BAYESIAN CONDITIONAL ESTIMATION OF THE WEIBULL PARAMETERS USING CENSORED DATA

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SUMMARY

Analagous to the Weibull distribution, the conditional probability density function of the Weibull parameters and the unknown sample size n, are obtained, Also the Bayesian estimators of the unknown parameters are derived using the squared error loss function.

1. INTRODUCTION

The general problem under consideration in this article is that of estimating the parameters of the Weibull distribution with density function given by

$$f(t,b,\theta)=(b/\theta)t^{b-1}\exp(-t^{b}/\theta), \quad \theta,t \geq 0 \tag{1.1}$$

where

b and θ are the shape and scale parameters respectively with $b \geq 0$ and $b \neq 1$, because if $b = 1$, (1.1) reduces to the one parameter exponential distribution

$$f(t;\theta) = (1/\theta)\exp(-t/\theta); \quad \theta \geq 0$$

In type II censored sample, n items are placed on a life test and the test is terminated after r failures have occured. Then the likelihood function of $t_{(1)},t_{(2)},\ldots t_{(r)}$ is given by

$$L = \frac{n!}{(n-r)!}\,[1-F(t_{(r)})]^{n-r}\prod_{i=1}^{r} f(t_{(i)}) \tag{1.2}$$

Using equation (1.1) in equation (1.2) we have

$$L = \frac{n!}{(n-r)!}\,(b/\theta)^{r}\prod_{i=1}^{r} t_{(i)}^{b-1}\exp\{-(1/\theta)T(b,n)\} \tag{1.3}$$

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where

$$T(b,n) = \sum_{i=1}^{r} t_{(i)}^{b} + (n-r)\, t_{(r)}^{b} \qquad (1.4)$$

If we have the following independent prior distribution function of θ,b and n

$$\pi(\theta) \propto \theta^{-(\alpha+1)} \exp(-\beta/\theta) , \quad \theta \geq 0$$

$$\pi(b) \propto 1 \quad \text{for all values of b}$$

$$\pi(n) \propto 1 \quad \text{" " " " n}$$

Then the posterior p.d.f. of θ,b and n given $T=(t_{(1)},t_{(2)},\ldots,t_{(r)})$ have the following form

$$f(\theta,b,n/T) \propto \frac{n!}{(n-r)!} b^{r} \theta^{-(\alpha+r+1)} \prod_{i=1}^{r} t_{(i)}^{b-1} \exp\{-(1/\theta)(T(b,n)+\beta)\}$$

$$r \leq n \leq m , \quad \theta \geq 0$$

$$\alpha_1 \leq b \leq \alpha_2$$

Where

α_1, α_2 are known positive constants, r is the number of failures and m is the largest sample size.

A great many papers have been written in the estimation of parameters for the unconditional distribution under the Bayesian and non-Bayesian framework. Gibbons and al (1981) survey most of the work done concerning point estimation of the Weibull parameter under the non-Bayesian view while under the Bayesian view, Soland (1969), Canavos and Tsokos (1977) discussed the problem of Bayesian estimation for the parameters of Weibull distribution. In the view of conditional estimation, Lowless (1973) used the conditional and unconditional confidence intervals for the parameters of the Weibull distribution under the non-Bayesian case, but in this work we shall find the following conditional distributions:

(i) The conditional probability density function of θ given b and n

(ii) The conditional probability density function of b given θ and n
(iii) The conditional probability density function of n given θ and b

From the above c.p.d.f's, we can derive the conditional Bayesian estimators θ^*, b^* and n^* of the unknown parameters θ, b and n respectively. Although variances and shape factors of the estimators are given in later.

2. THE c.p.d.f. OF θ GIVEN b AND n.

The c.p.d.f. of the scale parameter θ given the shape parameter, b, and the sample size, n, can be obtained as follows:

Since we have the joint p.d.f. of θ, b and n which is

$$f(\theta,b,n/T)\,\alpha \prod_{i=0}^{r-1}(n-i)b^r\theta^{-(\alpha+r+1)}\prod_{i=1}^{r} t_{(i)}^{b-1}\exp\{-(1/\theta)(T(b,n)+\beta)\}$$

and the c.p.d.f. of θ given b and n can be defined by the following equation:

$$f(\theta/b,n,T) = f(\theta,b,n/T)/f(b,n/T)$$

Where

$$f(b,n/T) = \int_0^\infty f(\theta,b,n/T)\,d\theta \;\alpha\; \{T(b,n)+\beta\}^{-(\alpha+r)}(\alpha+r-1)!$$

