Two-Sample Prediction of Odd Generalized Exponential Inverted Weibull Distribution with theApplication on COVID-19 Mortality Rate

تنبؤ بعينتين لتوزيع وايبل الأسي المعمم المرجح مع التطبيق على معدل وفيات كوفيد-١٩

Asmaa Abdelazim Mohamed Abdelazim
Statistics Department,
Faculty of Commerce,
AL-Azhar University (Girls' Branch),
Tafahna Al-Ashraf, Egypt;

Principal supervisor
Prof. Dr:
Ganat Ramdan Mohamed
Professor of Statistics
, Faculty of Commerce,
AL-Azhar University (Girls' Branch)

Co-supervisor
Prof.Dr:
Reda Mohamed Hassan
Assistant Professor of Statistics,
Faculty of Commerce,
AL-Azhar University (Girls' Branch)

المستخلص العربى

تناول هذا البحث دراسه عن التنبؤ الإحصائي وهو من أهم الموضوعات ذات الصلة الوثيقة بالتقديرات الإحصائية والتي من خلال دراستها يمكن الإستفادة مما تحتويه عينة الدراسة المعروفة من معلومات للتنبؤ بمعلومات تخص عينة مستقبلية من نفس مجتمع الدراسة ففي كثير من تجارب إختبارات الحياة أو تجارب إختبارات الصلاحية للمنتجات الصناعية يكون من المهم دراسة كيفية الإستفادة من معلومات متوفرة عن مجتمع ما للتنبؤ بمعلومات مستقبلية بغرض تحسين كفاءة هذا المنتج مستقبلاً. فإذا أجريت التجارب بناءً على عينة واحدة يسمى التنبؤ بعينة والذي يكون قد تم بالفعل تسجيل بعض القياسات لجزء من مفردات العينة تحت التجربة والمطلوب هو

التنبؤ بما تبقى من قياسات نفس العينة أما إذا أجريت التجربه بناءً على عينتين مستقلتين من نفس المجتمع العينة الأولى معروفة والعينة الثانية عينة مستقبلية فهذا ما يسمى بتنبؤ العينتين وهو محل هذه الدراسه.

تم الحصول على التنبؤات(بنقطة وبفترة) لمشاهدات مستقبلية تتبع العائلة الأسية المعمه المرجحه بإستخدام اسلوب التنبؤ بعينتين اعتمادا على عينات المراقبة من النوع الثانى وذلك باستخدام طريقة الإمكان الأكبرمن خلال التقدير بنقطة وفتره. وقد تم أيضاً إستنتاج حدى فترة التنبؤ للقراءة الصغرى المستقبلية وحدى فترة التنبؤ للقراءة العظمى المستقبلية وحدى فترة التنبؤ للقراءة الوسطى المستقبلية فى الحالة الفردية كحالات خاصة. وقد وجد من نتائج المحاكاة أن فترة التنبؤ للمشاهدة الأولى أقل من فترة التنبؤ للمشاهدة الأخيرة. ايضا تم التطبيق بإستخدم ثلاث مجموعات من بيانات فعلية (حقيقية) على هذا التوزيع وقد لوحظ ان النتائج متشابهه مع النتائج النظرية.

تم فى هذا البحث أيضا الحصول على تنبؤات بييز وأيضا تنبؤات توقع بييز (بنقطة وبفترة) لمشاهدات مستقبلية تتبع العائلة الأسية المعمه المرجحه بإستخدام أسلوب التنبؤ بعينتين اعتمادا على عينات المراقبة من النوع الثانى وبإستخدام دوال خسارة متماثلة مثل دالة خسارة مربع الخطأ المتوازنة وغير متماثلة مثل دالة الخسارة الأسية الخطية المتوازنة وقد تم إفتراض أن جميع معالم التوزيع غير معلومة. وأيضا تم إجراء مقارنات عددية لمختلف التقديرات الناتجة بإستخدام أسلوب المحاكاة وإستخدام مجموعة من البيانات الحقيقية لإيضاح الأهمية والكيفية التطبيقية للنتائج التى تم التوصل إليها. وقد وجد من نتائج المحاكاة أن طول فترة التنبؤللمشاهدة الأولى أقل من طول فترة التنبؤ للمشاهدة الأخيرة وهذا ما يتفق مع ما هو متوقع نظريا. أيضا طول الفترة لتنبؤات توقع بييز أقل من طول الفترة لتنبؤات بييز وهذا يدل على أن تنبؤات توقع بييز أكثر دقة من تنبؤات بييز. كما أن طول الفترة لتنبؤات توقع بييز بإستخدام دالة الخسارة الأسية الخطية المتوازنة تكون فى جميع الحالات أقل من طول الفترة لتنبؤات توقع بييز بإستخدام دالة خسارة مربع الخطأ المتوازنة. وأخيرا وجد أن طول الفترة لتنبؤات بييز بإستخدام دالة الخسارة الأسية الخطية المتوازنة تكون فى معظم الحالات أقل من طول الفترة لتنبؤات بييز بإستخدام دالة خسارة مربع الخطأ المتوازنة. كما لوحظ أن التنبؤات بنقطة لجميع الطرق المستخدمة تقع جميعها داخل حدود فترات التنبؤ.

الكلمات المفتاحية : توزيع وايبل الأسي المعمم المرجح- المراقبة من النوع الثاني - التنبؤ بعينتين – التنبؤ غير البييز- البييز – توقع البييز.

Abstract

One of the most crucial issues in life testing is statistical prediction, which has also been used in business, engineering, medicine, and other fields. When more information are available, a better choice will be promoted. When projecting business results, prediction is used to save time, effort, and money. The predictor might be either a point predictor or an interval predictor. The main aim of this research is to investigate the two-sample prediction problem from the odd generalized exponential inverted Weibull distribution based on Type II censored samples. Furthermore, point and interval predictions for future order statistics using non-Bayesian, Bayesian, and E-Bayesian models are looked at. Future order statistics point and interval projections are also offered using conditional, maximum likelihood, Bayesian, and E-Bayesian techniques. The Bayesian and E-Bayesian predictors are based on two different loss functions: the balanced squared error loss function, which is symmetric, and the balanced linear exponential loss function, which is asymmetric. The predictors are derived using uniform hyper prior distributions and gamma prior distributions. Results have been applied to real data sets (such as the COVID-19 death rate in different countries) as well as simulation studies to show the flexibility and potential applications of the distribution.

Keywords *Odd generalized exponential inverted Weibull distribution; Type-II censored samples; Two-sample prediction; Non-Bayesian; Bayesian and E-Bayesian prediction.*

Introduction

Growing interest in prediction, which is important in many domains, has been observed during the past few years. For instance, in the business world, an experimenter might attempt to estimate the lifespan of a future unseen unit using data from the current sample.

In order to make their items the center of consumers' attention and to achieve their desires, the experimenter or producer introduces their products to the market with more palatable warranty terms. A fundamental statistical issue is the prediction of future observations using past and present data. This issue arises in a variety of situations and has a wide range of solutions. Prediction techniques have been used for a variety of reasons by statisticians, engineers, and other applied sciences. One and two-sample predictions are frequently used. The prediction and its applications were examined by numerous scholars. For instance, with a Type-II censoring scheme, Kundu and Howlader (2010) deduced a Bayesian prediction for the inverted Weibull distribution. AL-Hussaini and Al-Awadhi (2010) constructed Bayes two-sample prediction and interval predictors of generalized order statistics. When the lifetime distribution of the experimental units is assumed to be a generalized exponential random variable, Asgharzadeh and Fallah (2011) proposed the problem of estimation and prediction for a family of exponentiated distributions. Valiollahi *et al.* (2017) obtained the *maximum likelihood* (ML) and Bayesian prediction (point and interval) of a future observation based on Type-I, Type-II, and hybrid censored samples.

Dey *et al.* (2018) introduced the one and two-sample prediction of the future samples for a weighted exponential distribution with Type-II progressive censoring. Additionally, Faizan and Sana (2018) considered prediction intervals for future observations of the two unknown parameters of Chen distribution based on upper record value. Abd El-Raheem (2019) investigated the intervals and one-sample Bayesian prediction of the generalized half-normal distribution under progressive Type-II censoring. Using the Bayesian methodology of the *Topp Leone* (TL) family of distributions, Arshad and Jamal (2019) forecasted future record values. Additionally, Okasha *et al.* (2020) developed the Bayesian and E-Bayesian prediction (point and interval) based on observed order statistics using two samples from a two-parameter Burr XII model using Type-II censored data. AL-Dayian *et al.* (2021) applied the two-sample prediction method to obtain the conditional ML, Bayesian and E-Bayesian prediction treats (point and interval) for future order statistics of the modified TL-Chen distribution based on progressive Type-II censored samples. Moreover, they obtained the predictors under symmetric and asymmetric loss functions assuming gamma prior density.

The Bayesian approach the unknown parameters as random variables and relies on knowledge about the parameters already known. Symmetric and asymmetric loss function functions are used to construct the Bayes estimators of the parameters.

Han (2007) introduced the *expected Bayesian* (E-Bayesian) estimation method which is very simple and it's a special Bayesian method used in the area related for the life testing of products with high reliability, small sample size, or censored data. It is now more widely accepted. The E-Bayesian approach was widely used by researchers to analyze a variety of distributions, for instance Reyad and Ahmed (2015), Gupta (2017), Han (2019), Algarni *et al.* (2020), Han (2020) and Rabie and Li (2020).

This paper discuss the *odd generalized exponential inverted Weibull* (OGEIW) distribution. The estimation based on Type-II censored samples and predicting future observations have not been studied in all the previous literature. The main objective of this paper is driving the prediction of the future observations from the OGEIW distribution. The two-sample prediction technique is used. Point and interval predictions are discussed using the ML, Bayesian, and E-Bayesian methods. The Bayesian and E-Bayesian predictors are considered based on two different loss functions, the *balanced squared loss* (BSEL) function; as a symmetric loss function and *balanced linear exponential* (BLL) function; as an asymmetric loss function.

