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FINITE INTEGRALS INVOLVING INCOMPLETE ALEPH FUNCTIONS, EXTENSION OF THE MITTAG-LEFFLER FUNCTION AND ELLIPTIC INTEGRAL OF FIRST SPECIES

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Abstract. In recent years, many authors have explored wide variety of integrals involving special functions. To continue the study, in the present paper, we evaluate the general finite integral involving the elliptic integral of first species, extension of Mittag-Leffler function and the incomplete Aleph-functions by application of known integral given by Brychkov. At the end, in consequence of general character of functions involved in this study, we shall see new results as several corollaries.

1. Introduction and Preliminaries

Integral formulas involving special functions play significant role to solve various problems of science and engineering. A number of authors have obtained interesting ordinary integrals as well as fractional integrals related to special functions, see [7, 8, 9, 10, 11, 12, 17, 20, 26]. Srivastava et al. [24] have studied the incomplete Gamma function (IGFs). Using the definition of IGFs, they introduced incomplete hypergeometric function. In 2018, Srivastava et al.[25] have introduced and studied the incomplete H-function and the incomplete $\overline{H}$-function . Several workers, Bansal et al.[2] Bansal and Kumar [1] and Bansal et al. [3] have introduced more generalized form of incomplete special functions. They studied the incomplete Aleph-function, the incomplete I-function and calculate the integrals about the incomplete H-function respectively. In continuation of the work of unified integrals, in this paper, we produce a generalized unified finite integral concerning the product of incomplete Aleph-function, extension of the Mittag-Leffler Function and the elliptic integral of first species by application of known integral given by

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Brychkov ([4], Eq.8 page 269).

The IGFs $\gamma(\alpha,x)$ and $\Gamma(\alpha,x)$ are defined in the following manner :

$$\gamma(\alpha,x)=\int_0^x u^{\alpha-1}e^{-u}\mathrm{d}u \quad (\Re(\alpha)>0;x\geq 0). \tag{1}$$

$$\Gamma(\alpha,x)=\int_x^\infty u^{\alpha-1}e^{-u}\mathrm{d}u \quad (x\geq 0;\Re(\alpha)>0 \text{ when } x=0). \tag{2}$$

Definitions (1) and (2) give following relation:

$$\Gamma(\alpha)=\gamma(\alpha,x)+\Gamma(\alpha,x) \quad (\Re(\alpha)>0). \tag{3}$$

Numerous generalizations of various special functions has been introduced time to time by different authors. The incomplete Aleph function is most generalized form of these special functions. Now, we give the expression of the incomplete Aleph-functions ${}^{(\Gamma)}\aleph^{m,n}_{p_i,q_i,\tau_i;r}(z)$ and ${}^{(\gamma)}\aleph^{m,n}_{p_i,q_i,\tau_i;r}(z)$ defined by Bansal *et al.*[2] :

$${}^{(\Gamma)}\aleph^{m,n}_{p_i,q_i,\tau_i;r}(z)={}^{(\Gamma)}\aleph^{m,n}_{p_i,q_i,\tau_i;r}\left(z\,\middle|\,\begin{matrix}(a_1,A_1,x),(a_j,A_j)_{2,n},[\tau_i(a_{ji},A_{ji})]_{n+1,p_i}\\(g_j,G_j)_{1,m},[\tau_i(g_{ji},G_{ji})]_{m+1,q_i}\end{matrix}\right)$$

$$=\frac{1}{2\pi\omega}\int_L\frac{\Gamma(1-a_1-A_1s,x)\prod_{j=2}^n\Gamma(1-a_j-A_js)\prod_{j=1}^m\Gamma(g_j+G_js)}{\sum_{i=1}^r\tau_i\left[\prod_{j=m+1}^{q_i}\Gamma(1-g_{ji}-G_{ji}s)\prod_{j=n+1}^{p_i}\Gamma(a_{ji}+A_{ji}s)\right]}z^{-s}\mathrm{d}s, \tag{4}$$

and

$${}^{(\gamma)}\aleph^{m,n}_{p_i,q_i,\tau_i;r}(z)={}^{(\gamma)}\aleph^{m,n}_{p_i,q_i,\tau_i;r}\left(z\,\middle|\,\begin{matrix}(a_1,A_1,x),(a_j,A_j)_{2,n},[\tau_i(a_{ji},A_{ji})]_{n+1,p_i}\\(g_j,G_j)_{1,m},[\tau_i(g_{ji},G_{ji})]_{m+1,q_i}\end{matrix}\right)$$

$$=\frac{1}{2\pi\omega}\int_L\frac{\gamma(1-a_1-A_1s,x)\prod_{j=2}^n\Gamma(1-a_j-A_js)\prod_{j=1}^m\Gamma(g_j+G_js)}{\sum_{i=1}^r\tau_i\left[\prod_{j=m+1}^{q_i}\Gamma(1-g_{ji}-G_{ji}s)\prod_{j=n+1}^{p_i}\Gamma(a_{ji}+A_{ji}s)\right]}z^{-s}\mathrm{d}s, \tag{5}$$

