M.V.U.E. OF THE RELIABILITY FUNCTION FOR AN EXPONENTIAL-LIFETIME DISTRIBUTION TRUNCATED FROM ABOVE

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ABSTRACT

The estimator of the reliability function of an exponential lifetime distribution truncated from above at a known point has been considered by Holla[3]. In this paper we derive a minimum variance unbiased estimator (M.V.U.E.) if the truncation point is unknown.

1. INTRODUCTION

In most studies of systems reliability, the failure distributions of these systems are considered such that the lifetimes of these systems take values from zero to infinity. In many applications, we may note that these lifetimes have limits which are neither zero nor infinity. So, it would be more realistic in these applications if these lifetimes assumed not to be less or greater than a given age. Hence, the idea of using truncated exponential distribution in describing the system lifetimes arised. RAIN[1] presented a procedure by which a uniformaly most powerful tests as well as an optimum exact confidence limits can be obtained for the failure-rate in an exponential distribution truncated from above (i.e. the system lifetime does not exceed a given point of time).

In a paper presented by Epestein and sobel,[2] the system lifetime was supposed to be greater than an unknown point of time. The reliability estimation of a system whose lifetime has an above truncated exponential distribution, was studied by Holla[3]. The point of truncation was known. In our study this point is assumed to be unknown.

Consider n components whose lifetimes follow an above truncated exponential distribution given by

$$f(v) = \frac{\lambda e^{-\lambda v}}{1 - e^{-\lambda A}} \qquad o < v < A \tag{1.1}$$

where λ and A are unknown . The reliability function for such a component is given by

$$R(t) = \int_t^A \frac{\lambda e^{-\lambda v}}{1-e^{-\lambda A}} dv = 1-F(t) \qquad o < t < A \quad (1.2)$$

where

$$F(t) = \frac{1 - e^{-\lambda t}}{1 - e^{-\lambda A}}$$

is the distribution function of (1.1).

Our interest is to get a minimum variance unbiased estimators (M.V.U.E.) for each of f(v) and F(t) defined in (1.1) and (1.2) respectively. The lifetimes observations of the tested components can be obtained in two different ways of sampling, namely Type I and Type II sampling. In Type I sampling, the lifetimes of the n tested components are observed. These components; may be tested sequentially (i.e. test one component at a time) or simultaneously (i.e. test all components together). In Type II sampling, only the observation of the lifetimes of the first $r \leq n$ components are available. In section 2, our estimators will be considered when using sampling of Type I. In section 3, these estimators will be studied when sampling of Type II is used.

2. THE CASE OF TYPE I SAMPLING

Let $X_1, X_2, \ldots X_n$ be lifetimes observations of n components tested independently. So the likelihood function of these observations is given by

$$L(X_1, X_2, \ldots, X_n) = \left(\frac{\lambda}{1-e^{-\lambda A}} \right)^n e^{-\lambda \sum_{i=1}^{n} x_i} \qquad (2.1)$$

Let $X_{(n)}$ be the largest value of the above given observations. Denote by V the set of observations after choosing $X_{(n)}$

namely $V = (V_1, V_2, \ldots, V_{n-1})$. It can be easily seen that the conditional density function of v_i given $X_{(n)}$ is

$$f(v_i / X_{(n)}) = \frac{\lambda\, e^{-\lambda\, v_i}}{1-e^{-\lambda\, X_{(n)}}} \qquad o < v_i < X_{(n)} \qquad (2.2)$$
$$i = (o, 1, \ldots, n-1)$$

From (2.2), it follows immediatly that $X_{(n)}$ and $T = \sum_{i=1}^{n-1} v_i$ are sufficient and complete statistics for A and λ respectively (see Smith[4] and Tukey[5]). According to BAIN[1], we may get the conditional density function of T given $X_{(n)}$ as

$$g(T/X_{(n)}) = \frac{1}{n-2!} \left(\frac{\lambda}{1-e^{-\lambda X_{(n)}}}\right)^{n-1} e^{-\lambda T} \sum_{k=o}^{k_o} (-1)^k \binom{n-1}{k} (T-k\, X_{(n)})^{n-2} \qquad (2.3)$$

where

$$k_o X_{(n)} < T < (k_o + 1)\, X_{(n)}, \quad k_o = (o, 1, \ldots\, n-2)$$

From (2.2) and (2.3), the conditional joint density function of v_i and t is

$$k(v, T/X_{(n)}) = f(v/X_{(n)}).\ g(T-v\ /\ X_{(n)})$$

$$= \frac{1}{n-3!} \left(\frac{\lambda}{1-e^{-\lambda X_{(n)}}}\right)^{n-1} e^{-\lambda T} \sum_{k=o}^{k_o} (-1)^k \binom{n-2}{k} (T-v-k\, X_{(n)})^{n-3}$$