The outline of the paper is as follows: in Section 2, presents a brief summary of the OGE-IW(α, β, ζ)distribution and descriptions of the main its properties. The *maximum likelihood* (ML) estimators of the parameters from the OGE-

IW(α, β, ζ) distribution based on Type-II censored samples are derived in Section 3. The conditional prediction (point and interval) is used as a non-Bayesian prediction for a future observation of the OGE-IW(α, β, ζ)distribution based on two-sample prediction scheme are obtained. In Section 4. Bayesian estimation and the description of the balanced loss functions are presented. In Section 5. Bayesian and E-Bayesian prediction (point and interval) for future observation of the OGE-IW(α, β, ζ) distribution based on two-sample prediction is studied in Section 6. A numerical example is given to illustrate the theoretical results and an application using real data sets are used to demonstrate how the results can be used in practice in Section 7. Finally, general conclusion is presented in Section 8.

٢. Odd Generalized Exponential Inverted Weibull Distribution

The inverted Weibull distributions have great importance due to their applicability in many areas such as engineering discipline of reliability, biological sciences, life test problems, medical, etc. The inverted Weibull distribution can be used to a diverse model of failure characteristics, such as infant mortality, age of production, and periods of erosion. The inverted Weibull distribution can also be used to determine the cost-effectiveness and maintenance periods of reliability centered maintenance activities.

Hassan *et al*. (2018) constructed a distribution with three parameters and presented some mathematical statistical properties of OGE-IW distribution such as quantile function, mode, moments, probability-weighted moments, incomplete moments, stress-strength model, moments of residual life function, and Rényi entropy. Also, they provided graphical illustrations of the dimensions of OGE-IW distribution and estimated the parameters using the ML method based on complete samples.

Assume a random variable X that follows the OGE-IW distribution, denoted by $X \sim \text{OGE-IW}(\alpha, \beta, \zeta)$. The pdf, cdf, sf, hrf and rhrf of OGE-IW distribution are respectively written as:

$$f(x; \underline{\Psi}) = \alpha\beta\zeta x^{-\alpha-1} e^{-\beta x^{-\alpha}} \left(1 - e^{-\beta x^{-\alpha}}\right)^{-2} \exp\left[-\left(\frac{\zeta}{e^{\beta x^{-\alpha}} - 1}\right)\right], \quad x > 0; \ \left(\underline{\Psi} > \underline{0}\right), \qquad (1)$$

and

$$F(x; \underline{\Psi}) = 1 - \exp\left[-\left(\frac{\zeta}{e^{\beta x^{-\alpha}} - 1}\right)\right], \quad x > 0; \left(\underline{\Psi} > \underline{0}\right), \qquad (2)$$

where $\underline{\Psi} = (\alpha, \beta, \zeta)$, α, β are the shape parameters and ζ is the scale parameter.

The pdf of OGE-IW distribution can be symmetric, unimodal and right skewed. For α =2, the OGE-IW $(\underline{\Psi})$distribution reduces to a new model named as odds generalized exponential inverse Rayleigh distribution. For $\alpha = 1$, the OGE-IW $(\underline{\Psi})$distribution reduces to another new model named as odds generalized exponential inverse exponential distribution.

The sf and hrf of OGE-IW $(\underline{\Psi})$distribution are, respectively, given by

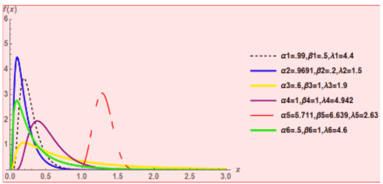
$$s(x; \underline{\Psi}) = \exp\left[-\left(\frac{\zeta}{e^{\beta x^{-\alpha}} - 1}\right)\right], \quad x > 0; \ \left(\underline{\Psi} > \underline{0}\right), \qquad (3)$$

$$h(x; \underline{\Psi}) = \zeta\alpha\beta x^{-\alpha-1} e^{-\beta x^{-\alpha}} \left(1 - e^{-\beta x^{-\alpha}}\right)^{-2}, x > 0; \ \left(\underline{\Psi} > \underline{0}\right), \qquad (4)$$

and

$$rh(x; \underline{\Psi}) = \frac{\alpha\beta\zeta x^{-\alpha-1} e^{-\beta x^{-\alpha}} \left(1 - e^{-\beta x^{-\alpha}}\right)^{-2}}{\left[\exp\left(\frac{\zeta}{e^{\beta x^{-\alpha}} - 1}\right) - 1\right]}, \ x > 0; \ \left(\underline{\Psi} > \underline{0}\right), \qquad (5)$$

The plots of the pdf, hrf and rhrf of OGE-IW $(\underline{\Psi})$ distribution are provided for different values of parameters in Figure 1 and Figure 6.



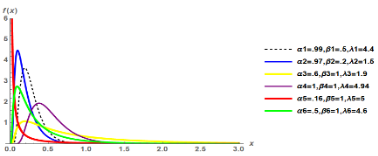


Figure 1

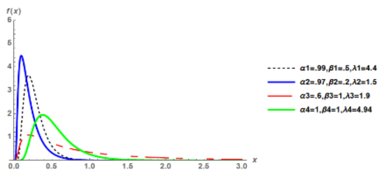


Figure 2

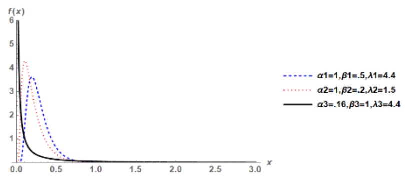
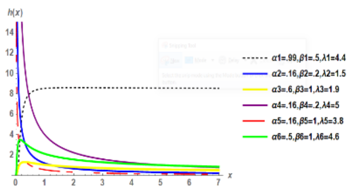


Figure٣ Figure ٤

The plots of the probability density functions at different values of the parameters



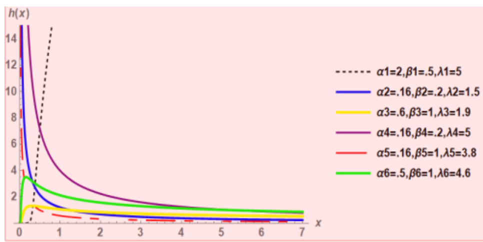


Figure 5 Figure 6

The plots of the hazard rate function at different values of the parameters

From Figure 1-4 displays OGE-IW $(\underline{\Psi})$ pdf for selected values of the parameters, where one can observe that the pdf of OGE-IW $(\underline{\Psi})$ distribution can be decreasing, unimodal or decreasing unimodal, and skewed to right (positive skewed). Also, from Figure 5-6 one can see that the plots of the hrf are a monotone decreasing, positive skewed and constant**.** This fact implies that the OGE-IW $(\underline{\Psi})$distribution is a flexible reliability mode and very useful for fitting data sets with various shapes. [For more details, see, Hassan *et al*. (2018)].

3. Estimation of Odd Generalized Exponential Inverted Weibull Distribution

In this section, the unknown parameters of the OGE-IW $(\underline{\Psi})$ distribution can be estimated using the ML method based on Type-II censored sample.

Suppose that $X_{(1)} \leq X_{(2)} \leq \cdots \leq X_{(r)}$ is a Type II censored sample of size r obtained from a life-test on n items whose lifetimes have an OGE-IW $(\underline{\Psi})$ distribution. Then the *likelihood function* (LF) in this case is given by (6) that is

$$L(\underline{\Psi};\underline{x}) = C\{\prod_{i=1}^{r} f(x_{(i)};\underline{\Psi})\}[s(x_{(r)};\underline{\Psi})]^{n-r}, \quad (6)$$

where

$$C = \frac{n!}{(n-r)!}, \quad (7)$$

$\underline{\Psi} = (\alpha,\beta,\zeta)'$ and $f(x_{(i)};\underline{\Psi})$, $s(x_{(r)};\underline{\Psi})$ are respectively given by (1) and (3).

By substituting (1) and (3) into (6) yields, the LF of OGE-IW $(\underline{\Psi})$ distribution is given by

$$L(\underline{\Psi};\underline{x}) \propto \left((\alpha\beta\zeta)^r \prod_{i=1}^{r}\left[x_i^{-\alpha-1}e^{-\beta x_i^{-\alpha}}\left(1-e^{-\beta x_i^{-\alpha}}\right)^{-2}\exp-\left(\frac{\zeta}{e^{\beta x_i^{-\alpha}}-1}\right)\right]\right)\times\left[\exp-\left(\frac{\zeta}{e^{\beta x_r^{-\alpha}}-1}\right)\right]^{n-r}, \quad (8)$$

The natural logarithm of $L(\underline{\Psi};\underline{x})$ is given by

$$\ell \propto r\ln(\alpha) + r\ln(\beta) + r\ln(\zeta) - (\alpha+1)\sum_{i=1}^{r}\ln x_i - \beta\sum_{i=1}^{r} x_i^{-\alpha} - \sum_{i=1}^{r}\left(\frac{\zeta}{e^{\beta x_i^{-\alpha}}-1}\right) - 2\sum_{i=1}^{r} ln\left(1-e^{-\beta x_i^{-\alpha}}\right) -(n-r)ln\left(\frac{\zeta}{e^{\beta x_r^{-\alpha}}-1}\right) \quad (9)$$