Therefore the c.p.d.f. $f(\theta/b,n,T)$ can be obtained as

$$f(\theta/b,n,T) = \{T(b,n)+\beta\}^{\alpha+r}\,\Gamma^{-1}(\alpha+r)\;\theta^{-(\alpha+r+1)}\exp\{-(1/\theta)(T(b,n)+\beta)\}$$

$$\theta \geq 0$$

With the r-th non-central moment equal to

$$\mu_k' = E(\theta^k/b,n) = \{T(b,n)+\beta\}^k\,(\alpha+r-k-1)!/(\alpha+r-1)!$$

Especially if k = 1,2,3,4 we have

$\mu_1' =$ the conditional mean $= E(\theta/b,n) = \theta^*$

$$= (T(b,n)+\beta)/(\alpha+r-1) \quad (2.1)$$

$$\mu_2' = \{T(b,n)+\beta\}^2 / \{(\alpha+r-1)(\alpha+r-2)\} \quad (2.2)$$

$$\mu_3' = \{T(b,n)+\beta\}^3 / \{(\alpha+r-1)(\alpha+r-2)(\alpha+r-3)\} \quad (2.3)$$

$$\mu_4' = \{T(b,n)+\beta\}^4 / \{(\alpha+r-1)(\alpha+r-2)(\alpha+r-3)(\alpha+r-4)\} \quad (2.4)$$

Then based on (2.1) and (2.2) the Bayesian conditional variance of θ is the variance of the conditional estimator of θ i.e.

$$\text{var}(\theta^*/b,n) = \{T(b,n)+\beta\}^2/(\alpha+r-1)^2(\alpha+r-2) \quad (2.5)$$

Sometimes the measure of skewness can be measured by the 3-rd central moment and the symmetrical distribution can be shown to have $\mu_3=0$, therefore we have the 3-rd moment about the mean equal to

$$\mu_3 = \mu_3' - 3\mu_1'\mu_2' + 2\mu_1'^3 = \frac{4\{T(b,n)+\beta\}^3}{(\alpha+r-1)^3(\alpha+r-2)(\alpha+r-3)} \quad (2.6)$$

Therefore the coefficient of skewness can be calculated by using equations (2.5) and (2.6) to have

$$\alpha_1' = \frac{4(\alpha+r-2)^{\frac{1}{2}}}{(\alpha+r-3)}$$

The quantity $S = \frac{\text{mean-median}}{\text{standard deviation}}$ provides an alternative measure of skewness and it can be proved that $-1 \leq S \leq 1$. To compute the quantity S we have the mean which is defined by (2.1) and the standard deviation which is the positive square root of (2.5) while the median can be defined such that:

$$\int_0^{\text{median}} f(\theta/b,n)d\theta = 0.5$$

i.e. the median must be equal to $\frac{2\{T(b,n)+\beta\}}{\chi^2_{2(\alpha+r)}(0.5)}$ and the measure S can be defined as

$$S = \frac{\chi^{2(0.5)}_{2(\alpha+r)}[\alpha+r-2]^{\frac{1}{2}}-2(\alpha+r-1)}{\chi^{2(0.5)}_{2(\alpha+r)}}$$

Where the quality $\chi^{2(0.5)}_{2(\alpha+r)}$ is the precentage point of Chi-square distribution with degrees of freedom equal to $2(\alpha+r)$.

Also the fourth moment about the mean is sometimes used as a measure of Kurtosis which is the degree of flatness of a density near its centre. Now the 4-th moment about mean can be calculated from the equations (2.1),(2.2),(2.3) and (2.4) to have

$$\mu_4 = \frac{3(\alpha+r-5)\ \{T(b,n)+\beta\}^4}{(\alpha+r-1)^4\ (\alpha+r-2)(\alpha+r-3)(\alpha+r-4)}$$

Therefore the coefficient of Kurtosis is calculated in the form

$$\alpha_2' = \frac{3(\alpha+r-5)\ \{T(b,n)+\beta\}^2}{(\alpha+r-1)^4(\alpha+r-3)(\alpha+r-4)}$$

The coefficients α_1' and α_2' can be used to investegate the distribution of θ.

3. THE c.p.d.f. OF b GIVEN θ AND n

The c.p.d.f. of the shape parameter, b, given the scale parameter θ and the sample size n, can be obtained by using the following defenition:

$$f(b/\theta,n,T) = \frac{f(\theta,b,n/T)}{f(\theta,n/T)}$$

$$= I_1^{-1}\ b^r \prod_{i=1}^{r} t_{(i)}^{b-1} \exp\{-(1/\theta)(T(b,n)+\beta\}, \alpha_1 < b < \alpha_2$$

Where

$$I_1 = \int_{\alpha_1}^{\alpha_2} b^r \prod_{i=1}^{r} t_{(i)}^{b-1} \exp\{-(1/\theta)(T(b,n)+\beta\}\ db$$

The Bayesian estimator, b^*, can be obtained by using the squared error loss function $L(b,b^*) = C(b-b^*)^2$, where C is constant that may is equal 1. Then we have b^* given as

$$b^{*} = E(b/\theta ,n,T) = I_2/T_1$$

Where

$$I_2 = \int_{\alpha_1}^{\alpha_2} b^{r+1} \prod_{i-1}^{r} t_{(i)}^{b-1} \exp\{ -(1/\theta)(T(b,n)+\beta)\} db \quad (3.2)$$

and the variance of b^{*} is the Bayesian variance of b which is given as

$$\text{var}(b^{*}/\theta ,n) = E(b^2/\theta,n)-E^2(b/\theta,n \quad (3.3)$$

$$= I_3/I_1-(I_2/I_1)^2$$

Where

$$I_3 = \int_{\alpha_1}^{\alpha_2} b^{r+2} \prod_{i=1}^{r} t_{(i)}^{b-1} \exp\{ -(1/\theta)(T(b,n)+\beta) \}db \quad (3.4)$$

and I_1 and I_2 are defined by (3.1) and (3.2)

The numerical integration can be used to solve equations (3.1), (3.2) and (3.3) and so to obtain's the numerical values of b* and the variance of it.