The incomplete $\aleph$-functions ${}^{(\Gamma)}\aleph^{m,n}_{p_i,q_i,\tau_i;r}(z)$ and ${}^{(\gamma)}\aleph^{m,n}_{p_i,q_i,\tau_i;r}(z)$ defined above exits for $x\geq 0$ and the following validities conditions.
The contour L is in the s-plane and run from $\sigma-i\infty$ to $\sigma+i\infty$ where σ if is a real number with loop, if necessary to ensure that the poles of $\Gamma(1-a_j-A_js), j=2,\cdots,n$ to the right of the contour L and the poles of $\Gamma(g_j+G_js), j=1,\cdots,m$ to the left of the contour L. The parameters τ_i, m,n,p_i,q_i are positive numbers satisfying $0\leq n\leq p_i$, $0\leq m\leq q_i$ and a_j,g_j,a_{ji},g_{ji} are complex numbers. The poles of the integrand are simple. Following conditions hold:

$$\Omega_i>0,|\arg(z)|<\frac{\pi}{2}\Omega_i,\ i=1,\cdots,r \tag{6}$$

$$\Omega_j\geq 0,|\arg(z)|<\frac{\pi}{2}\Omega_i \text{ and } \Re(\zeta_i)+1<0 \tag{7}$$

where

$$\Omega_i=\sum_{j=1}^n A_j+\sum_{j=1}^m G_j-\tau_i\left(\sum_{j=n+1}^{p_i}A_{ji}+\sum_{j=m+1}^{q_i}G_{ji}\right) \tag{8}$$

and

$$\zeta_i=\sum_{j=1}^{m}g_j-\sum_{j=1}^{n}a_j+\tau_i\left(\sum_{j=m+1}^{q_i}g_{ji}-\sum_{j=n+1}^{p_i}a_{ji}\right)+\frac{p_i-q_i}{2},\ i=1,\cdots,r \tag{9}$$

We can have following relation:

$$^{(\Gamma)}\aleph^{m,n}_{p_i,q_i,\tau_i;r}(z)+^{(\gamma)}\aleph^{m,n}_{p_i,q_i,\tau_i;r}(z)=\aleph^{m,n}_{p_i,q_i,\tau_i;r}(z). \tag{10}$$

Where $\aleph^{m,n}_{p_i,q_i,\tau_i;r}(z)$ is the Aleph-function defined by Südland [27]. Let's see the special cases.

Taking $\tau_i\to 1$, then the incomplete Aleph-functions $^{(\Gamma)}\aleph^{m,n}_{p_i,q_i,\tau_i;r}(z)$ and $^{(\gamma)}\aleph^{m,n}_{p_i,q_i,\tau_i;r}(z)$ reduce respectively to incomplete I-functions $^{(\Gamma)}I^{m,n}_{p_i,q_i;r}(z)$ and $^{(\gamma)}I^{m,n}_{p_i,q_i;r}(z)$ where:

$$^{(\Gamma)}I^{m,n}_{p_i,q_i;r}(z)={}^{(\Gamma)}I^{m,n}_{p_i,q_i;r}\left(z\,\middle|\,\begin{matrix}(a_1,A_1,x),(a_j,A_j)_{2,n},(a_{ji},A_{ji})_{n+1,p_i}\\(g_j,G_j)_{1,m},(g_{ji},G_{ji})_{m+1,q_i}\end{matrix}\right)$$

$$=\frac{1}{2\pi\omega}\int_L\frac{\Gamma(1-a_1-A_1s,x)\prod_{j=2}^{n}\Gamma(1-a_j-A_js)\prod_{j=1}^{m}\Gamma(g_j+G_js)}{\sum_{i=1}^{r}\left[\prod_{j=m+1}^{q_i}\Gamma(1-g_{ji}-G_{ji}s)\prod_{j=n+1}^{p_i}\Gamma(a_{ji}+A_{ji}s)\right]}z^{-s}\mathrm{d}s, \tag{11}$$

and

$$^{(\gamma)}I^{m,n}_{p_i,q_i;r}(z)={}^{(\gamma)}I^{m,n}_{p_i,q_i;r}\left(z\,\middle|\,\begin{matrix}(a_1,A_1,x),(a_j,A_j)_{2,n},(a_{ji},A_{ji})_{n+1,p_i}\\(g_j,G_j)_{1,m},(g_{ji},G_{ji})_{m+1,q_i}\end{matrix}\right)$$

$$=\frac{1}{2\pi\omega}\int_L\frac{\gamma(1-a_1-A_1s,x)\prod_{j=2}^{n}\Gamma(1-a_j-A_js)\prod_{j=1}^{m}\Gamma(g_j+G_js)}{\sum_{i=1}^{r}\left[\prod_{j=m+1}^{q_i}\Gamma(1-g_{ji}-G_{ji}s)\prod_{j=n+1}^{p_i}\Gamma(a_{ji}+A_{ji}s)\right]}z^{-s}\mathrm{d}s, \tag{12}$$

under the same conditions that above with $\tau_i\to 1$. Now, we suppose $r=1$, the incomplete I-functions $^{(\Gamma)}I^{m,n}_{p_i,q_i;r}(z)$ and $^{(\gamma)}I^{m,n}_{p_i,q_i;r}(z)$ reduce respectively to incomplete H-functions $^{(\Gamma)}H^{m,n}_{p,q}(z)$ and $^{(\gamma)}H^{m,n}_{p,q}(z)$ where

$$^{(\Gamma)}H^{m,n}_{p,q}(z)={}^{(\Gamma)}H^{m,n}_{p,q}\left(z\,\middle|\,\begin{matrix}(a_1,A_1,x),(a_j,A_j)_{2,p}\\(g_j,G_j)_{1,q}\end{matrix}\right)$$

$$=\frac{1}{2\pi\omega}\int_L\frac{\Gamma(1-a_1-A_1s,x)\prod_{j=2}^{n}\Gamma(1-a_j-A_js)\prod_{j=1}^{m}\Gamma(g_j+G_js)}{\prod_{j=m+1}^{q}\Gamma(1-g_j-G_js)\prod_{j=n+1}^{p}\Gamma(a_j+A_js)}z^{-s}\mathrm{d}s, \tag{13}$$

and

$$^{(\gamma)}H^{m,n}_{p,q}(z)={}^{(\gamma)}H^{m,n}_{p,q}\left(z\,\middle|\,\begin{matrix}(a_1,A_1,x),(a_j,A_j)_{2,p}\\(g_j,G_j)_{1,q}\end{matrix}\right)$$