The ML estimators of $\underline{\Psi}$ can be obtained by differentiating ℓ in (9) with respect to α,β and ζ and then setting to zeros. Hence

$$\frac{\partial\ell}{\partial\zeta} = \frac{r}{\hat{\zeta}} - \sum_{i=1}^{r}\frac{1}{\left(e^{\hat{\beta} x_i^{-\hat{\alpha}}}-1\right)} - (n-r)\left(\frac{1}{\hat{\zeta}}\right) = 0\,, \quad (10)$$

$$\frac{\partial \ell}{\partial \alpha} = \frac{r}{\hat{\alpha}} + \sum_{i=1}^{r} \ln x_i + \hat{\beta} \sum_{i=1}^{r} x_i^{-\hat{\alpha}} \ln x_i$$

$$- \sum_{i=1}^{r} \left(\frac{\hat{\zeta} \hat{\beta} x_i^{-\hat{\alpha}} \ln x_i e^{\hat{\beta} x_i^{-\hat{\alpha}}}}{\left(e^{\hat{\beta} x_i^{-\hat{\alpha}}} - 1\right)^2} \right) - 2 \sum_{i=1}^{r} \left(\frac{\hat{\beta} x_i^{-\hat{\alpha}} \ln x_i e^{-\hat{\beta} x_i^{-\hat{\alpha}}}}{\left(1 - e^{-\hat{\beta} x_i^{-\hat{\alpha}}}\right)} \right)$$

$$+ (n-r) \left[\frac{\left(e^{\hat{\beta} x_r^{-\hat{\alpha}}} - 1\right) \hat{\beta} x_r^{-\hat{\alpha}} \ln x_r e^{\hat{\beta} x_r^{-\hat{\alpha}}}}{\hat{\zeta}} \right] = 0, \qquad (11)$$

and

$$\frac{\partial \ell}{\partial \beta} = \frac{r}{\hat{\beta}} - \sum_{i=1}^{r} x_i^{-\hat{\alpha}} + \sum_{i=1}^{r} \left(\frac{\hat{\zeta} x_i^{-\hat{\alpha}} e^{\hat{\beta} x_i^{-\hat{\alpha}}}}{\left(e^{\hat{\beta} x_i^{-\hat{\alpha}}} - 1\right)^2} \right)$$

$$-2 \sum_{i=1}^{r} \left(\frac{x_i^{-\hat{\alpha}} e^{-\hat{\beta} x_i^{-\hat{\alpha}}}}{\left(1 - e^{-\hat{\beta} x_i^{-\hat{\alpha}}}\right)} \right)$$

$$-(n-r) \left[\frac{\left(e^{\hat{\beta} x_r^{-\hat{\alpha}}} - 1\right) x_r^{-\hat{\alpha}} e^{\hat{\beta} x_r^{-\hat{\alpha}}}}{\hat{\zeta}} \right] = 0. \qquad (12)$$

The solution of the system of non-linear Equations (10), (11) and (12) can be solved numerically using Newton-Raphson method, to obtain the ML estimates of the parameters α, β and ζ.

4. Non-Bayesian Prediction for Odd Generalized Exponential Inverted Weibull Distribution Based on Two-Sample Prediction.

In this section, the conditional prediction (point and interval) for a future observation $Y_{(s)}$, of the OGE-IW $(\underline{\Psi})$distribution based on Type II censored data under two-sample prediction method are discussed.

Considering that $X_{(1)} \leq X_{(2)} \leq \cdots \leq X_{(r)}$ are the first r ordered life times in a random sample of n components Type II censoring whose failures times are identically distributed as a random variable X's having OGE-IW$(\underline{\Psi})$ of distribution; informative sample, and that $Y_{(1)}, Y_{(2)}, \dots, Y_{(m)}$ is a second independent random sample of size m of future observables from the same distribution, the future sample. Our aim is to predict the s^{th} order statistic in the future sample based on the informative sample.

For the future sample of size m, let $Y_{(s)}$ denotes the s^{th} order statistic, $1 < s < m$, the conditional density function of $Y_{(s)}$, given the vector of the parameters $\underline{\Psi}$, just by replacing $x_{(i)}$ by $y_{(s)}$ as follows is given by

$$h(y_{(s)}|\underline{\Psi}) = D(s)[F(y_{(s)}|\underline{\Psi})]^{s-1}.[1 - F(y_{(s)}|\underline{\Psi})]^{m-s} .f(y_{(s)}|\underline{\Psi}), \qquad y_{(s)} > 0 \qquad (13)$$

where s is the order statistic of the predicted future observation in the future sample,

$$D(s) = s\binom{m}{s} = \frac{m!}{(s-1)!(m-s)!} = \frac{1}{B(s,m-s+1)}, \text{and} \quad s = 1,2, \dots, m, \qquad (14)$$

Using the binomial expansion, for the term $[1 - F(y_{(s)}|\underline{\Psi})]^{m-s}$, yields then it can be expressed as follows

$$[1 - F(y_{(s)})]^{m-s} = \sum_{\ell_1=0}^{g-s} \binom{m-s}{\ell_1} (-1)^{\ell_1} [F(y_{(s)})]^{\ell_1},$$

Then, the $h(y_{(s)}|\underline{\Psi})$ in (13) is expressed

$$h(y_{(s)}|\underline{\Psi}) = D(s) f(y_{(s)}|\underline{\Psi}) \sum_{\ell_1=0}^{m-s} \binom{m-s}{\ell_1} (-1)^{\ell_1} [F(y_{(s)}|\underline{\Psi})]^{s+\ell_1-1}. \qquad (15)$$

Substituting (1) and (2) in (15) yields

$$h(y_{(s)}|\underline{\Psi}) = D(s) \sum_{\ell_2,\ell_3=0}^{\infty} \sum_{\ell_1=0}^{m-s} \binom{m-s}{\ell_1} (-1)^{\ell_1+\ell_2} \frac{\alpha\beta\zeta^{\ell_2+1}}{\ell_2!} \frac{\Gamma(\ell_2+2+\ell_3)}{\Gamma(\ell_2+2)\ell_3!} \times y_{(s)}{}^{-\alpha-1} e^{-\beta y_{(s)}{}^{-\alpha}(\ell_2+\ell_3+1)} \times \left[1 - \exp - \zeta\left(\frac{e^{-\beta y_{(s)}{}^{-\alpha}}}{1-e^{-\beta y_{(s)}{}^{-\alpha}}}\right)\right]^{s+\ell_1-1}, \quad (\underline{\Psi} > \underline{0}), \qquad (16)$$

By using binomial expansion

$$\left[1-\exp-\zeta\left(\frac{e^{-\beta y_{(s)}{}^{-\alpha}}}{1-e^{-\beta y_{(s)}{}^{-\alpha}}}\right)\right]^{s+\ell_1-1}=\sum_{\ell_4=0}^{s+\ell_1-1}(-1)^{\ell_4}\binom{s+\ell_1-1}{\ell_4}$$

$$\times\exp-\zeta\ell_4\left(\frac{e^{-\beta y_{(s)}{}^{-\alpha}}}{1-e^{-\beta y_{(s)}{}^{-\alpha}}}\right), \qquad (17)$$

Also, by using the exponential expansion

$$\exp-\zeta\ell_4\left(\frac{e^{-\beta y_{(s)}{}^{-\alpha}}}{1-e^{-\beta y_{(s)}{}^{-\alpha}}}\right)=\sum_{\ell_5=0}^{\infty}(-1)^{\ell_5}\frac{(\zeta\ell_4)^{\ell_5}}{\ell_5!}\left(\frac{e^{-\beta y_{(s)}{}^{-\alpha}}}{1-e^{-\beta y_{(s)}{}^{-\alpha}}}\right)^{\ell_5}, \qquad (18)$$

then

$$\left[1-\exp-\zeta\left(\frac{e^{-\beta y_{(s)}{}^{-\alpha}}}{1-e^{-\beta y_{(s)}{}^{-\alpha}}}\right)\right]^{s+\ell_1-1}$$

$$=\sum_{\ell_4=0}^{s+\ell_1-1}\sum_{\ell_5=0}^{\infty}(-1)^{\ell_4+\ell_5}\binom{s+\ell_1-1}{\ell_4}\frac{(\zeta\ell_4)^{\ell_5}}{\ell_5!}\left(\frac{e^{-\beta y_{(s)}{}^{-\alpha}}}{1-e^{-\beta y_{(s)}{}^{-\alpha}}}\right)^{\ell_5}.$$

For real value of β, using the binomial expansion

$$(1+z)^{-\beta}=\sum_{i=0}^{\infty}(-1)^i\binom{\beta+i-1}{i}z^i=\sum_{j=0}^{\infty}\frac{\Gamma(\beta+j)z^j}{\Gamma(\beta)j!},$$

$$|z|<1,\beta>0, \qquad (19)$$

for $|z|<1$ and β is a positive real non – integer.

Then, the previous $\left[1-\exp-\zeta\left(\frac{e^{-\beta y_{(s)}{}^{-\alpha}}}{1-e^{-\beta y_{(s)}{}^{-\alpha}}}\right)\right]^{s+\ell_1-1}$ is written as follows

$$\left[1-\exp-\zeta\left(\frac{e^{-\beta y_{(s)}{}^{-\alpha}}}{1-e^{-\beta y_{(s)}{}^{-\alpha}}}\right)\right]^{s+\ell_1-1}$$

$$=\sum_{\ell_4=0}^{s+\ell_1-1}\sum_{\ell_6,\ell_5=0}^{\infty}(-1)^{\ell_4+\ell_5}\binom{s+\ell_1-1}{\ell_4}\frac{(\zeta\ell_4)^{\ell_5}}{\ell_5!}\frac{\Gamma(\ell_6+\ell_5)}{\Gamma(\ell_5)\ell_6!}$$

$$\times\exp-\beta y_{(s)}{}^{-\alpha}(\ell_6+\ell_5). \qquad (20)$$

Substituting (20) in (16) yields

$h(y_{(s)}|\underline{\Psi})=$

$D(s)\sum_{(\ell_5,\ell_6,\ell_2,\ell_3=0)}^{\infty}\sum_{\ell_4=0}^{s+\ell_1-1}\sum_{\ell_1=0}^{m-s}\binom{m-s}{\ell_1}\binom{s+\ell_1-1}{\ell_4}(-1)^{\ell_1+\ell_2+\ell_4+\ell_5}$

$$\times \frac{\alpha\beta\zeta^{\ell_5+\ell_2+1}{\ell_4}^{\ell_5}}{\ell_2!\ell_5!\ell_6!}\frac{\Gamma(\ell_2+2+\ell_3)}{\Gamma(\ell_2+2)\ell_3!}\times\frac{\Gamma(\ell_6+\ell_5)}{\Gamma(\ell_5)\ell_6!}$$

$$\times\, {y_{(s)}}^{-\alpha-1}e^{-\beta {y_{(s)}}^{-\alpha}(\ell_2+\ell_3+1+\ell_6+\ell_5)}\,,\quad y_{(s)}>0;(\underline{\Psi}>\underline{0}),\quad (21)$$