(2.4)

Since $k_o X_{(n)} + v < T < (k_o+1) X_{(n)} + v$, $k_o = (o,1,\dots n-3)$
Using (2.4), the conditional density function of T given $X_{(n)}$ can be rewritten in the following form

$$g(T/X_{(n)}) = \frac{1}{n-2!}\left(\frac{\lambda}{1-e^{-\lambda X_{(n)}}}\right)^{n-1} e^{-\lambda T} \begin{cases} \sum_{k=o}^{k_o} (-1)^k \binom{n-1}{k} (T-k\,X_{(n)})^{n-2} & k_o X_{(n)} < T < (k_o+1) X_{(n)} \\ \sum_{k=o}^{k_o+1} (-1)^k \binom{n-1}{k} (T-k\,X_{(n)})^{n-2} & (k_o+1) X_{(n)} < T < (k_o+2) X_{(n)} \end{cases}$$

$$k_o = o,1,\dots, n-3 \qquad (2.5)$$

From (2.4) and (2.5), the conditional density function of v given T and $X_{(n)}$ is

$$f_1(v/T, X_{(n)}) = \frac{k(v, T/X_{(n)})}{g(T/X_{(n)})}$$

$$= \begin{cases} (n-2)\dfrac{\sum_{k=o}^{k_o} (-1)^k \binom{n-2}{k} (T-v-k\,X_{(n)})^{n-3}}{\sum_{k=o}^{k_o} (-1)^k \binom{n-1}{k} (T-k\,X_{(n)})^{n-2}} = I(k_o) & o < v < T-k_o X_{(n)} \\ (n-2)\dfrac{\sum_{k=o}^{k_o-1} (-1)^k \binom{n-2}{k} (T-v-k\,X_{(n)})^{n-3}}{\sum_{k=o}^{k_o} (-1)^k \binom{n-1}{k} (T-k\,X_{(n)})^{n-2}} = I(k_o) & T-(k_o)X_{(n)} < v < X_{(n)} \end{cases}$$

$$I(-1) = I(n-3) = o \qquad k_o = o,1,\dots, n-3 \qquad (2.6)$$

Theorem (1):

The statistic $\frac{n-1}{n}\ f_1(v/T_1\ X_{(n)})$ is a minimum variance unbiased estimator for the probability density function f(v) defined in (1.1).

Proof:

We have

$$E_{(T,X_{(n)})}[f_1(v/T,X_{(n)})] = \int_{(T,X_{(n)})} f_o(X_{(n)})\, g(T/X_{(n)})\, f_1(v/T,X_{(n)})\, dT dx_{(n)} \tag{2.7}$$

where

$$f_o(X_{(n)}) = \frac{n!}{n-1!}\left[\frac{1-e^{-\lambda X_{(n)}}}{1-e^{-\lambda A}}\right]^{n-1} \frac{\lambda e^{-\lambda X_{(n)}}}{1-e^{-\lambda A}}$$

i.e.

$$E_{(T,X_{(n)})}[f_1(v/T,X_{(n)})] = \int_{X_{(n)}=o}^{A} f_o(X_{(n)})\left[\int_T g(T/X_{(n)})\, f_1(v/T,X_{(n)})\, dT\right] \cdot dX_{(n)}$$

