Since $q_o = 1$, we can add and subtract the term $(2r)^{-1} E\,|S_r|$ to the first and second sums respectively, which yields the required identity (3.5)

Thus, the inductive proof is complete.
going back to equation (3.4) and applying the inductive formula (3.5), we get :

$$E(M_n) = \sum_{1}^{n} \sum_{k=1}^{r} (2k)^{-1} q_{r-k}\, q_{n-r}\, E\,|S_k|$$

$$= \sum_{k=1}^{n} (2k)^{-1} \; E\,|S_k| \; \sum_{i=0}^{n-k} q_i \, q_{n-k-i}$$

$$= \sum_{k=1}^{n} (2k)^{-1} \; E\,|S_k| \,, \quad \text{in virtue of lemma(3).}$$

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