$$=\frac{1}{2\pi\omega}\int_L\frac{\gamma(1-a_1-A_1s,x)\prod_{j=2}^{n}\Gamma(1-a_j-A_js)\prod_{j=1}^{m}\Gamma(g_j+G_js)}{\prod_{j=m+1}^{q}\Gamma(1-g_j-G_js)\prod_{j=n+1}^{p}\Gamma(a_j+A_js)}z^{-s}\mathrm{d}s, \tag{14}$$

under the same conditions verified by the incomplete I-functions with $r=1$. By using the formula (11) and (12), we have the following relations:

$$^{(\Gamma)}I^{m,n}_{p_i,q_i;r}(z)+^{(\gamma)}I^{m,n}_{p_i,q_i,r}(z)=I^{m,n}_{p_i,q_i;r}(z) \tag{15}$$

the function $I^{m,n}_{p_i,q_i,r}(z)$ being the function defined by Saxena [22] and

$$^{(\Gamma)}H^{m,n}_{p,q}(z) +^{(\gamma)} H^{m,n}_{p,q}(z) = H^{m,n}_{p,q}(z) \tag{16}$$

The complete elliptic integrals of first species are defined by, (see Whittaker and Watson [28], p. 515).

$$K(k) = \int_0^{\frac{\pi}{2}} \frac{\mathrm{d}\theta}{\sqrt{1-k^2\sin^2\theta}} = \int_0^1 \frac{\mathrm{d}t}{\sqrt{(1-t^2)(1-k^2t^2)}} \tag{17}$$

The extended Euler's Beta function is defined by Chaudhry *et al.* [5], see also Chaudhry and Zubair [6] for more information.

$$B_p(x,y) = B(x,y;p) = \int_0^1 w^{x-1}(1-w)^{y-1}\exp\left[-\frac{p}{w(1-w)}\right]\mathrm{d}w \tag{18}$$

where $\Re(p), \Re(x), \Re(y) > 0$.

The Mittag-Leffler (M-L) function and its various extensions provide solutions to certain problems established in terms of fractional order differential, integral and difference equations. Many authors have introduced a number of generalizations of this function due to significant applications in fractional calculus and related areas. Recently Mittal [16] has introduced and studied the following extended form of M-L function by applying the definition (18) of extended Euler's Beta function, which is defined as follows:

$$E^{(\gamma,c);\mathbf{q}}_{\alpha,\beta}(x;\mathbf{p}) = \sum_{l=0}^{\infty} \frac{B_{\mathbf{p}}(\gamma+l\mathbf{q}, c-\gamma)}{B(\gamma,c-\gamma)} \frac{(c)_{l\mathbf{q}}}{\Gamma(\alpha l+\beta)} \frac{x^l}{l!} \tag{19}$$

where $\alpha,\beta,\gamma,c,\mathbf{p} \in \mathbb{C}, \Re(\mathbf{p}), \Re(c), \Re(\alpha), \Re(\beta), \Re(\gamma) > 0, \mathbf{q} < \Re(\alpha)+1$, for the sake of brevity, we will use the notations

$$A^{(\gamma,c);\mathbf{q}}_{\alpha,\beta}(\mathbf{p};l) = \frac{B_{\mathbf{p}}(\gamma+l\mathbf{q}, c-\gamma)}{B(\gamma,c-\gamma)} \frac{(c)_{l\mathbf{q}}}{\Gamma(\alpha l+\beta)} \frac{1}{l!} \tag{20}$$

Now, we consider the extented M-L function defined by Özarslan and Yilmaz [18],

$$E^{(\gamma;c)}_{\alpha,\beta}(x;\mathbf{p}) = \sum_{l=0}^{\infty} \frac{B_{\mathbf{p}}(\gamma+l, c-\gamma)}{B(\gamma,c-\gamma)} \frac{(c)_l}{\Gamma(\alpha l+\beta)} \frac{x^l}{l!} \tag{21}$$

We note

$$A^{(\gamma,c)}_{\alpha,\beta}(\mathbf{p};l) = \frac{B_{\mathbf{p}}(\gamma+l, c-\gamma)}{B(\gamma,c-\gamma)} \frac{(c)_l}{\Gamma(\alpha l+\beta)} \frac{1}{l!} \tag{22}$$

where $\alpha,\beta,\gamma,c,\mathbf{p} \in \mathbb{C}, \Re(\mathbf{p}), \Re(c), \Re(\gamma) > 0$.

Taking $\frac{B_{\mathbf{p}}(\gamma+l,c-\gamma)}{B(\gamma,c-\Gamma)} \frac{(c)_l}{\Gamma(\alpha l+\beta)} \frac{1}{l!} \to \frac{(\gamma)_l}{(c)_l}$, we have the generalized M-L function, see Salim [21].

$$E^{\gamma,c}_{\alpha,\beta}(z) = \sum_{l=0}^{\infty} \frac{(\gamma)_l}{\Gamma(\alpha k+\beta)} \frac{z^l}{(c)_l} \tag{23}$$

where $\alpha,\beta,\gamma,c \in \mathbb{C}, \Re(\alpha), \Re(\beta), \Re(c), \Re(\gamma) > 0$.
We use the notations

$$B^{\gamma,c}_{\alpha,\beta}(l) = \frac{(\gamma)_l}{\Gamma(\alpha l+\beta)(c)_l} \tag{24}$$

Let $c \to 1$, we obtain, Generalized M-L function defined by Prabhakar [19],

$$E^{\gamma}_{\alpha,\beta}(z) = \sum_{l=0}^{\infty} \frac{(\gamma)_l}{\Gamma(\alpha l+\beta)} \frac{z^l}{l!} \tag{25}$$

where $\alpha, \beta, \gamma \in \mathbb{C}, \Re(\alpha) > 0$.
We will use the notation:

$$A^{\gamma}_{\alpha,\beta}(l) = \frac{(\gamma)_l}{\Gamma(\alpha l+\beta)} \frac{1}{l!} \tag{26}$$

Taking $\gamma \to 1$, we have the function defined by Wiman [29, 30], also see [31].