Using (2.5) & (2.6)

$$\int_T g(T/X_{(n)})\ f_1(v/T,X_{(n)})\ dT = e^{-\lambda v}\ \frac{\lambda^{n-2}(B_1 + B_2)}{(1-e^{-\lambda X_{(n)}})^{n-1}}$$

where

$$B_1 = \sum_{k_o=o}^{n-3} \sum_{k=o}^{k_o} (-1)^k \binom{n-2}{k} \sum_{j=o}^{n-4} \frac{X_{(n)}^{n-3-j}}{n-3-j!} \cdot \frac{e^{-\lambda k_o X_{(n)}}}{\lambda^j}$$

$$\cdot\{(k_o-k)^{n-3-j} - (k_o-k+i)^{n-3-j}\, e^{-\lambda X_{(n)}}\}$$

and

$$B_2 = \sum_{k_o=o}^{n-3} \sum_{k=o}^{k_o} (-1)^k \binom{n-2}{k} \frac{e^{-\lambda k_o X_{(n)}}}{\lambda^{n-3}} (1-e^{-\lambda X_{(n)}})$$

It may be shown that

$$B_1 = o$$

$$B_2 = \lambda^{-(n-3)} (1-e^{-\lambda X_{(n)}})^{n-2}$$

and hence,

$$E_{(T,X_{(n)})}[f_1(v/T,X_{(n)})] = \lambda e^{-\lambda v} \int_{x_{(n)}=o}^{A} f_o(X_{(n)}) \frac{d X_{(n)}}{(1-e^{-\lambda X(n)})} \quad (2.8)$$

Substituting for $f_o(X_o)$ from (2.7) and performing the integration in (2.8) we obtain

$$E_{(T,X_{(n)})}[f_1(v/T,X_{(n)})] = \frac{n}{n-1} \frac{\lambda e^{-\lambda v}}{1-e^{-\lambda A}} \qquad o < v < A$$

from which it follows that $\frac{n-1}{n} f_1(v/T, X_{(n)})$ is an unbiased estimator of $f(v) = \frac{\lambda e^{-\lambda v}}{1-e^{-\lambda A}}$. Since $(T, X_{(n)})$ is sufficient for (λ,A) it follows by the Blackwell-Rao theorem that this estimat is M.V.U.E.

<u>Theorem (2):</u>

Let v denotes a random variable having the probability density function f_1 defined in (2.6). For any specified t

define the random variable Z as

$$Z = \psi(v) = \begin{cases} o & \text{for} \quad v > t \\ 1 & \text{for} \quad v < t \end{cases}$$

then

$$E_v(Z/T, X_{(n)}) = \begin{cases} \dfrac{\sum_{k=o}^{k_o}(-1)^k\binom{n-2}{k}\{(T-k\,X_{(n)})^{n-2}-(k_o X_{(n)}-k\,X_{(n)})^{n-2}\}}{\sum_{k=o}^{k_o}(-1)^k\binom{n-1}{k}\,(T-k\,X_{(n)})^{n-2}} = Z_1(k_o) \\ \qquad k_o X_{(n)} < T < k_o\,X_{(n)}+t \\[2ex] \dfrac{\sum_{k=o}^{k_o}(-1)^k\binom{n-2}{k}\{(T-k\,X_{(n)})^{n-2}-(T-t-k\,X_{(n)})^{n-2}\}}{\sum_{k=o}^{k_o}(-1)^k\binom{n-1}{k}\,(T-k\,X_{(n)})^{n-2}} = Z_2(k_o) \\ \qquad k_o X_{(n)}+t < T < (k_o+1)X_{(n)} \\[2ex] \dfrac{\sum_{k=o}^{k_o-1}(-1)^k\binom{n-2}{k}\{[(k_o+1)X_{(n)}-k\,X_{(n)}]^{n-2}-(T-t-kX_{(n)})^{n-2}\}}{\sum_{k=o}^{k_o}(-1)^k\,(n-1)\,(T-k\,X_{(n)})^{n-2}} \\ \qquad = Z_3(k_o-1) \\ \qquad (k_o)X_{(n)} < T < (k_o)X_{(n)}+t \end{cases}$$

$$k_o = o,\ 1, \ldots,\ n-3 \qquad (2.9)$$

$$Z_1(n-3) = Z_2(n-3) = o \quad \& \quad Z_3(-1) = o$$

and the statistic

$\frac{n-1}{n} E_v(Z/T, X_{(n)})$ is M.V.U.E. for F(t)

<u>Proof</u>:

