USE OF CONFIDENCE INTERVALS IN QUEUEING SYSTEM SIMULATION - WITH APPLICATION TO A MAINTENANCE SHOP

MOHAMED A. SHALABY*

ABSTRACT

This paper focuses on using confidence interval estimation while simulating a queueing system. The approach relies on the existence of regenerative points in the queueing process being simulated that produces a set of n independent observations. An application is given to a maintenance shop which is modeled as a multiserver queueing system. The shop is assumed to serve two classes of customers; passenger cars and cargo trucks. Comparison of results obtained from simulation using three different estimators is given. The approach can be applied to any queueing system that posses a recurrent steady state stochastic process.

1. Introduction:

Queueing system applications are evident in many problem areas. A queueing system is simply a service system where customers demanding service arrive according to a given probability distribution. Each arriving customer waits for service, receives the requested

* Assistant Professor. Faculty of Engineering-Cairo University.

service, and then departs from the system, Here, customers may represent production orders in a production facility, automotives requesting repair at a maintenance shop, or trucks requesting unloading at a warehouse. Customers may also represent ships arriving at a seaport, areoplanes landing in an airport, emergency call requests at a fire station, or persons requesting service at a bank or any public service center. For systems of this nature, a researcher would be interested in analysing and evaluating few measures of system performance for purposes of system design and control. These measures include the expected number of customers waiting in the system, the expected waiting time per customer, optimal number of servers, and server utilization.

Simulation has become of particular importance among the alternative techniques to solve such queueing problems. Exact analytical approaches are not suitable since they give rise to sophistcated mathematical models that can not be solved difficiently. Simulation on the other hand would provide average measures that usually have a 50% chance of being true. However, when a simulation model is coupled with confidence interval estimation, estimated lower and upper bounds for the exact values of the measures of performance can be obtained. The probability that the exact value will be within these bounds can be made as high 0.9 or even 0.99.

It is our intent in this paper to show how confidence interval estimation can be implemented within a queueing system simulation model. First the necessary simulation frame work is introduced to collect independent observations for a relevant set of random variables. System measures of performance are expressed as ratioes of such random variables. Second, several confidence interval estimators for ratioes of random variables are presented for purposes of comparison. Finally, to illustrate the applicability of the approach, an application for an automotive maintenance shop is given. The paper is concluded with few remarks that ought to be considered while simulating a queueing system.

2. Simulation Framework:

When a queueing system is simulated as a continuous stochastic process observations regarding the number of customers in the system can be recorded (see Fig. (1)). If these observations are independent and identically distributed, then classical statistical inference tools can be readily applied. This concept is the motivation for the regenerative method suggested originally by Cox [1] and demonstrated in [2], [3], [5], [6] and [8].

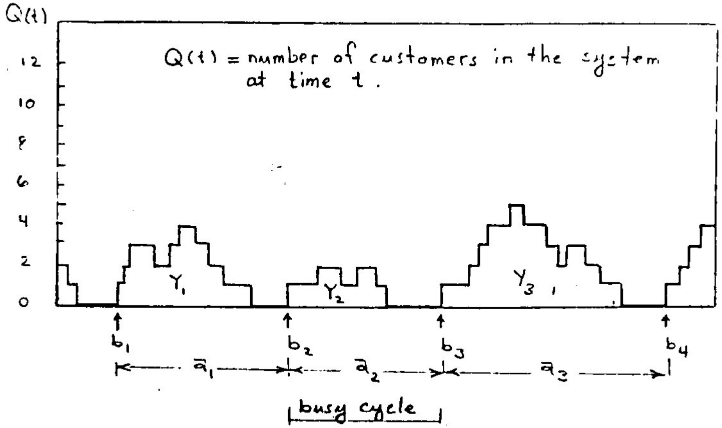

Fig (1) : The number of customers in the system stochastic process.

In the regenerative method, we look for cycles of lengthes a_i in the stochastic process which produces independent and identically distributed (i.i.d.) blocks of data. If we define quantities Y_i that can be observed and recorded in each cycle, then the pairs (Y_i, a_i), $i=1,\ldots,n$ are i.i.d. Clearly, such pairs will provide n independent observations that enable using statistical estimation tools. A typical examples will be to represent a_i as a sequence of busy cycles[1], since upon starting a new busy cycle the system will behave independently of its past history. For the analysis we need to represent the simulated queueing