So that, (٢1) can be written as

$$h(y_{(s)}|\underline{\Psi})=\sum\nolimits_{ai}^{****}\Omega_{2(\ell_1+\ell_2+\ell_3+\ell_4+\ell_5+\ell_6)}(\alpha,\beta,\zeta)$$

$$\times\, {y_{(s)}}^{-\alpha-1}e^{-\beta(\ell_2+\ell_3+1+\ell_6+\ell_5){y_{(s)}}^{-\alpha}}.$$

$$y_{(s)}>0;(\underline{\Psi}>\underline{0}),\qquad (22)$$

where $\Omega_{2(\ell_1+\ell_2+\ell_3+\ell_4+\ell_5+\ell_6)}(\alpha,\beta,\zeta)$ is a constant and

$$\Omega_{2(\ell_1+\ell_2+\ell_3+\ell_4+\ell_5+\ell_6)}(\alpha,\beta,\zeta) =$$

$$D(s)\binom{m-s}{\ell_1}\binom{s+\ell_1-1}{\ell_4}(-1)^{\ell_1+\ell_2+\ell_4+\ell_5}\frac{\alpha\beta\zeta^{\ell_5+\ell_2+1}{\ell_4}^{\ell_5}}{\ell_2!\ell_5!\ell_6!}\frac{\Gamma(\ell_2+2+\ell_3)}{\Gamma(\ell_2+2)\ell_3!}\times\frac{\Gamma(\ell_6+\ell_5)}{\Gamma(\ell_5)\ell_6!}.$$

and

$$\sum\nolimits_{ai}^{****}=\sum\nolimits_{(\ell_5,\ell_6,\ell_2,\ell_3=0)}^{\infty}\sum\nolimits_{\ell_4=0}^{s+\ell_1-1}\sum\nolimits_{\ell_1=0}^{m-s}\qquad (23)$$

The conditional density functions of $y_{(s)}$ is given by (22), can be used to find both the conditional point and interval predictors of $y_{(s)}$.

The *conditional predictor* (CP) (point and interval) for a future observation $Y_{(s)}$, given the informative sample of the OGE-IW($\underline{\Psi}$)distribution based on two-sample prediction scheme are derived.

Assuming that the parameters $\underline{\Psi}$ are unknown and independent, then the CP of $y_{(s)}$ given $\underline{\hat{\Psi}}_{ML}$ can be obtained using the conditional pdf of the s^{th} order statistic of the future sample, and by using the invariance property of the ML estimators, the vector of the parameters $\underline{\Psi}=(\alpha,\beta,\zeta)'$ will be replaced by their ML estimators $\alpha_{ML},\beta_{ML},\zeta_{ML}$ which was obtained from (10)-(12) as follows:

$$h_1(y_{(s)}|\underline{\hat{\Psi}}_{ML})=\sum\nolimits_{ai}^{****}\Omega_{2(\ell_1+\ell_2+\ell_3+\ell_4+\ell_5+\ell_6)}(\hat{\alpha},\hat{\beta},\hat{\zeta})\times$$

$${y_{(s)}}^{-\hat{\alpha}-1}\,e^{-\hat{\beta}\,(\ell_2+\ell_3+1+\ell_6+\ell_5){y_{(s)}}^{-\hat{\alpha}}},\qquad y_{(s)}>0;(\underline{\hat{\Psi}}_{ML}>\underline{0}),\qquad (24)$$

where

$\Omega_{2(\ell_1+\ell_2+\ell_3+\ell_4+\ell_5+\ell_6)}(\hat{\alpha},\hat{\beta},\hat{\zeta})$ is a constant and

$$\Omega_{2(\ell_1+\ell_2+\ell_3+\ell_4+\ell_5+\ell_6)}(\hat{\alpha},\hat{\beta},\hat{\zeta}) = D(s)\binom{m-s}{\ell_1}\binom{s+\ell_1-1}{\ell_4}(-1)^{\ell_1+\ell_2+\ell_4+\ell_5} \times \frac{\hat{\alpha}\hat{\beta}\hat{\zeta}^{\ell_5+\ell_2+1}\ell_4{}^{\ell_5}}{\ell_2!\ell_5!\ell_6!}\frac{\Gamma(\ell_2+2+\ell_3)}{\Gamma(\ell_2+2)\ell_3!}\times\frac{\Gamma(\ell_6+\ell_5)}{\Gamma(\ell_5)\ell_6!}. \qquad (25)$$

a. Point prediction based on conditional prediction

The CP of $y_{(s)}$for a future observation of the OGE-IW($\underline{\Psi}$) distribution based on Type II censoring samples can be derived using (24).

Assuming that the parameters $\underline{\Psi}$ are unknown and independent, then the CP of $y_{(s)}$ as follows:

$$\hat{y}_{(s)(ML)} = E(y_{(s)};\underline{\hat{\Psi}}_{ML}) = \int_{y_{(s)}}^{} y_{(s)}.h_1\left(y_{(s)}|\underline{\hat{\Psi}}_{ML}\right)dy_{(s)}$$
$$= \sum_{ai}^{****}\int_{y_{(s)}}^{} y_{(s)}{}^{-\hat{\alpha}} \times e^{-\hat{\beta}\,(\ell_2+\ell_3+1+\ell_6+\ell_5)y_{(s)}{}^{-\hat{\alpha}}} \times \Omega_{2(\ell_1+\ell_2+\ell_3+\ell_4+\ell_5+\ell_6)}(\hat{\alpha},\hat{\beta},\hat{\zeta})\,dy_{(s)}$$
$$=\sum_{ai}^{****}\Omega_{3(\ell_1+\ell_2+\ell_3+\ell_4+\ell_5+\ell_6)}(\hat{\alpha},\hat{\beta},\hat{\zeta}), \qquad (26)$$

where

$$\Omega_{3(\ell_1+\ell_2+\ell_3+\ell_4+\ell_5+\ell_6)}(\hat{\alpha},\hat{\beta},\hat{\zeta}) = \int_{y_{(s)}}^{} y_{(s)}{}^{-\hat{\alpha}} \times e^{-\hat{\beta}\,(\ell_2+\ell_3+1+\ell_6+\ell_5)y_{(s)}{}^{-\hat{\alpha}}} \times \Omega_{2(\ell_1+\ell_2+\ell_3+\ell_4+\ell_5+\ell_6)}(\hat{\alpha},\hat{\beta},\hat{\zeta})\,dy_{(s)}. \qquad (27)$$

and $D(s)$, s are defined in (14)and $\Omega_{2(\ell_1+\ell_2+\ell_3+\ell_4+\ell_5+\ell_6)}(\hat{\alpha},\hat{\beta},\hat{\zeta})$ is given by (25) and $\sum_{ai}^{****}$ is given by (23).

Numerical method can be used to obtain $\hat{y}_{(s)}$ in a closed form.

Special cases of the conditional predictors, $\hat{y}_{(s)}$, are:

- If $s = 1$, in (26), one can predict the minimum observable, $Y_{(1)}$, which represents the first failure time in a future sample of size m.

- If $s = m$, in (26), one can predict the maximum observable, $Y_{(m)}$, which represents the largest failure time in a future sample of size m.
- If $s = \frac{m+1}{2}$, in (26), one can predict the median observable if m is odd, $Y_{\left(\frac{m+1}{2}\right)}$, which represents the median failure time in a future sample of size m.

b. Interval prediction based on conditional prediction

A $(1-\tau)100$ % *conditional predictive bounds* (CPB) for the future observation $y_{(s)}$, such that $P\left((L_{S1}(\underline{x}) < y_{(s)} < U_{S1}(\underline{x}))\middle|\underline{x}\right) = (1-\tau)$, where, the lower and upper bounds $L_{S1}(\underline{x})$, $U_{S1}(\underline{x})$) can be obtained by evaluating from, In general are as follows:

$$P\left((y_{(s)} > L_{S_1}(\underline{x}))\middle|\underline{x}\right) = \int_{L_{S_1}(\underline{x})}^{\infty} h_1\left(y_{(s)}\middle|\underline{\hat{\Psi}}_{ML}\right) dy_{(s)} = 1 - \frac{\tau}{2} \quad (28)$$

and

$$P\left(\left(y_{(s)} > U_{S_1}(\underline{x})\right)\middle|\underline{x}\right) = \int_{U_{S_1}(\underline{x})}^{\infty} h_1\left(y_{(s)}\middle|\underline{\hat{\Psi}}_{ML}\right) dy_{(s)} = \frac{\tau}{2}. \quad (29)$$

Substituting (24) in (28) and (29), then the CPB are obtained as given below:

$$P\left((y_{(s)} > L_{S1}(\underline{x}))\middle|\underline{\hat{\Psi}}_{ML}\right) = D(s).\sum_{ai}^{****} \Omega_{2(\ell_1+\ell_2+\ell_3+\ell_4+\ell_5+\ell_6)}(\hat{\alpha},\hat{\beta},\hat{\zeta}) \times \int_{L_{S1}(\underline{x})}^{\infty} y_{(s)}^{-\hat{\alpha}-1} e^{-\hat{\beta}(\ell_2+\ell_3+1+\ell_6+\ell_5)y_{(s)}^{-\hat{\alpha}}} dy_{(s)} = 1 - \frac{\tau}{2}, \quad (30)$$

and

$$P\left((y_{(s)} > U_{S1}(\underline{x}))\middle|\underline{\hat{\Psi}}_{ML}\right) = D(s).\sum_{ai}^{****} \Omega_{2(\ell_1+\ell_2+\ell_3+\ell_4+\ell_5+\ell_6)}(\hat{\alpha},\hat{\beta},\hat{\zeta}) \times \int_{U_{S1}(\underline{x})}^{\infty} y_{(s)}^{-\hat{\alpha}-1} e^{-\hat{\beta}(\ell_2+\ell_3+1+\ell_6+\ell_5)y_{(s)}^{-\hat{\alpha}}} dy_{(s)} = \frac{\tau}{2}, \quad (31)$$

where (s), s are defined in (14) and $\Omega_{2(\ell_1+\ell_2+\ell_3+\ell_4+\ell_5+\ell_6)}(\hat{\alpha},\hat{\beta},\hat{\zeta})$ is given by (25) and $\sum_{ai}^{****}$ ⬚ is given by (23).