We have

$$E_v(Z/T,X_{(n)}) = \Pr(v < t/T, X_{(n)})$$

$$= \begin{cases} \int_o^{T-k_oX_{(n)}} f_1(v/T,X_{(n)})dv & k_oX_{(n)}<T<k_oX_{(n)}+t \\ \int_o^t f_1(v/T,X_{(n)})dv & k_oX_{(n)}+t<T<(k_o+1)X_{(n)} \\ \int_{T-(k_o)X_{(n)}}^t f_1(v/T,X_{(n)})dv & (k_o)X_{(n)}<T<(k_o)X_{(n)}+t \end{cases}$$

which are readily verified to reduce to the results given above. Furthermore, following the method of proof given in theorem 2.1, it may be shown that

$$E_{(T,X_{(n)})}\{E_v(Z/T,X_{(n)})\} = \frac{n}{n-1}\,\frac{1-e^{-\lambda t}}{1-e^{-\lambda A}} .$$

Hence $\frac{n-1}{n} E_v(Z/T,X_{(n)})$ is an unbiased estimate of the distribution function F(t) and by the Blackwell-Rao theorem it is a M.V.U.E. of F(t). It further follows from this result and (1.2) that

$$1 - \frac{n-1}{n} E_v(Z/T, X_{(n)})$$

is a M.V.U.E. of the reliability function R(t).

3. THE CASE OF TYPE II SAMPLING

Suppose that on a test of n components whose lifetimes follows the exponential distribution defined in (1.1), the first r failure times are $X_1 \leq X_2 \leq \ldots \leq X_r$. Then the joint density of $X_1, X_2, \ldots, X_r$ is

$$f(X_1, X_2, \ldots, X_r) = \frac{n!}{n-r!} \left(\frac{\lambda}{1-e^{-\lambda A}}\right)^r e^{-\lambda \sum_{i=1}^{r} X_i} \left[\frac{e^{-\lambda X_r} - e^{-\lambda A}}{1 - e^{-\lambda A}}\right]^{n-r} \tag{3.1}$$

$$o \leq X_1 \leq X_2 \leq \ldots \leq X_r \leq A$$

From which it follows immediately that $(X_r, \sum_{i=1}^{r-1} X_i)$ is a sufficient statistic for (A, λ) [see also smith[5] and Takey[4]].

Lemma:

Let $y_1 = (r-1)X_1$, $y_2 = (r-2)(X_2-X_1), \ldots \ldots, y_{r-1} = (X_{r-1}-X_{r-}$

Conditional upon $X_r = x_r$, the random variables $y_1, y_2, \ldots,$ y_{r-1} are independent and each has the truncated exponential distribution

$$f(y/X_r) = \frac{\lambda e^{-\lambda y}}{1-e^{-\lambda X_r}} \qquad o < y < X_r \tag{3.2}$$

This lemma is easily proved by adopting proof of similar lemma by Epstein and Sobel[2].

Setting

$$W = \sum_{i=1}^{r-2} y_i$$

it follows from Bain and weeks[1] (1964) that conditional upon X_r, the density of W is

$$f(W/X_r) = \frac{1}{r-3!} \left(\frac{\lambda}{1-e^{-\lambda X_r}}\right)^{r-2} e^{-\lambda w} \sum_{k=o}^{k_o} (-1)^k \binom{r-2}{k} (W-k\, X_r)^{r-3} \tag{3.3}$$

$$k_o X_r \leq W \leq (k_o+1)X_r, \quad k_o = o, 1, \ldots, r-3$$

Now, let $T_r = W + Y_{r-1}$ and $V = Y_{r-1}$. Noting that W and Y_{r-1} are independent and using (3.2) and (3.3) it follows that the joint density of T_r and v given X_r is,

$$k(v, T_r/X_r) = \frac{1}{r-3!} \left(\frac{\lambda}{1-e^{-\lambda X_r}}\right)^{r-1} e^{-\lambda T_r} \sum_{k=o}^{k_o} (-1)^k \binom{r-2}{k} (T_r - v - k\, X_r)^{r-3} \tag{3.4}$$