4- THE c.p.d.f. OF n GIVEN θ AND b

The c.p.d.f. of the sample size n can bd evaluated from the following equation

$$f(n/\theta ,b) = \frac{\prod_{i=0}^{r-1} (n-i)b^{r}\theta^{-(\alpha+r+1)} \prod_{i=1}^{r} t_{(i)}^{b-1} \exp\{-(1/\theta)(T(b,n)+\beta)\}}{\int_{r}^{m} \prod_{i=0}^{r-1} (n-i)b^{r}\theta^{-(\alpha+r+1)} \prod_{i=1}^{r} t_{(i)}^{b-1} \exp\{-(1/\theta)(T(b,n)+\beta)\}dn}$$

Now define

$$\prod_{i=0}^{r-1} (n-i) = \sum_{s=0}^{r-1} (-1)^{s} \ell_s n^{r-s}$$

Where

$$\ell_0 = 1,$$

$$\ell_1 = \sum_{i=1}^{r-1} i, \ell_2 = \sum_{i \leq j}^{r-1} ij, \ldots, \ell_{r-1} = \prod_{i=1}^{r-1} i$$

Therefore we have

$$f(n/\theta,b) = \frac{\prod_{i=0}^{r-1} (n-i) \exp\{-(1/\theta)(T(b,n)+\beta)\}}{\int_r^m \sum_{s=0}^{r-1} (-1)^s \ell_s n^{r-s} \exp\{-(1/\theta)(T(b,n)+\beta)\} dn}$$

$$= \frac{\prod_{i=0}^{r-1} (n-i) \exp\{-(1/\theta)(T(b,n)+\beta)\}}{e^{T(b)} \sum_{s=0}^{r-1} (-1)^s \ell_s \int_r^m n^{r-s} \exp\{-(n/\theta)t^b_{(r)}\} dn}$$

where $T(b) = (\sum_{i=1}^{r} t^b_{(i)} - rt^b_{(r)} + \beta/\theta$

Using the expansion of $\exp\{-(1/\theta)t^b_{(r)}\}$ to have

$$f(n/\theta,b) = [\text{const}]^{-1} \prod_{i=0}^{r-1} (n-i) \exp(-nt^b_{(r)}/\theta).$$

Where

$$\text{const.} = \sum_{j=0}^{\infty} (-t^b_{(r)}/\theta)^j/j! \sum_{s=0}^{r-1} (-1)^s \ell_s (m^{r-s+j} - r^{r-s+j})/(r-s+j),$$

$$r \leq n \leq m \quad , \quad r > 0$$

The Bayesian estimator ,n*, of n can be found as

$$n^* = E(n/\theta,b) = [\text{const}]^{-1} \int_r^m \frac{nn!}{(n-r)!} \exp(-nt^b_{(r)}/\theta)dn$$

$$=[\text{const}]^{-1} \sum_{j=0}^{\infty} (-t^b_{(r)}/\theta)^j/j! \sum_{s=0}^{r-1} (-1)^s \ell_s (m^{r-s+j+1} - r^{r-s+j+1}) /(r-s+j+1) \quad (4.1)$$

The Bayesian variance of n is the variance of the Bayesian estimator, n*, i.e.

$$v(n^*/b,\theta) = E(n^2/b,\theta) - E^2(n/b,\theta) \quad (4.2)$$

Where

$$E(n^2/b,\theta) = [\text{const}]^{-1} \sum_{j=0}^{\infty} (-t_{(r)}^{b}/\theta)^j/j! \sum_{s=0}^{r-1} (-1)^s \ell_s (m^{r-s+j+2} - r^{r-s+j+2})/(r-s+j+2) \quad (4.3)$$

Therefore the variance of the Bayesian estimator (4.2) can be calculated by using (4.1) and (4.3). To simplfy the calculation of the shape factors of the c.p.d.f. of n, we will derive the following k-th non-central moment which is giveh by

$$\mu_k' = E(n^k/b,\theta) = [\text{const}]^{-1} \sum_{j=0}^{\infty} (-t_{(r)}^{b}/\theta)^j/j! \sum_{s=0}^{r-1} (-1)^s \ell_s (m^{r-s+j+k} - r^{r-s+j+k})/(r-s+j+k), \quad \text{for } k = 1,2,3,\ldots\ldots$$

Finally the c.p.d.f.'s of θ ,b and n in the case of complete sample size can be obtained as a special case from our results under the condition that r=n and n is continuous.

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