Equations (30) and (31) can be solved numerically.

5. Bayesian Estimation Based on the Balanced Loss Function

In this section, the Bayesian estimation of the OGE-IW $(\underline{\Psi})$ distribution can be obtained based on Type-II censored sample and balanced loss functions.

5.1 Bayesian estimation

Let $X_{(1)} \leq X_{(2)} \leq, \dots, \leq X_{(r)}$ denote Type II censored sample of size r obtained from a life-test on n items with lifetimes from OGEIW$(\underline{\Psi})$ distribution.

Considering the prior knowledge of the vector of parameters $\underline{\Psi} = (\alpha, \beta, \zeta)'$, is adequately represented by gamma priors are assumed as independent prior distributions for the parameters Then the joint prior distribution of all the unknown parameters has a joint pdf given by

$$\pi(\underline{\Psi}) = \prod_{j=1}^{3} \pi(\Psi_j), \tag{32}$$

where

$\Psi_j \sim$ gamma (a_j, b_j) and a_j, b_j are the hyper-parameters of the prior distribution for

$j = 1,2,3,$ with the following pdf

$$\pi(\underline{\Psi}_j; a_j, b_j) = \frac{{b_j}^{a_j}}{\Gamma(a_j)} {\Psi_j}^{a_j - 1} \exp(-b_j \Psi_j),$$

$$\underline{\Psi}_j > \underline{0};\ (a_j, b_j) > \underline{0}, \quad j = 1,2,3, \tag{33}$$

where $\Psi_1 = \alpha, \Psi_2 = \beta$ and $\Psi_3 = \zeta$, $\Psi_j \sim$ gamma (a_j, b_j),

$\underline{\Psi} = (\alpha, \beta, \zeta)'$.Then the joint prior distribution of all the unknown parameters has a joint pdf given by

$$\pi(\underline{\Psi}; \underline{a}, \underline{b}) \propto (\alpha^{a_1 - 1} \beta^{a_2 - 1} \zeta^{a_3 - 1}) \exp[-(b_1\, \alpha + b_2\, \beta + b_3\, \zeta)],$$

$$\underline{\Psi} > \underline{0};\ (\underline{a}, \underline{b}) > \underline{0}. \tag{34}$$

Combining the LF in (8) can be written as follows:

$$L(\underline{\Psi}|\underline{x}) \propto (\alpha\beta\zeta)^r \exp\left[-\alpha\sum_{i=1}^{r}\ln\left(x_{(i)}\right) - \beta\sum_{i=1}^{r}x_i^{-\alpha}\right]$$
$$\times \exp\left[(n-r)ln\left(\frac{\zeta}{e^{\beta x_r^{-\alpha}}-1}\right) - \sum_{i=1}^{r}\left(\frac{\zeta}{e^{\beta x_i^{-\alpha}}-1}\right)\right.$$
$$\left.-2\sum_{i=1}^{r}ln\left(1-e^{-\beta x_i^{-\alpha}}\right)\right] \qquad (35)$$

and the joint prior distribution given by (34), then the joint posterior distribution of the parameters, for $\underline{\Psi} = (\alpha,\beta,\zeta)'$ can be obtained as follows:

$$\pi(\underline{\Psi}|\underline{x}) \propto L(\underline{\Psi}|\underline{x})\pi(\underline{\Psi};\underline{a},\underline{b}),$$
$$\propto \left(\alpha^{a_1+r-1}\beta^{a_2+r-1}\zeta^{a_3+r-1}\right)\exp\left[-\alpha\sum_{i=1}^{r}\ln\left(x_{(i)}\right) - \beta\sum_{i=1}^{r}x_i^{-\alpha}\right]$$
$$\times \exp\left[-\sum_{i=1}^{r}\left(\frac{\zeta}{e^{\beta x_i^{-\alpha}}-1}\right) - 2\sum_{i=1}^{r}ln\left(1-e^{-\beta x_i^{-\alpha}}\right)\right]$$
$$\times \exp\left[(n-r)ln\left(\frac{\zeta}{e^{\beta x_r^{-\alpha}}-1}\right) - (b_1\,\alpha + b_2\,\beta + b_3\,\zeta)\right], \qquad (36)$$

The joint posterior distribution given by (36) can be written as follows:

$$\pi(\underline{\Psi}|\underline{x}) = K_1 L(\underline{\Psi}|\underline{x})\pi(\underline{\Psi};\underline{a},\underline{b}),$$
$$= K_1\left(\alpha^{a_1+r-1}\beta^{a_2+r-1}\zeta^{a_3+r-1}\right)\exp\left[-\alpha\sum_{i=1}^{r}\ln\left(x_{(i)}\right) - \beta\sum_{i=1}^{r}x_i^{-\alpha}\right]$$
$$\times \exp\left[-\sum_{i=1}^{r}\left(\frac{\zeta}{e^{\beta x_i^{-\alpha}}-1}\right) - 2\sum_{i=1}^{r}ln\left(1-e^{-\beta x_i^{-\alpha}}\right)\right]$$
$$\times \exp\left[(n-r)ln\left(\frac{\zeta}{e^{\beta x_r^{-\alpha}}-1}\right) - (b_1\,\alpha + b_2\,\beta + b_3\,\zeta)\right], \qquad (37)$$

where K_1 is a normalizing constant.
Then

$$K_1^{-1} = \int_{\underline{\Psi}}\left(\alpha^{a_1+r-1}\beta^{a_2+r-1}\zeta^{a_3+r-1}\right)\exp\left[-\alpha\sum_{i=1}^{r}\ln\left(x_{(i)}\right) - \beta\sum_{i=1}^{r}x_i^{-\alpha}\right]$$
$$\times \exp\left[-\sum_{i=1}^{r}\left(\frac{\zeta}{e^{\beta x_i^{-\alpha}}-1}\right) - 2\sum_{i=1}^{r}ln\left(1-e^{-\beta x_i^{-\alpha}}\right)\right]$$
$$\times \exp\left[(n-r)ln\left(\frac{\zeta}{e^{\beta x_r^{-\alpha}}-1}\right) - (b_1\,\alpha + b_2\,\beta + b_3\,\zeta)\right] d\underline{\Psi}, \qquad (38)$$

where

$$\int_{\underline{\Psi}} = \int_{\alpha}\int_{\beta}\int_{\zeta}, d\underline{\Psi} = d\alpha\, d\beta\, d\zeta, \text{ and } \underline{\Psi} = (\alpha,\beta,\zeta). \qquad (39)$$

The marginal posterior distributions of the parameters, $\underline{\Psi} = (\alpha,\beta,\zeta)$ are obtained from (37) as follows:

$$\pi\left(\Psi_\tau \middle| \underline{x}\right) = \int_{\underline{\Psi}_j} \pi\left(\underline{\Psi} \middle| \underline{x}\right) d\underline{\Psi}_j, \qquad \tau \neq j, \quad \tau, j = 1,2,3. \tag{40}$$

The Bayes estimators for the parameters of the OGE-IW$(\underline{\Psi})$ distribution are considered under the BLF. The estimator of a function using BLF is a mixture of the ML estimator, least squares estimators or any other estimator and the Bayes estimator using any loss function.

5.2 Balanced loss functions

Ahmadi *et al.* (2009) suggested the use of the *balanced loss function* (BLF), which was originated by Zellner (1994), to be of the form

$$L^*\left(\Psi, \tilde{\Psi}\right) = \omega\, l\left(\Psi, \hat{\Psi}\right) + (1-\omega)\, l\left(\Psi, \tilde{\Psi}\right), \tag{41}$$

where $l\left(\Psi, \tilde{\Psi}\right)$ is an arbitrary loss function, $\hat{\Psi}$ is a chosen target estimator of Ψ and the weight $\omega \epsilon [0, 1]$.

The BLF specializes to various choices of loss functions as a symmetric loss function and as an asymmetric loss function such as the absolute error loss, entropy, *linear exponential* (LINEX) and *squared error loss* (SEL) functions and generalizes SEL function.

The Bayes estimator of Ψ, using the BSEL function is given by

$$\tilde{\Psi}_{\text{BSE}} = \omega\, \hat{\Psi}_{\text{ML}} + (1-\omega)\, \tilde{\Psi}_{\text{SE}}, \tag{42}$$

where $\hat{\Psi}_{\text{ML}}$ is the ML estimator of Ψ and $\tilde{\Psi}_{\text{SE}}$ is its Bayes estimator using SEL function. Also, the Bayes estimator using the BLL function of Ψ is obtained as follows:

$$\tilde{\Psi}_{BL} = \frac{-1}{v} \ln\left\{\omega\, exp\left(-v\hat{\Psi}_{ML}\right) + (1-\omega)\, E\left(exp(-v\Psi)\, \middle| \underline{x}\right)\right\}, \tag{43}$$

where $v \neq 0$ is the shape parameter of BLL function.

Many authors used the symmetric and asymmetric distributions to construct Bayes estimators for various other distributions BLF. To learn more, [see Deniz (2006), AL-Hussaini and Hussein (2012), Abushal and AL-Zaydi (2017)].

The Bayes and E-Bayes prediction are considered under the BSEL function; as a symmetric loss function and BLL function; as an asymmetric loss function.

6. Bayesian and E-Bayesian Prediction for Odd Generalized Exponential Inverted Weibull Distribution based on Two-Sample Prediction.

This section, two -sample Bayesian and E-Bayesian prediction of the future ordered failure, $Y_{(s)}$, from OGEIW$(\underline{\Psi})$ distribution based on Type II censoring scheme is derived under two different loss functions, the BSEL function; as a symmetric loss function and BLL function; as an asymmetric loss function . Moreover, credible interval of $Y_{(s)}$ is obtained.

6.1 Bayesian prediction

In this section, Bayesian prediction (point and interval) for a future observation $y_{(s)}$, of the OGEIW$(\underline{\Psi})$ distribution based on two-sample prediction scheme are considered. Assuming that the parameters $\underline{\Psi}$ are unknown and independent, then the *Bayesian predictive density* (BPD) of $y_{(s)}$ given $\underline{x}$ based on informative prior can be obtained, using the following equation

$$q(y_{(s)}|\underline{x}) = \int_{\underline{\Psi}} h(y_{(s)}|\underline{\Psi})\, \pi(\underline{\Psi}|\underline{x}) d\underline{\Psi}, \quad y_{(s)} > \underline{0}; (\underline{\Psi} > \underline{0}), \quad (44)$$

where $\pi(\underline{\Psi}|\underline{x})$,$\int_{\underline{\Psi}} d\underline{\Psi}$ and $h(y_{(s)}|\underline{\Psi})$ are given by (37), (39) and (22) respectively.