where

$$k_o X_r + v < T_r < (k_o+1)X_r + v, \quad k_o = o, 1, \ldots, r-3$$

Integrating out v we thus get the conditional density of T_r as

$$g(T_r/X_r)=\begin{cases} \dfrac{1}{r-2!}\left(\dfrac{\lambda}{1-e^{-\lambda X_r}}\right)^{r-1}\displaystyle\sum_{k=o}^{k_o}(-1)^k\binom{r-1}{k}(T_r-k\,X_r)^{r-2}\,e^{-\lambda T_r} \\ \qquad\qquad k_oX_r < T_r < (k_o+1)\,X_r \\ \\ \dfrac{1}{r-2!}\left(\dfrac{\lambda}{1-e^{-\lambda X_r}}\right)^{r-1}\displaystyle\sum_{k=o}^{k_o+1}(-1)^k\binom{r-1}{k}(T_r-k\,X_r)^{r-2}\,e^{-\lambda T_r} \\ \qquad\qquad (k_o+1)X_r < T_r < (k_o+2)\,X_r \end{cases}$$

$$k_o = o,1,\ldots,r-3 \qquad (3.5)$$

It follows that the conditional density of v given T and X_r is

$$f_1(v/T_r,X_r)=\frac{k(v,T_r/X_r)}{g(T_r/X_r)}=\begin{cases} (r-2)\,\dfrac{\sum_{k=o}^{k_o}(-1)^k\binom{r-2}{k}(T_r-v-k\,X_r)^{r-3}}{\sum_{k=o}^{k_o}(-1)^k\binom{r-1}{k}(T_r-k\,X_r)^{r-2}}=Z(k_o) \\ \qquad\qquad o < v < T_r-k_oX_r \\ \\ (r-2)\,\dfrac{\sum_{k=o}^{k_o-1}(-1)^k\binom{r-2}{k}(T_r-v-k\,X_r)^{r-3}}{\sum_{k=o}^{k_o}(-1)^k\binom{r-1}{k}(T_r-k\,X_r)^{r-2}}=Z(k_o- \\ \qquad\qquad T_r-(k_o)X_r < v < X_r \end{cases}$$

$$k_o = o,1,\ldots\ldots,r-3$$

$$Z(r-3)=o \quad \& \quad Z(-1)=o \qquad (3.6)$$

Theorem (1)`:

The statistic $\frac{r-1}{n} f_1(v/T_r, X_r)$ is a minimum variance unbiased estimator of the probabilty density function defined in (1.1).

Proof:

This theorem can be proved similarly as theorem (1).

Theorem (2)`:

Let v denotes a random variable with the probability density function $f(V/T_r, A_r)$ defined in (3.6). For any specified t define the random variable,

$$Z = \psi(v) = \begin{cases} o & \text{for} \quad v > t \\ 1 & \text{for} \quad v < t \end{cases}$$

then

$$E_v(Z/T_r, X_r) = \begin{cases} \dfrac{\sum_{k=o}^{k_o} (-1)^k \binom{r-2}{k} [(T_r - k X_r)^{r-2} - (k_o X_r - k X_r)^{r-2}]}{\sum_{k=o}^{k_o} (-1)^k \binom{r-1}{k} (T_r - k X_r)^{r-2}} = I_1(k_o) & k_o X_r < T_r < k_o X_r + t \\[2ex] \dfrac{\sum_{k=o}^{k_o} (-1)^k \binom{r-2}{k} [(T_r - k X_r)^{r-2} - (T_r - t - k X_r)^{r-2}]}{\sum_{k=o}^{k_o} (-1)^k \binom{r-1}{k} (T_r - k X_r)^{r-2}} = I_2(k_o) & k_o X_r + t < T_r < (k_o+1) X_r \\[2ex] \dfrac{\sum_{k=o}^{k_o - 1} (-1)^k \binom{r-2}{k} [((k_o+1) X_r - k X_r)^{r-2} - (T_r - t - k X_r)^{r-2}]}{\sum_{k=o}^{k_o} (-1)^k \binom{r-1}{k} (T_r - k X_r)^{r-2}} = I_3(k_o) & (k_o) X_r < T_r < (k_o) X_r + t \end{cases}$$

where $I_1(r-2) = I_2(r-2) = 0$ & $I_3(-1) = o$

$$k_o = o, 1, \ldots, r-3 \tag{3.7}$$

and the statistic $\frac{r-1}{n} E_v(Z/T_r, X_r)$ is M.V.U.E, for F(t).

Proof:

See the proof of theorem (2).

Note: Putting r = n in (3-7) we get immediatly (2.9).

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