$$E_{\alpha,\beta}(z) = \sum_{l=0}^{\infty} \frac{z^l}{\Gamma(\alpha l+\beta)} \tag{27}$$

where $\alpha, \beta \in \mathbb{C}, \Re(\alpha) > 0$.
We will pose

$$A_{\alpha,\beta}(l) = \frac{1}{\Gamma(\alpha l+\beta)} \tag{28}$$

Let $\beta \to 1$, we obtain the function defined and studied by Mittag-leffler [13, 14, 15],

$$E_{\alpha}(z) = \sum_{l=0}^{\infty} \frac{z^l}{\Gamma(\alpha l+1)} \tag{29}$$

where $\alpha \in \mathbb{C}, \Re(\alpha) > 0$.
We will pose

$$A_{\alpha}(l) = \frac{1}{\Gamma(\alpha l+1)} \tag{30}$$

2. **Required Integral**

In this section, we give a general finite integral given by Brychkov, see ([4], Eq.8, page 269).

Lemma 1.

$$\int_0^1 \frac{x^{s-1}}{\sqrt{1+a^2x}} \ln(a\sqrt{x} + \sqrt{(1+a^2x)} K(\sqrt{1-x}) \mathrm{d}x = \frac{\pi a \Gamma^2\left(s+\frac{1}{2}\right)}{2\Gamma^2\left(s+1\right)} {}_4F_3\left(\begin{array}{c} 1, 1, s+\frac{1}{2}, s+\frac{1}{2} \\ \frac{3}{2}, s+1, s+1 \end{array} \middle| -a^2 \right) \tag{31}$$

where ${}_pF_q$ is generalized hypergeometric function and $\Re(s) > -\frac{1}{2}, |\arg(1+a^2)| < \pi$.

3. **Main Integral**

In this section, on the basis of known integral (31), we study a generalization of the finite integral involving the product of elliptic integral of first species, extended form of M-L function and incomplete Aleph-funtion.

Theorem 1.

$$\int_0^1 \frac{x^{s-1}}{\sqrt{1+a^2x}} \ln(a\sqrt{x}+\sqrt{(1+a^2x)}K(\sqrt{1-x})\ E_{\alpha,\beta}^{(\gamma,c);\mathbf{q}}(Zx^B;\mathbf{p})^{(\Gamma)}\aleph_{p_i,q_i,\tau_i;r}^{m,n}(zx^A)\mathrm{d}x$$
$$= \frac{\pi a}{2} \sum_{l,n'=0}^{\infty} \frac{n'!}{\left(\frac{3}{2}\right)_{n'}} (-a^2)^{n'} A_{\alpha,\beta}^{(\gamma,c);\mathbf{q}}(\mathbf{p};l)\ Z^l$$
$$\times\ {}^{(\Gamma)}\aleph_{p_i+2,q_i+2,\tau_i;r}^{m,n+2}\left(z \middle| \begin{array}{c} (a_1,A_1,x'),E_1,E_1,(a_j,A_j)_{2,n},[\tau_i(a_{ji},A_{ji})]_{n+1,p_i} \\ (g_j,G_j)_{1,m},[\tau_i(g_{ji},G_{ji})]_{m+1,q_i},E_2,E_2 \end{array}\right) \tag{32}$$

where $E_1 = \left(\frac{1}{2}-s-Bl-n';A\right), E_2 = (-s-Bl-n';A)$
provided $\Re(s) > -\frac{1}{2}, |\arg(1+a^2)| < \pi,\ \Re(s+lB)+A\min_{1\leq j\leq m}\Re\left(\frac{g_j}{G_j}\right) > -\frac{1}{2}$, $\Omega_i > 0, |\arg(z)| < \frac{\pi}{2}\Omega_i,\ i=1,\cdots,r$ or $\Omega_i \geq 0, |\arg(z)| < \frac{\pi}{2}\Omega_i$ and $\Re(\zeta_i)+1 < 0$, Ω_i and ζ_i is defined by (8) and (9) respectively. $A,B > 0$. $\alpha,\beta,\gamma,c,\mathbf{p} \in \mathbb{C}, \Re(\mathbf{p}),\Re(c),\Re(\gamma) > 0, \mathbf{q} < \Re(\alpha)+1$.

Proof. To prove the theorem, expressing the extension of M-L function defined by Mittal [16] in series with the help of (19) the modified incomplete Gamma Aleph-function in Mellin-Barnes contour integral with the help of (4) and interchange the order of integrations and series which is valid due to absolute convergence of the integral. Collecting the power of x, we obtain I

$$I = \int_0^1 \frac{x^{s-1}}{\sqrt{1+a^2x}} \ln(a\sqrt{x}+\sqrt{(1+a^2x)}K(\sqrt{1-x})$$
$$\times E_{\alpha,\beta}^{(\gamma,c);\mathbf{q}}(Zx^B;\mathbf{p})^{(\Gamma)}\aleph_{p_i,q_i,\tau_i;r}^{m,n}(zx^A)\mathrm{d}x$$
$$= \sum_{l=0}^{\infty} A_{\alpha,\beta}^{(\gamma,c);\mathbf{q}}(\mathbf{p};l)Z^l$$
$$\times \frac{1}{2\pi\omega}\int_L \frac{\Gamma(1-a_1-A_1t,x')\prod_{j=2}^{n}\Gamma(1-a_j-A_jt)\prod_{j=1}^{m}\Gamma(g_j+G_jt)}{\sum_{i=1}^{r}\tau_i\left[\prod_{j=m+1}^{q_i}\Gamma(1-g_{ji}-G_{ji}t)\prod_{j=n+1}^{p_i}\Gamma(a_{ji}+A_{ji}t)\right]}z^{-t}$$
$$\times \int_0^1 \frac{x^{s+Bl-tA-1}}{\sqrt{1+a^2x}} \ln(a\sqrt{x}+\sqrt{(1+a^2x)}K(\sqrt{1-x})\mathrm{d}x\mathrm{d}t, \tag{33}$$