(1) Busy cycle is defined as the time elapsed between two consecutive arrivals of customers who find the system empty (empty ≡ number of customers in the systems is zero)

system as a stochastic process $\{X(t), t \geq o\}$ with a finite state space. Let $o \leq b_1 \leq b_2 \ldots \leq b_n$ be an increasing sequence of regenerative times (see Fig (1)) such that the portions $X(t) : b_i \leq t \leq b_{i+1}$ are i.i.d. replica. In each such portions define the random variable:

$$Y_i = \int_{b_i}^{b_{i+1}} f[X(t)] \, dt,$$

where $f[.]$ is a general real valued function and $a_i = b_{i+1} - b_i$ is the length of the i^{th} cycle. The key result is that the sequence (Y_i, a_i), $i \geq 1$ consists of i.i.d. random vectors and that $E[f(X)] = E[Y_i] / E[a_i]$, the reader is referred to [5]. Among $f[.]$ which are of practical importance we have the steady state waiting time, the average number of customers in the system, and penalty or cost functions. Hence, we need to estimate the ratio $r = E[f(X)] = E[Y_i] / E[a_i]$ with the pairs (Y_i, a_i) i.i.d. random variables.

3. Ratio Estimators:

The problem of deriving a confidence interval for the ratio r is treated in different ways. Let $U_i = (Y_i, a_i)$ be a column vector, $i=1,2,\ldots,n$ and n is the sample size, denote the sample mean $\bar{U}=(\bar{Y}/\bar{a})=\frac{1}{n} \Sigma U_i$, and the sample covariance by

$$S = \begin{bmatrix} s_{11} & s_{12} \\ s_{12} & s_{22} \end{bmatrix} = \frac{1}{n-1} \Sigma\ (U_i - \bar{U})(U_i - \bar{U})',$$

Let $k = z^2_{1-v/2}/n$, $z_{1-v/2} = \Phi^{-1}\ (1-v/2)$, where Φ is the distribution function of a mean zero and variance 1 normal random variable. Reference [5] presented the following estimators with their 100 (1-v) percent confidence interval for the ratio r:

(1) Fieller estimator:

$$\hat{r} = \frac{\bar{Y}\ \bar{a} - k\ s_{12}}{\bar{a}^2 - k\ s_{22}},$$

$$(r - \frac{D^{\frac{1}{2}}}{(\bar{a}^2 - k\ s_{22})} < r < \hat{r} + \frac{D^{\frac{1}{2}}}{(\bar{a}^2 - k\ s_{22})}$$

where $\bar{D} = (\bar{Y}\ \bar{a} - k\ s_{12})^2 - (\bar{a}^2 - k\ s_{22})(\bar{Y}^2 - k\ s_{11})$

(2) Classical estimator:

$$\hat{r} = \frac{\bar{Y}}{\bar{a}},$$

$$(\hat{r} - z_{1-v/2}S_c/\bar{a}n^{\frac{1}{2}} < r < \hat{r} + z_{1-v/2}S_c n^{\frac{1}{2}})$$

where $S_c = (s_{11} - 2\ \hat{r}\ s_{12} + \hat{r}^2\ s_{22})^{\frac{1}{2}}$

(3) Jacknife estimator:

$$\hat{r} = \frac{1}{n} \sum_{i=1}^{n} w_i,$$

$$(\hat{r} - z_{1-v/2}\, S_j / r^{\frac{1}{2}} < r < \hat{r} + z_{1-v/2}\, S_j / n^{\frac{1}{2}})$$

$$\text{where } w_i = n(\bar{Y}/\bar{a}) - (n-1)\left(\sum_{j\neq i} Y_j / \sum_{j\neq i} a_i\right),$$

$$S_j = \left[\sum_{i=1}^{n} (w_i - r) \,/\, (n-1)\right]^{\frac{1}{2}}$$

References [3], [5] include a list of references for the development of the above estimators, as well as applications to the M/M/1 queueing system, periodic review inventory model, and the repairman problem. A comparison of results ranked the Jacknife first in terms of small bias.

4. Application to a Maintenance Shop:

Consider an automotive maintenance shop that receives two classes of customers requesting service: Passenger cars (class 1, and cargo trucks (class 2). The receipt of customer requests follow a composite interarrival time distribution with rate λ, and an arriving customer belongs to either one of two classes with probability p and 1-p respectively. The shop has a parallel service stations, each station is capable of performing the required service. Service times for each class follow an exponential distribution with service rate μ_1 and μ_2 for the two classes. An arriving class 2 customer is not admitted into the shop if the number of class 2 orders (trucks) in the system is k, ($k \leq s$). In queueing theory notation, this system is denoted by G/M M_2/s/N system, where N is the maximum number of customers of both classes allowed in the system. This system has an exact steady state solution which is available only for small s,N,k since it involves matrix operations. Our intended simulation should provide a 100 (1-v) percent confidence interval for the expected number of customers in the system when λ, μ_1, μ_2, p, s, k are known.