Substituting (37) and (22) into (44), then the BPD of $y_{(s)}$ given $\underline{x}$ is given by

$$\begin{aligned} q(y_{(s)}|\underline{x}) = K_1 \textstyle\sum_{ai}^{****} \int_{\underline{\Psi}} \Omega^{**}_{2(\ell_1+\ell_2+\ell_3+\ell_4+\ell_5+\ell_6)}(\alpha,\beta,\zeta) \\ \times {y_{(s)}}^{-\alpha-1} \times \ exp\,[-\alpha \textstyle\sum_{i=1}^{r} ln(x_{(i)}) \\ -\beta \textstyle\sum_{i=1}^{r} {x_i}^{-\alpha} \ - \beta(\ell_2+\ell_3+1+\ell_6+\ell_5){y_{(s)}}^{-\alpha}] \\ \times exp\,[-\textstyle\sum_{i=1}^{r}\left(\frac{\zeta}{e^{\beta {x_i}^{-\alpha}}-1}\right) - 2\textstyle\sum_{i=1}^{r} ln\left(1-e^{-\beta {x_i}^{-\alpha}}\right)] \\ \times \exp\left[(n-r) ln\left(\frac{\zeta}{e^{\beta {x_r}^{-\alpha}}-1}\right) - (b_1\,\alpha + b_2\,\beta + b_3\,\zeta)\right] d\underline{\Psi}, \end{aligned}$$

$$y_{(s)} > \underline{0}; (\underline{\Psi} > \underline{0}), \quad (45)$$

where $D(s)$, s are defined in (14), K_1^{-1} is given by (38), $\int_{\underline{\Psi}}$, $d\underline{\Psi}$ and $\underline{\Psi}$ are given by (39), $\sum_{ai}^{****}$ is given by (23) and

$$\Omega^{**}{}_{2(\ell_1+\ell_2+\ell_3+\ell_4+\ell_5+\ell_6)}(\alpha,\beta,\zeta) = D(s)\binom{m-s}{\ell_1}\binom{s+\ell_1-1}{\ell_4}(-1)^{\ell_1+\ell_2+\ell_4+\ell_5} \times \frac{\alpha^{a_1+r}\beta^{a_2+r}\zeta^{\ell_5+\ell_2+a_3+r}\ell_4{}^{\ell_5}}{\ell_2!\ell_5!\ell_6!} \times \frac{\Gamma(\ell_2+2+\ell_3)}{\Gamma(\ell_2+2)\ell_3!} \times \frac{\Gamma(\ell_6+\ell_5)}{\Gamma(\ell_5)\ell_6!}. \quad (46)$$

a. Point prediction based on Bayesian prediction

The Bayes point predictor is derived under two types of loss functions BSEL function and BLL function.

1. Balanced squared error loss function

The *Bayes predictor* (BP) for the future observation $Y_{(s)}$, under BSEL function can be derived using (42) and (45) as given below

$$\hat{y}_{(s)(BBSE)} = \omega\hat{y}_{(s)(ML)} + (1-\omega)\int_{y_{(s)}} y_{(s)}\, q\left(y_{(s)}\middle|\underline{x}\right) dy_{(s)}$$

$$= \omega\hat{y}_{(s)(ML)} + (1-\omega)K_1\sum_{ai}^{****}\int_{\underline{\Psi}_*} y_{(s)}{}^{-\alpha}$$

$$\times\, \Omega^{**}{}_{2(\ell_1+\ell_2+\ell_3+\ell_4+\ell_5+\ell_6)}(\alpha,\beta,\zeta)$$

$$\times\, exp\left[-\alpha\sum_{i=1}^{r} ln\left(x_{(i)}\right) - \beta\sum_{i=1}^{r} x_i{}^{-\alpha} - \beta(\ell_2+\ell_3+1+\ell_6+\ell_5)\right]y_{(s)}{}^{-\alpha}$$

$$\times \exp\left[-\sum_{i=1}^{r}\left(\frac{\zeta}{e^{\beta x_i{}^{-\alpha}}-1}\right) - 2\sum_{i=1}^{r} ln\left(1-e^{-\beta x_i{}^{-\alpha}}\right)\right]$$

$$\times \exp\left[(n-r)ln\left(\frac{\zeta}{e^{\beta x_r{}^{-\alpha}}-1}\right) - (b_1\,\alpha + b_2\,\beta + b_3\,\zeta)\right] d\underline{\Psi}_*, \quad (47)$$

where $\hat{y}_{(s)(ML)_4}$ is the ML predictor for the future observation $y_{(s)}$, $D(s)$, s are defined in (14), K_1^{-1} is given by (38), $\sum_{ai}^{****}$ is given by (23), $\Omega^{**}{}_{2(\ell_1+\ell_2+\ell_3+\ell_4+\ell_5+\ell_6)}(\alpha,\beta,\zeta)$ is given by (46) and

$$\int_{\underline{\Psi}_*} = \int_{y_{(s)}}\int_{\alpha}\int_{\beta}\int_{\zeta},\ d\underline{\Psi}_* = d\alpha\, d\beta\, d\zeta\, dy_{(s)}. \quad (48)$$

2. Balanced linear exponential loss function

The BPE for the future observation $Y_{(s)}$, under BLL function can be derived using (43) and (45) as follows:

$$\hat{y}_{(s)(\mathrm{BBL})} = \frac{-1}{v}\ln\ \omega \exp\left(-v\hat{y}_{(s)(ML)}\right)$$
$$+(1-\omega)\int_{y_{(s)}} \exp\left(-v\, y_{(s)}\right)\ q\left(y_{(s)}\middle|\underline{x}\right) dy_{(s)}$$
$$= \frac{-1}{v}\ln\ \omega \exp\left(-v\hat{y}_{(s)(ML)}\right)$$
$$+(1-\omega)K_1 \sum_{ai}^{****} \int_{\underline{\Psi_*}} \Omega^{**}{}_{2(\ell_1+\ell_2+\ell_3+\ell_4+\ell_5+\ell_6)}(\alpha,\beta,\zeta)$$
$$\times exp\left[-\alpha \sum_{i=1}^{r} ln\left(x_{(i)}\right) - \beta \sum_{i=1}^{r} x_i{}^{-\alpha} - \beta(\ell_2+\ell_3+1+\ell_6+\ell_5) y_{(s)}{}^{-\alpha}\right]$$
$$\times exp\left[-v y_{(s)} - \sum_{i=1}^{r}\left(\frac{\zeta}{e^{\beta x_i{}^{-\alpha}}-1}\right) - 2\sum_{i=1}^{r} ln\left(1-e^{-\beta x_i{}^{-\alpha}}\right)\right] \times y_{(s)}{}^{-\alpha-1}$$
$$\times exp\left[(n-r)ln\left(\frac{\zeta}{e^{\beta x_r{}^{-\alpha}}-1}\right) - (b_1\,\alpha + b_2\,\beta + b_3\,\zeta)\right] d\underline{\Psi_*}, \quad (49)$$

where $\hat{y}_{(s)(ML)_4}$ is the ML predictor for the future observation $y_{(s)}$, $D(s)$, s are defined in (14), $\mathrm{K}_1{}^{-1}$ is given by (38), $\sum_{ai}^{****}$ is given by (23), $\Omega^{**}{}_{2(\ell_1+\ell_2+\ell_3+\ell_4+\ell_5+\ell_6)}(\alpha,\beta,\zeta)$ is given by (46), $\int_{\underline{\Psi_*}}$ and $d\underline{\Psi_*}$ are defined in (48).

Special cases of $\hat{y}_{(s)(BBSE)}$ and $\hat{y}_{(s)(\mathrm{BBL})}$ are:

I. When $s=1$, in (47), (49), the BP of the first observation in the future sample can be obtained.

II. When the future sample size is odd and by setting $s=\frac{m+1}{2}$, in (47), (49), the BP of the median observation in the future sample can be obtained.

III. When $s=m$, in (47), (49), the BP of the last observation in the future sample is obtained.

b. Bayesian prediction bounds

A $(1-\tau)100\%$ *Bayesian prediction bounds* (BPBs), for the future observation $y_{(s)}$, such that

$P\left((\mathrm{L}_{SB1}(\underline{x}) < y_{(s)} < \mathrm{U}_{SB1}(\underline{x}))|\underline{x}\right) = (1-\tau)$, the lower and upper bounds $\mathrm{L}_{SB1}(\underline{x})$, $\mathrm{U}_{SB1}(\underline{x})$ can be obtained by evaluating from (45), In general are as follows:

$$P\left((y_{(s)} > \mathrm{L}_{SB1}(\underline{x}))|\underline{x}\right) = \int_{\mathrm{L}_{SB1}(\underline{x})}^{\infty} h_2\left(y_{(s)}|\underline{x}\right) dy_{(s)} = 1 - \frac{\tau}{2}, \quad (50)$$

and

$$P\left((y_{(s)} > \mathrm{U}_{SB1}(\underline{x}))|\underline{x}\right) = \int_{\mathrm{U}_{SB1}(\underline{x})}^{\infty} h_2\left(y_{(s)}|\underline{x}\right) dy_{(s)} = \frac{\tau}{2}, \quad (51)$$