We use the lemma (31), this gives

$$I = \int_0^1 \frac{x^{s-1}}{\sqrt{1+a^2x}} \ln(a\sqrt{x}+\sqrt{(1+a^2x)}K(\sqrt{1-x})$$
$$\times E_{\alpha,\beta}^{(\gamma,c);\mathbf{q}}(Zx^B;\mathbf{p})^{(\Gamma)}\aleph_{p_i,q_i,\tau_i;r}^{m,n}(zx^A)\mathrm{d}x$$
$$= \frac{\pi a}{2}\sum_{l=0}^{\infty} A_{\alpha,\beta}^{(\gamma,c);\mathbf{q}}(\mathbf{p};l)Z^l$$
$$\times \frac{1}{2\pi\omega}\int_L \frac{\Gamma(1-a_1-A_1t,x')\prod_{j=2}^{n}\Gamma(1-a_j-A_jt)\prod_{j=1}^{m}\Gamma(g_j+G_jt)}{\sum_{i=1}^{r}\tau_i\left[\prod_{j=m+1}^{q_i}\Gamma(1-g_{ji}-G_{ji}t)\prod_{j=n+1}^{p_i}\Gamma(a_{ji}+A_{ji}t)\right]}z^{-t}$$
$$\times \frac{\Gamma^2\left(s+Bl-At+\frac{1}{2}\right)}{\Gamma^2\left(s+Bl-At+1\right)}\,{}_4F_3\left(\begin{array}{c} 1,1,s+Bl-At+\frac{1}{2},s-At+Bl+\frac{1}{2} \\ \frac{3}{2},s+Bl-At+1,s-At+Bl+1 \end{array}\middle| -a^2\right)\mathrm{d}t$$

We replace the Gauss hypergeometric function by the series $\sum_{n=0}^{\infty}$,(see Slater [23]), under the hypothesis, we can interchange this series and the t-integrals, we have:

$$I = \int_0^1 \frac{x^{s-1}}{\sqrt{1+a^2x}} \ln(a\sqrt{x} + \sqrt{(1+a^2x)}K(\sqrt{1-x}) \times E_{\alpha,\beta}^{(\gamma,c);\mathbf{q}}(Zx^B;\mathbf{p})^{(\Gamma)}\aleph_{p_i,q_i,\tau_i;r}^{m,n}(zx^A)\mathrm{d}x$$

$$= \frac{\pi a}{2}\sum_{l,n'=0}^{\infty}\frac{n'!}{(\frac{3}{2})_{n'}}(-a^2)^{n'}A_{\alpha,\beta}^{(\gamma,c);\mathbf{q}}(\mathbf{p};l)Z^l \times \frac{1}{2\pi\omega}\int_L \frac{\Gamma(1-a_1-A_1t,x')\prod_{j=2}^{n}\Gamma(1-a_j-A_jt)\prod_{j=1}^{m}\Gamma(g_j+G_jt)}{\sum_{i=1}^{r}\tau_i\left[\prod_{j=m+1}^{q_i}\Gamma(1-g_{ji}-G_{ji}t)\prod_{j=n+1}^{p_i}\Gamma(a_{ji}+A_{ji}t)\right]}z^{-t} \times \frac{\Gamma^2\left(s+Bl-At+\frac{1}{2}\right)}{\Gamma^2\left(s+Bl-At+1\right)}\frac{(s+Bl-At+\frac{1}{2})_{n'}^2}{(s-At+Bl+1)_{n'}^2}(-a^2)^{n'}\mathrm{d}t \tag{34}$$

Now we apply the relation $\Gamma(a)(a)_n = \Gamma(a+n)$, this gives:

$$I = \int_0^1 \frac{x^{s-1}}{\sqrt{1+a^2x}} \ln(a\sqrt{x} + \sqrt{(1+a^2x)}K(\sqrt{1-x}) \times E_{\alpha,\beta}^{(\gamma,c);\mathbf{q}}(Zx^B;\mathbf{p})^{(\Gamma)}\aleph_{p_i,q_i,\tau_i;r}^{m,n}(zx^A)\mathrm{d}x$$

$$= \frac{\pi a}{2}\sum_{l,n'=0}^{\infty}\frac{n'!}{(\frac{3}{2})_{n'}}(-a^2)^{n'}A_{\alpha,\beta}^{(\gamma,c);\mathbf{q}}(\mathbf{p};l)Z^l \times \frac{1}{2\pi\omega}\int_L \frac{\Gamma(1-a_1-A_1t,x')\prod_{j=2}^{n}\Gamma(1-a_j-A_jt)\prod_{j=1}^{m}\Gamma(g_j+G_jt)}{\sum_{i=1}^{r}\tau_i\left[\prod_{j=m+1}^{q_i}\Gamma(1-g_{ji}-G_{ji}t)\prod_{j=n+1}^{p_i}\Gamma(a_{ji}+A_{ji}t)\right]}z^{-t} \times \frac{\Gamma^2\left(s+Bl-At+\frac{1}{2}+n'\right)}{\Gamma^2\left(s+Bl-At+1+n'\right)}\mathrm{d}t \tag{35}$$

We interpret this contour integral of the incomplete Gamma Aleph-function, we obtain the desired result.
We have the same result with the incomplete gamma Aleph-function.
Theorem 2.