Let Q(t) be the stochastic process representing the number of customers in the system at time t, t>o, and E(Q) is its expected values, then

$$Y_i = \int_{b_i}^{b_i+1} Q(t)\, dt,$$

$$E(Q) = E(Y_i) \ / \ E(a_i)$$

where a_i is the i^{th} busy cycle, $a_i=b_{i+1}-b_i$. The graph of Q(t) process is shown in Fig (1). During the period of simulation, each busy cycle defines a single indepe-ndent observation for the pairs (Y_i,a_i) that can be recorded. Collecting n independent observations makes it possible to apply the previous analysis and calcu-late a confidence interval for the E(Q).

To implement this approach, a system with:

- number of paraller service stations	s	=	5
- customer arrival rate/unit time	λ	=	6
- service rate for class 1	μ_1	=	1
- service rate for class 2	μ_2	=	.25
- system maximum capacity	N	=	5
- maximum number of class 2 allowed	k	=	2
- probability of an arriving class 1	p	=	.4

is simulated.

For a sample of size 300, table (1) shows a comparison of E(Q) as estimated using the Classical, the Fieller, and the Jacknife estimators. The 90% confidence interval for each estimator is given between parenthesis, also estimates and confidence interval for the cumulative distribution function (CDF) of the number of customers in the system termed CDF(i). The last column is the table gives the exact steady state values for E(Q) and CDF(i) for purposes of comparison with estimated results. Table (2) exibits similar analysis when the interarrival time distribution is Gamma with coefficient of variation 0.447.

5. Conclusions:

We have presented an application of the regenerative method to queueing system simulation of a maintence shop. The essence of this method is to construct a simulation model such that a set of n independent observations can be recorded. This enables a researcher to apply the classical statistical estimation tools to obtain point and confidence interval estimators for the concerned measures of system performance (as E(Q) and CDF(i)). The approach can be extended to other queueing systems. Few remarks are in order:

1. The three different estimators converge to the exact value, and they all compare favourably. Hence it is better to use the classical estimator since it requires the least calculations.

2. All confidence intervals contain the exact values as they supposed to. A confidence interval can be made arbitrary tighter by increasing the sample size.

3. Experience shows that if the busy cycle (time for one observation) is long, as in systems with high traffic intensity, a smaller sample size is needed. (see [7] for a rough estimate of n).

4. If the busy cycle is too long, or there are no regenerative points, then a physical sampling unit must be selected. For example, one day of a service station, or a fixed number of hours for the system sperations can be used to generate a single observation. The method used is known as Batch means [4].

Table (1): Results for the expected number of customers E(Q), and the probability of i or less customers CDF(i) in the system when interarrival time distribution is exponential.

Sample size	Estimated measure	Classical estimator	Fieller estimator	Jacknife estimator	Exact value
300	E(Q)	3.642 (3.634,3.650)	3.642 (3.634,3.650)	3.642 (3.634,3.650)	3.645
300	CDF(Q)	.0011 (.0010,.0012)	.0011 (.0010,.0012)	.0011 (.0010,.0021)	.0010
300	CDF(1)	.0175 (.0169,0180)	.0175 (.0170,.0180)	.0175 (.0169,.0180)	.0170
300	CDF(2)	.1529 (.1505,.1553)	.153 (.1506,.1554)	.1529 (.1505,.1553)	.1522
300	CDF(3)	.4338 (.4302,.4373)	.4338 (.4303,.4373)	.4337 (.4302,.4373)	.4324
300	CDF(4)	.7523 (.7497,.7548)	.7497 (.7497,.7548)	.7522 (.7497,.7548)	.7515

Table (2): Results for the expected number of customers E(Q), and the probability of i or less customers in the system CDF(i) when interarrival time distribution is Gamma.

Sample size	Estimated measure	Classical estimator	Fieller estimator	Jacknife estimator	Exact value
100	E(Q)	3.618 (3.609,3.626)	3.618 (3.609,3.626)	3.618 (3.609,3.626)	3.612
100	CDF(0)	.0004 (.0003,.0005)	.0004 (.0003,.0005)	.0004 (.0003,.0005)	.0004
100	CDF(1)	.0123 (.0119,.0128)	.0123 (.0119,.0128)	.0123 (.0119,.0128)	.0121
100	CDF(2)	.1417 (.1393,.1440)	.1416 (.1392,.1440)	.1417 (.1393,.1440)	.1432
100	CDF(3)	.4416 (.4378,.4453)	.4414 (.4376,.4453)	.4416 (.4373,.4454)	.4447
100	CDF(4)	.7859 (.7831,.7887)	.7858 (.7829,.7886)	.7860 (.7831,.7881)	.7872

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