Substituting (45) in (50) and (51), then the BPB are obtained as given below:

$$P\left((y_{(s)} > \mathrm{L}_{SB1}(\underline{x}))|\underline{x}\right) = K_1 \sum_{ai}^{****} \int_{\mathrm{L}_{SB1}(\underline{x})}^{\infty} \int_{\underline{\Psi}} \Omega^{**}{}_{2(\ell_1+\ell_2+\ell_3+\ell_4+\ell_5+\ell_6)}(\alpha,\beta,\zeta)$$
$$\times y_{(s)}{}^{-\alpha-1} \times \ exp\left[-\alpha \sum_{i=1}^{r} ln(x_{(i)}) - \beta \sum_{i=1}^{r} x_i{}^{-\alpha} - \beta(\ell_2+\ell_3+1+\ell_6+\ell_5) y_{(s)}{}^{-\alpha}\right]$$
$$\times exp\left[-\sum_{i=1}^{r}\left(\frac{\zeta}{e^{\beta x_i^{-\alpha}}-1}\right) - 2\sum_{i=1}^{r} ln\left(1-e^{-\beta x_i^{-\alpha}}\right)\right]$$
$$\times \exp\left[(n-r) ln\left(\frac{\zeta}{e^{\beta x_r^{-\alpha}}-1}\right) - (b_1\,\alpha + b_2\,\beta + b_3\,\zeta)\right] d\underline{\Psi} dy_{(s)}$$
$$= 1 - \frac{\tau}{2}, \quad (52)$$

and

$$P\left((y_{(s)} > \mathrm{U}_{SB1}(\underline{x}))|\underline{x}\right) = K_1 \sum_{ai}^{****} \int_{\mathrm{U}_{SB1}(\underline{x})}^{\infty} \int_{\underline{\Psi}} \Omega^{**}{}_{2(\ell_1+\ell_2+\ell_3+\ell_4+\ell_5+\ell_6)}(\alpha,\beta,\zeta)$$
$$\times y_{(s)}{}^{-\alpha-1} \times \ exp\left[-\alpha \sum_{i=1}^{r} ln(x_{(i)}) - \beta \sum_{i=1}^{r} x_i{}^{-\alpha} - \beta(\ell_2+\ell_3+1+\ell_6+\ell_5) y_{(s)}{}^{-\alpha}\right]$$
$$\times exp\left[-\sum_{i=1}^{r}\left(\frac{\zeta}{e^{\beta x_i^{-\alpha}}-1}\right) - 2\sum_{i=1}^{r} ln\left(1-e^{-\beta x_i^{-\alpha}}\right)\right]$$
$$\times \exp\left[(n-r) ln\left(\frac{\zeta}{e^{\beta x_r^{-\alpha}}-1}\right) - (b_1\,\alpha + b_2\,\beta + b_3\,\zeta)\right] d\underline{\Psi} dy_{(s)}$$
$$= \frac{\tau}{2}, \quad (53)$$

where (s), s are defined in (14) , $\mathrm{K}_1{}^{-1}$ is given by (38) , $\sum_{ai}^{****}$ is given by (23) , $\Omega^{**}{}_{2(\ell_1+\ell_2+\ell_3+\ell_4+\ell_5+\ell_6)}(\alpha,\beta,\zeta)$ is given by (46) , $\int_{\underline{\Psi_*}}$ and $d\underline{\Psi_*}$ are defined in (48).

Equations (52) and (53) can be solved numerically.

6.2 E-Bayesian prediction

This subsection, the E-Bayesian prediction (point and interval) for a future observation $y_{(s)}$, of the OGEIW$(\underline{\Psi})$ distribution based on two-sample prediction scheme are obtained.

According to Han (2007), the hyper-parameters a_j and b_j should be selected to guarantee that $\pi(\theta_j; a_j, b_j)$, are decreasing functions of Ψ_j, ($j = 1,2,3$).
The derivative of $\pi(\underline{\Psi}_j; a_j, b_j)$ with respect to $\underline{\Psi}_j$ is given below

$$\frac{d\,\pi(\underline{\Psi}_j;a_j,b_j)}{d\underline{\Psi}_j} = \frac{{b_j}^{a_j}}{\Gamma(a_j)}\,{\Psi_j}^{a_j-2}\,exp(-b_j\Psi_j)\,[(a_j-1)-b_j\Psi_j],$$

$$j = 1,2,3, \qquad (54)$$

for $0 < a_j < 1$ and $b_j > 0$, then $\frac{d\,\pi(\underline{\Psi}_j;a_j,b_j)}{d\underline{\Psi}_j} < 0$, which means that $\pi(\underline{\Psi}_j; a_j, b_j)$ can be decreasing functions of $\underline{\Psi}_j$.

The E-Bayes estimators of the parameters are obtained based on three different distributions of the hyper-parameters a_j and b_j. These distributions are used to investigate the effect of different prior distributions on the E-Bayesian estimation of $\underline{\Psi}_j$.

Assuming that the hyper-parameters a_j and b_j are independent with bivariate density functions

$$\pi_\hbar(a_j, b_j) = \pi_\hbar(a_j)\,\pi_\hbar(b_j), \quad j = 1,2,3 \;\; \hbar = 1,2,\dots,6. \qquad (55)$$

Then, the bivariate uniform hyper prior distributions are:

$$\pi_\hbar(a_j, b_j) = \frac{2(c_j-b_j)}{c_j^2}, \qquad 0 < a_j < 1, 0 < b_j < c_j, \quad (56)$$

$$\pi_\hbar(a_j, b_j) = \frac{1}{c_j}, \qquad 0 < a_j < 1, \; 0 < b_j < c_j, \quad (57)$$

$$\pi_\hbar(a_j, b_j) = \frac{2b_j}{c_j^2}, \qquad 0 < a_j < 1, \; 0 < b_j < c_j. \quad (58)$$

The E-Bayes estimators of $\underset{\sim}{\Psi}_{j}$ (expectation of the Bayes estimators of $\underset{\sim}{\Psi}_{j}$) can be derived as follows:

$$\underset{\sim}{\tilde{\Psi}}_{jEB} = E_{\pi_{\hbar}}\left(\underset{\sim}{\tilde{\Psi}}_{jB}(a_j, b_j)\right) = \iint_D \underset{\sim}{\tilde{\Psi}}_{jB}(a_j, b_j)\, \pi_{\hbar}(a_j, b_j)\; da_j\, db_j,$$

$$j = 1,2,3, \hbar = 1,2, \dots ,6, \quad (59)$$

where $E_{\pi_{\hbar}}(\hbar = 1,2, \dots ,6)$ stands for the expectation of the bivariate hyper prior distributions, D is the domain of the function $\pi_{\hbar}(a_j, b_j)$ and $\underset{\sim}{\tilde{\Psi}}_{jB}(a_j, b_j)$ are the Bayes estimators of the parameters $\underset{\sim}{\Psi}_{j}, j = 1,2,3$ based on BSEL and BLL functions.

a. Point prediction based on E- Bayesian prediction

The Bayes point predictor is derived under two types of loss functions, BSEL function and BLL function.

- **Balanced squared error loss function**

The three *E-Bayes predictors* (EBPs) for the future observation $Y_{(s)}$, under BSEL function can be obtained by substituting (47) and (56)-(58) in (59) as given below

$$\hat{y}_{(s)EBBS\hbar} = E_{\pi_{\hbar}}\left(\hat{y}_{(s)BBSE}\right) = \int_0^c \int_0^1 \hat{y}_{(s)BBSE}\, \pi_{\hbar}(a, b)\, dadb\,,$$

$$\hbar = 1,2,3. \quad (60)$$

- **Balanced linear exponential loss function**

The three EBPs for the future observation $Y_{(s)}$, under BLL function can be derived by substituting (49) and (56)-(58) in (59) as follows:

$$\hat{y}_{(s)EBBL\hbar} = E_{\pi_{\hbar}}\left(\hat{y}_{(s)BBL}\right) = \int_0^c \int_0^1 \hat{y}_{(s)BBL}\, \pi_{\hbar}(a, b)\, dadb\,,$$

$$\hbar = 1,2,3. \quad (61)$$

b. E- Bayesian prediction bounds based on two-sample prediction

A $(1-\tau)100\,\%$ *E-Bayesian prediction bounds* (EBPBs) for the future observation $y_{(s)}$, such that $P\left((L_{SB2}(\underset{\sim}{x}) < y_{(s)} < U_{SB2}(\underset{\sim}{x}))\middle|\underset{\sim}{x}\right) = (1-\tau)$, the lower and upper bounds $L_{SB2}(\underset{\sim}{x})$, $U_{SB2}(\underset{\sim}{x})$ can be obtained by substituting (52) and (56)-(58) in (59) , (53) and (56)-(58) in (59) , respectively.

Special cases of $\hat{y}_{(s)EBBS\hbar}$ and $\hat{y}_{(s)EBBL\hbar}$ are:

- If $s = 1$, in (60) and (61), one can predict the minimum observable, $Y_{(1)}$, which represents the first failure time in a future sample of size m.
- If $s = m$, in (60) and (61), one can predict the maximum observable,$Y_{(m)}$, which represents the largest failure time in a future sample of size m.
- If $s = \frac{m+1}{2}$, in (60) and (61), one can predict the median observable if n is odd, $Y_{\left(\frac{m+1}{2}\right)}$, which represents the median failure time in a future sample of size m.

7. Numerical Illustration

This section aims to investigate the precision of the theoretical results of prediction on the basis of simulated and real data sets.

7.1 Simulation algorithm

In this subsection, the conditional ML, Bayes and E-Bayes predictors (point and interval) for a future observation from the OGEIW$(\underline{\Psi})$ distribution based on Type II censored data are computed for the two-sample case. Simulation study and real data are applied to obtain the ML predictors using Mathematica 11 and to calculate the Bayes and E-Bayes predictors, R programming language is used.

7.1.1 Maximum likelihood prediction

The steps of the simulation procedure based on Type II censored data are as follows:

Step 1: For given values of α, β and ζ, random samples of size n are generated from the OGEIW$(\underline{\Psi})$ distribution.

- The transformation between the uniform distribution and OGEIW$(\underline{\Psi})$ distribution is obtained from Hassan *et al.* (2018) as follows:

$$x_u = \left[\frac{1}{\beta} ln\left[\frac{-\lambda}{ln(1-u)} + 1\right]\right]^{\frac{-1}{\alpha}}, \quad 0 < u < 1.$$

Step 2: For each sample size n, sort the x_i`s, such that

$x_{(1)} \leq x_{(2)} \leq \cdots \leq x_{(n)}$.

Step 3: The number of failures r are chosen to be less than or equal the sample size n.