$$\int_0^1 \frac{x^{s-1}}{\sqrt{1+a^2x}} \ln(a\sqrt{x} + \sqrt{(1+a^2x)}K(\sqrt{1-x})\ E_{\alpha,\beta}^{(\gamma,c);\mathbf{q}}(Zx^B;\mathbf{p})^{(\gamma)}\aleph_{p_i,q_i,\tau_i;r}^{m,n}(zx^A)\mathrm{d}x = \frac{\pi a}{2}\sum_{l,n'=0}^{\infty}\frac{n'!}{\left(\frac{3}{2}\right)_{n'}}(-a^2)^{n'}A_{\alpha,\beta}^{(\gamma,c);\mathbf{q}}(\mathbf{p};l)\ Z^l \times {}^{(\gamma)}\aleph_{p_i+2,q_i+2,\tau_i;r}^{m,n+2}\left(z\,\middle|\,\begin{matrix}(a_1,A_1,x'),E_1,E_1,(a_j,A_j)_{2,n},[\tau_i(a_{ji},A_{ji})]_{n+1,p_i}\\(g_j,G_j)_{1,m},[\tau_i(g_{ji},G_{ji})]_{m+1,q_i},E_2,E_2\end{matrix}\right) \tag{36}$$

under the same conditions that the Theorem 1. E_1 and E_2 are same as of Theorem 1.
In the following section, we cite several particular cases.

4. **Special Cases**

In this section, in consequence of generality of incomplete Aleph functions and M-L functions, we cite certain special cases of our main results. First, we consider the incomplete I-functions defined by the equations (11) and (12).

Corollary 1.

$$\int_0^1 \frac{x^{s-1}}{\sqrt{1+a^2x}}\ln(a\sqrt{x}+\sqrt{(1+a^2x)}K(\sqrt{1-x})\ E_{\alpha,\beta}^{(\gamma,c);\mathbf{q}}(Zx^B;\mathbf{p})^{(\Gamma)}I_{p_i,q_i;r}^{m,n}(zx^A)\mathrm{d}x$$
$$=\frac{\pi a}{2}\sum_{l,n'=0}^{\infty}\frac{n'!}{\left(\frac{3}{2}\right)_{n'}}(-a^2)^{n'}A_{\alpha,\beta}^{(\gamma,c);\mathbf{q}}(\mathbf{p};l)\ Z^l$$
$$\times\,^{(\Gamma)}I_{p_i+2,q_i+2;r}^{m,n+2}\left(z\left|\begin{array}{c}(a_1,A_1,x'),E_1,E_1,(a_j,A_j)_{2,n},[\tau_i(a_{ji},A_{ji})]_{n+1,p_i}\\ (g_j,G_j)_{1,m},[\tau_i(g_{ji},G_{ji})]_{m+1,q_i},E_2,E_2\end{array}\right.\right) \tag{37}$$

provided $\Re(s)>-\frac{1}{2},|\arg(1+a^2)|<\pi$, $\Re(s+lB)-A\min\limits_{1\leq j\leq m}\Re\left(\frac{g_j}{G_j}\right)>-\frac{1}{2}$, $\Omega_i>0,|\arg(z)|<\frac{\pi}{2}\Omega_i$, or $\Omega_i\geq 0,|\arg(z)|<\frac{\pi}{2}\Omega_i$ and $\Re(\zeta_i)+1<0$, $A,B>0$; $\alpha,\beta,\gamma\in\mathbb{C};\Re(\alpha)>0$ where

$$\Omega_i=\sum_{j=1}^{n}A_j+\sum_{j=1}^{m}G_j-\left(\sum_{j=n+1}^{p_i}A_{ji}+\sum_{j=m+1}^{q_i}G_{ji}\right) \tag{38}$$

and

$$\zeta_i=\sum_{j=1}^{m}g_j-\sum_{j=1}^{n}a_j+\left(\sum_{j=m+1}^{q_i}g_{ji}-\sum_{j=n+1}^{p_i}a_{ji}\right)+\frac{p_i-q_i}{2}, \tag{39}$$

E_1 and E_2 are same as of Theorem 1.

Corollary 2.

$$\int_0^1 \frac{x^{s-1}}{\sqrt{1+a^2x}}\ln(a\sqrt{x}+\sqrt{(1+a^2x)}K(\sqrt{1-x})\ E_{\alpha,\beta}^{(\gamma,c);\mathbf{q}}(Zx^B;\mathbf{p})^{(\gamma)}I_{p_i,q_i;r}^{m,n}(zx^A)\mathrm{d}x$$
$$=\frac{\pi a}{2}\sum_{l,n'=0}^{\infty}\frac{n'!}{\left(\frac{3}{2}\right)_{n'}}(-a^2)^{n'}A_{\alpha,\beta}^{(\gamma,c);\mathbf{q}}(\mathbf{p};l)\ Z^l$$
$$\times\,^{(\gamma)}I_{p_i+2,q_i+2;r}^{m,n+2}\left(z\left|\begin{array}{c}(a_1,A_1,x'),E_1,E_1,(a_j,A_j)_{2,n},[\tau_i(a_{ji},A_{ji})]_{n+1,p_i}\\ (g_j,G_j)_{1,m},[\tau_i(g_{ji},G_{ji})]_{m+1,q_i},E_2,E_2\end{array}\right.\right) \tag{40}$$

under the conditions cited in the corollary 1. E_1 and E_2 are same as of Theorem 1. Due to the definitions (13) and (14) of incomplete H-functions, we obtain:

Corollary 3.

$$\int_0^1 \frac{x^{s-1}}{\sqrt{1+a^2x}}\ln(a\sqrt{x}+\sqrt{(1+a^2x)}K(\sqrt{1-x})\ E_{\alpha,\beta}^{(\gamma,c);\mathbf{q}}(Zx^B;\mathbf{p})^{(\Gamma)}H_{p,q}^{m,n}(zx^A)\mathrm{d}x$$
$$=\frac{\pi a}{2}\sum_{l,n'=0}^{\infty}\frac{n'!}{\left(\frac{3}{2}\right)_{n'}}(-a^2)^{n'}A_{\alpha,\beta}^{(\gamma,c);\mathbf{q}}(\mathbf{p};l)\ Z^l$$
$$\times\,^{(\Gamma)}H_{p+2,q+2}^{m,n+2}\left(z\left|\begin{array}{c}(a_1,A_1,x'),\left(\frac{1}{2}-s-Bl-n';A\right),\left(\frac{1}{2}-s-Bl-n';A\right),(a_j,A_j)_{2,p}\\ (g_j,G_j)_{1,q},(-s-Bl-n';A),(-s-Bl-n';A)\end{array}\right.\right) \tag{41}$$

provided $\Re(s) > -\frac{1}{2}, |\arg(1+a^2)| < \pi,\ \Re(s+lB) - A \min_{1\le j\le m} \Re\left(\frac{g_j}{G_j}\right) > -\frac{1}{2},\ \Omega > 0, |\arg(z)| < \frac{\pi}{2}\Omega$, or $\Omega \geq 0, |\arg(z)| < \frac{\pi}{2}\Omega$ and $\Re(\zeta)+1 < 0,\ A, B > 0;\ \alpha, \beta, \gamma, \in \mathbb{C}; \Re(\alpha) > 0$ where

$$\Omega = \sum_{j=1}^{n} A_j + \sum_{j=1}^{m} G_j - \left(\sum_{j=n+1}^{p} A_j + \sum_{j=m+1}^{q} G_j \right) \tag{42}$$

and

$$\zeta = \sum_{j=1}^{m} g_j - \sum_{j=1}^{n} a_j + \left(\sum_{j=m+1}^{q} g_j - \sum_{j=n+1}^{p} a_j \right) + \frac{p-q}{2}. \tag{43}$$

Corollary 4.

$$\int_0^1 \frac{x^{s-1}}{\sqrt{1+a^2x}} \ln(a\sqrt{x} + \sqrt{(1+a^2x)}K(\sqrt{1-x})\ E_{\alpha,\beta}^{(\gamma,c);\mathbf{q}}(Zx^B;\mathbf{p})^{(\gamma)}H_{p,q}^{m,n}(zx^A)\mathrm{d}x$$
$$= \frac{\pi a}{2} \sum_{l,n'=0}^{\infty} \frac{n'!}{\left(\frac{3}{2}\right)_{n'}} (-a^2)^{n'} A_{\alpha,\beta}^{(\gamma,c);\mathbf{q}}(\mathbf{p};l)\ Z^l$$
$$\times\, {}^{(\gamma)}H_{p+2,q+2}^{m,n+2}\left(z \,\middle|\, \begin{array}{c} (a_1,A_1,x'), \left(\frac{1}{2}-s-Bl-n';A\right), \left(\frac{1}{2}-s-Bl-n';A\right), (a_j,A_j)_{2,p} \\ (g_j,G_j)_{1,q}, (-s-Bl-n';A), (-s-Bl-n';A) \end{array}\right) \tag{44}$$

under the conditions cited in the corollary 3.

In the following we consider several particular cases due to the various definitions of M-L type functions recorded in section 1. First, we consider the extended M-L function defined by Özarslan and Yilmaz [18], this gives:

Corollary 5.

$$\int_0^1 \frac{x^{s-1}}{\sqrt{1+a^2x}} \ln(a\sqrt{x} + \sqrt{(1+a^2x)}K(\sqrt{1-x})\ E_{\alpha,\beta}^{(\gamma;c)}(Zx^B;\mathbf{p})^{(\Gamma)}\aleph_{p_i,q_i,\tau_i;r}^{m,n}(zx^A)\mathrm{d}x$$
$$= \frac{\pi a}{2} \sum_{l,n'=0}^{\infty} \frac{n'!}{\left(\frac{3}{2}\right)_{n'}} (-a^2)^{n'} A_{\alpha,\beta}^{(\gamma;c)}(\mathbf{p};l)\ Z^l$$
$$\times\, {}^{(\Gamma)}\aleph_{p_i+2,q_i+2,\tau_i;r}^{m,n+2}\left(z \,\middle|\, \begin{array}{c} (a_1,A_1,x'), E_1, E_1, (a_j,A_j)_{2,n}, [\tau_i(a_{ji},A_{ji})]_{n+1,p_i} \\ (g_j,G_j)_{1,m}, [\tau_i(g_{ji},G_{ji})]_{m+1,q_i}, E_2, E_2 \end{array}\right) \tag{45}$$

under the conditions of Theorem 1. E_1 and E_2 are same as of Theorem 1 with $\alpha, \beta, \gamma, p \in \mathbb{C}; \Re(\mathbf{p}), \Re(c), \Re(\gamma) > 0$.

If we apply $E_{\alpha,\beta}^{\gamma,c}(z)$ defined by Salim [21]. We get,
Corollary 6.

$$\int_0^1 \frac{x^{s-1}}{\sqrt{1+a^2x}}\ln(a\sqrt{x}+\sqrt{(1+a^2x)}K(\sqrt{1-x})\ E_{\alpha,\beta}^{\gamma,c}(Zx^B)^{(\Gamma)}\aleph_{p_i,q_i,\tau_i;r}^{m,n}(zx^A)\mathrm{d}x$$
$$= \frac{\pi a}{2}\sum_{l,n'=0}^{\infty}\frac{n'!}{\left(\frac{3}{2}\right)_{n'}}(-a^2)^{n'}B_{\alpha,\beta}^{\gamma,c}(l)\ Z^l$$
$$\times\ {}^{(\Gamma)}\aleph_{p_i+2,q_i+2,\tau_i;r}^{m,n+2}\left(z\ \middle|\ \begin{array}{c}(a_1,A_1,x'),E_1,E_1,(a_j,A_j)_{2,n},[\tau_i(a_{ji},A_{ji})]_{n+1,p_i}\\ (g_j,G_j)_{1,m},[\tau_i(g_{ji},G_{ji})]_{m+1,q_i},E_2,E_2\end{array}\right) \tag{46}$$

under the conditions verified by Theorem 1. E_1 and E_2 are same as of Theorem 1 with $\alpha,\beta,\gamma,\in\mathbb{C};\Re(c),\Re(\gamma)>0$.
On considering M-L function defined by Prabhakar [19], we have:
Corollary 7.

$$\int_0^1 \frac{x^{s-1}}{\sqrt{1+a^2x}}\ln(a\sqrt{x}+\sqrt{(1+a^2x)}K(\sqrt{1-x})\ E_{\alpha,\beta}^{\gamma}(Zx^B)^{(\Gamma)}\aleph_{p_i,q_i,\tau_i;r}^{m,n}(zx^A)\mathrm{d}x$$
$$= \frac{\pi a}{2}\sum_{l,n'=0}^{\infty}\frac{n'!}{\left(\frac{3}{2}\right)_{n'}}(-a^2)^{n'}A_{\alpha,\beta}^{\gamma}(l)\ Z^l$$
$$\times\ {}^{(\Gamma)}\aleph_{p_i+2,q_i+2,\tau_i;r}^{m,n+2}\left(z\ \middle|\ \begin{array}{c}(a_1,A_1,x'),E_1,E_1,(a_j,A_j)_{2,n},[\tau_i(a_{ji},A_{ji})]_{n+1,p_i}\\ (g_j,G_j)_{1,m},[\tau_i(g_{ji},G_{ji})]_{m+1,q_i},E_2,E_2\end{array}\right) \tag{47}$$

under the conditions verified by Theorem 1, E_1 and E_2 are same as of Theorem 1 with $\alpha,\beta,\gamma,\in\mathbb{C};\Re(\alpha)>0$.
The function (25) reduces to function defined by Wiman [28, 29], we get:
Corollary 8.

$$\int_0^1 \frac{x^{s-1}}{\sqrt{1+a^2x}}\ln(a\sqrt{x}+\sqrt{(1+a^2x)}K(\sqrt{1-x})\ E_{\alpha,\beta}(Zx^B)^{(\Gamma)}\aleph_{p_i,q_i,\tau_i;r}^{m,n}(zx^A)\mathrm{d}x$$
$$= \frac{\pi a}{2}\sum_{l,n'=0}^{\infty}\frac{n'!}{\left(\frac{3}{2}\right)_{n'}}(-a^2)^{n'}A_{\alpha,\beta}(l)\ Z^l$$
$$\times\ {}^{(\Gamma)}\aleph_{p_i+2,q_i+2,\tau_i;r}^{m,n+2}\left(z\ \middle|\ \begin{array}{c}(a_1,A_1,x'),E_1,E_1,(a_j,A_j)_{2,n},[\tau_i(a_{ji},A_{ji})]_{n+1,p_i}\\ (g_j,G_j)_{1,m},[\tau_i(g_{ji},G_{ji})]_{m+1,q_i},E_2,E_2\end{array}\right) \tag{48}$$

under the conditions verified by Theorem 1. E_1 and E_2 are same as of Theorem 1 with $\alpha,\beta\in\mathbb{C};\Re(\alpha)>0$.
Now, we arrive to M-L function, we obtain:

Corollary 9

$$\int_0^1 \frac{x^{s-1}}{\sqrt{1+a^2x}} \ln(a\sqrt{x}+\sqrt{(1+a^2x)}K(\sqrt{1-x})\ E_\alpha(Zx^B)^{(\Gamma)}\aleph^{m,n}_{p_i,q_i,\tau_i;r}(zx^A)\mathrm{d}x$$
$$= \frac{\pi a}{2}\sum_{l,n'=0}^{\infty}\frac{n'!}{\left(\frac{3}{2}\right)_{n'}}(-a^2)^{n'}A_\alpha(l)\ Z^l$$
$$\times\ {}^{(\Gamma)}\aleph^{m,n+2}_{p_i+2,q_i+2,\tau_i;r}\left(z\left|\begin{array}{c}(a_1,A_1,x'),E_1,E_1,(a_j,A_j)_{2,n},[\tau_i(a_{ji},A_{ji})]_{n+1,p_i}\\(g_j,G_j)_{1,m},[\tau_i(g_{ji},G_{ji})]_{m+1,q_i},E_2,E_2\end{array}\right.\right) \tag{49}$$

under the conditions verified by Theorem 1, E_1 and E_2 are same as of Theorem 1 with $\alpha\in\mathbb{C};\Re(\alpha)>0$.

Remarks.
Considering last five corollaries, we obtain the same formulas with the incomplete gamma Aleph function ${}^{(\gamma)}\aleph^{m,n}_{p_i,q_i,\tau_i;r}(z)$, the incomplete Gamma I-function ${}^{(\Gamma)}I^{m,n}_{p_i,q_i;r}(z)$, the incomplete gamma I-function, ${}^{(\gamma)}I^{m,n}_{p_i,q_i;r}(z)$, the incomplete H-functions ${}^{(\Gamma)}H^{m,n}_{p,q}(z)$ and ${}^{(\gamma)}H^{m,n}_{p,q}(z)$. We have the same generalized finite integrals involving the extension of M-L function , with the aleph-function defined by Südland [27], the I-function defined by Saxena [22] and the Fox's H-function.

5. Conclusion

The significance of the results presented in this paper lies in their manifold generality. By giving special values to the parameters as well as variable in the incomplete aleph functions ${}^{(\Gamma)}\aleph^{m,n}_{p_i,q_i,\tau_i,r}(z)$ and ${}^{(\gamma)}\aleph^{m,n}_{p_i,q_i,\tau_i,r}(z)$, we obtain so many new results involving remarkably wide variety of well known and useful special functions, which are expressible in terms of I-function defined by Saxena [22], H-function, E-function, Meijer's G-function, and hypergeometric function of one variable. Next, by specializing the parameters of the elliptic integral of first species $K(\cdot)$, we can get a large number of integrals involving the incomplete Aleph-functions. Further, by giving specialized values to the parameters and variable of the extension of the M-L function involved in this paper, we obtain a large number of new integrals.

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