Step 4: The ML estimates for the parameters α, β and ζ are computed based on Type II censored scheme.

Step 5: Substituting the conditional ML estimates of the parameters in the equation of $\hat{y}_{(s)(ML)}$ and for given values for s, the conditional MLP for the future observation $Y_{(s)}$, can be computed based on 80% level of Type II censored sample.

Step 6: Using the ML estimates for the parameters and a certain value of s, the conditional MLPB for the future observation $Y_{(s)}$, can be computed under Type II censored sample.

Step 7: Repeat all the previous steps N=10000 times.

7.1.2 Bayesian and E-Bayesian prediction

Step 1: Generate a_j and b_j from the bivariate uniform hyperprior distributions; $\pi_\hbar(a_j, b_j)$, $j = 1,2,3, \hbar = 1,2, \dots, 9$, given in (56)-(58).

Step 2: For given values of a_j and b_j, generate α, β and ζ from the gamma prior distributions.

Step 3: Applying the previous generation steps, Type II censored sample can be generated from the OGEIW$(\underline{\Psi})$ distribution.

Step 4: Calculate the joint posterior distribution for the parameters based on Type II censored sample from the OGEIW$(\underline{\Psi})$ distribution.

Step 5: The BPD of the future observation $Y_{(s)}$, can be obtained.

Step 6: The BP is calculated based on BSEL and BLL functions. Also, the BPB is evaluated.

Step 7: Using the BP, the EBPs for a future observation from the OGEIW$(\underline{\Psi})$ distribution based on BSEL and BLL functions are calculated based on 80% and 60% level of Type II censored sample. Similarly, using the BPB, the EBPB are evaluated.

Step 8: Repeat all the previous steps N=10000 times.

Table 1-٢ present the conditional prediction of the future observations from OGEIW$(\underline{\Psi})$ distribution and the bounds of the conditional interval of the future observations along with their lengths based on the two-sample prediction scheme.

Tables ٦-٩ present the Bayes and E-Bayes prediction based on BSEL and BLL functions of the future observations from OGEIW$(\underline{\Psi})$ distribution along with their lengths based on the two-sample prediction scheme.

7.2 Some applications

The main aim of this subsection is to demonstrate how the proposed methods can be used in practice. Three real lifetime data sets are used for this purpose. The OGEIW$(\underline{\Psi})$ distribution is fitted to the three real data using Kolmogorov-Smirnov goodness of fit test through R programming language.

Application 1

The first data introduced by Liu *et al.* (2021). The data refer to the survival times of patients suffering from the COVID-19 epidemic in China. The considered data set representing the survival times of patients from the time admitted to the hospital until death. Among them, a group of fifty-three (53) COVID-19 patients were found in critical condition in hospital from January to February 2020. The data set can be retrieved from https://www.worldometers.info/coronavirus/ and is given by: 0.054, 0.064, 0.704, 0.816, 0.235, 0.976, 0.865, 0.364, 0.479, 0.568, 0.352, 0.978, 0.787, 0.976, 0.087, 0.548, 0.796, 0.458, 0.087, 0.437, 0.421, 1.978, 1.756, 2.089, 2.643, 2.869, 3.867, 3.890, 3.543, 3.079, 3.646, 3.348, 4.093, 4.092, 4.190, 4.237, 5.028, 5.083, 6.174, 6.743, 7.274, 7.058, 8.273, 9.324, 10.827, 11.282, 13.324, 14.278, 15.287, 16.978, 17.209, 19.092 and 20.083.

Application 2

The second data is given by Murthy et al. (2004). The data refers to the time between failures for a repairable item: 1.43, 0.11, 0.71, 0.77, 2.63, 1.49, 3.46, 2.46, 0.59, 0.74, 1.23, 0.94, 4.36, 0.40, 1.74, 4.73, 2.23, 0.45, 0.70, 1.06, 1.46, 0.30, 1.82, 2.37, 0.63, 1.23, 1.24, 1.97, 1.86 and 1.17.

Application 3:

The third data set is given by Singh and Maddala (1976) the data represents the strength of 1.5 cm glass fibers for 60 devices. The data set is:0.636, 0.252, 0.157, 0.187, 2.771, 0.209, 0.617, 2.078, 1.013, 0.499, 0.431, 0.642, 0.46, 0.749,0.205, 0.576, 0.439, 0.471, 0.262, 0.387, 0.324, 0.424, 0.548, 1.794, 1.233, 0.915, 0.702, 0.417,0.337, 0.435, 0.359, 0.293, 0.147, 0.87, 0.608, 0.153, 0.098, 0.557, 0.415, 0.122, 0.912, 0.341,0.725, 0.364, 0.24, 0.594, 0.325, 0.416, 0.08, 0.582, 1.257, 1.575, 0.48, 0.909, 0.17, 0.319, 0.09,0.154, 2.248 and 0.292.

The Kolmogorov–Smirnov goodness of fit test is applied to check the validity of the fitted model. The p- values are given, respectively, 0.9746, 0.9578 and 0.9867. The p value given in each case showed that the model fits the data very well.

Table ٣-٥presents the ML two-sample predictors of the real data sets based on 80% level of Type II censored sample. Also, Tables 1٠ -١٥ display the Bayes and E-Bayes two-sample predictors of the real data sets under BSEL and BLL functions based on 80% level of Type II censored sample.

7.3 Concluding remark

- The results in Tables 1-15 indicate that the length of the interval of the first future order statistic is smaller than the length of the interval of the last future order statistic.
- The ML, Bayes and E-Bayes intervals include the predictive values (between the *lower limit* (LL) and *upper limit* (UL)).
- The lengths of the intervals of the E-Bayes predictors are less than the lengths of the intervals of the Bayes predictors, so the E-Bayesian prediction method is better than the Bayesian prediction method.
- In most cases, the lengths of the intervals of E-Bayes predictors under BLL function are less than the lengths of the intervals of the E-Bayes predictors under BSEL function.
- The lengths of the intervals of the Bayes predictors under BLL function in most cases are less than the lengths of the intervals of the Bayes predictors under BSEL function.
- The lengths of the intervals of the conditional ML, Bayes and E-Bayes predictors increase when s increases.

8. General Conclusion

In this research, the two-sample prediction technique is applied to obtain the conditional ML, Bayesian and E-Bayesian prediction (point and interval) for future order statistics of the OGEIW$(\underline{\Psi})$ distribution based on Type II censored samples. The predictors are considered under two different loss functions, the BSEL function; as a symmetric loss function and BLL function; as an asymmetric loss function. The predictors are obtained based on conjugate gamma prior and uniform hyper prior distributions. A numerical example is given to illustrate the theoretical results and three applications using real data sets are used to demonstrate how the results can be used in practice. In general, numerical computations showed that the length of the interval of the first future order statistic is smaller than the length of the interval of the last future order statistic. The ML, Bayes and E-Bayes intervals include the predictive values. Also, the lengths of the interval of the E-Bayes predictors are less than the lengths of the interval of the Bayes predictors, so the E-Bayes prediction technique is better than the Bayes prediction technique. The Bayesian and E-Bayesian prediction (point and interval) for future order statistics of the OGEIW$(\underline{\Psi})$ distribution under different type of loss functions such as general entropy and precautionary loss functions would be useful as a basis for further researches in distribution theory.

References

Abd El-Raheem, A. M. (2019). Inference and optimal design of multiple constant-stress testing for generalized half-normal distribution under Type-II progressive censoring. *Journal of Statistical Computation and Simulation*. Vol. 89, No. 2, pp. 1-30.

Abushal, T. A. and AL-Zaydi, A. M. (2017). Inference on constant-partially accelerated life tests for mixture of Pareto distributions under progressive Type-II censoring. *Open Journal of Statistics.* Vol. 7, pp. 323-346. https://doi.org/10.4236/ojs.2017.72024.

Ahmadi, J., Jozani, M. J., Marchand, E. and Parsian, A. (2009). Bayes estimation based on k-record data from a general class of distributions under balanced type loss functions. *Journal* *of Statistical Planning and Inference*. Vol.139, No.3, pp. 1180-1189. http://dx.doi.org/10.1016/j.jspi.2008.07.008.

AL-Dayian, G. R., EL-Helbawy, A. A., AL-Sayed, N. T. and Swielum, E. M. (2021). Prediction for modified Topp Leone-Chen distribution based on progressive Type-II censoring scheme. *Journal of Advances in Mathematics and Computer Science.* Vol. 36, No. 3, pp. 33-57.

Algarni, A., Almarashi, A. M., Okasha, H. and Ng, H. K. T. (2020). E-Bayesian estimation of Chen distribution based on Type-I censoring scheme. *Entropy*. Vol. 22, No. 6, pp. 1-14. https://doi.org/10.3390/e22060636

AL-Hussaini, E. K. and Al-Awadhi, F. (2010). Bayes two-sample prediction of generalized order statistics with fixed and random sample size. *Journal of Statistical Computation and Simulation*, Vol. 1, pp. 80, pp. 13-28.

AL-Hussaini, E. K. and Hussein, M. (2012). Estimation under a finite mixture of exponentiated exponential components model and balanced square error loss. *Open Journal of Statistics*. Vol. 2, pp. 28-38. https://doi.org/10.4236/ojs.2012.21004.

Arshad, M. and Jamal, Q. A. (2019). Statistical inference for Topp–Leone-generated family of distributions based on records. *Journal of Statistical Theory and Applications.* Vol. 18, No. 1, pp. 65–78.

Asgharzadeh, A. and Fallah, A. (2011). Estimation and prediction for exponentiated family of distributions based on records. *Comm. Statist. Theory Methods,* No 40, pp 68–83.

Deniz, E. G. (2006). On the use of the weighted balanced loss function to obtain credibility premiums. *International Conference on Mathematical and Statistical Modeling in Honor of Enrique Castillo.* pp. 1-12.

Dey, S., Kayal, T., & Tripathi, Y. M. (2018). Statistical inference for the weighted exponential distribution under progressive Type-II censoring with binomial removal. *American*

Journal of Mathematical and Management Sciences, Vol. 2, No. 37, pp. 188-208.

Faizan, M. and Sana (2018). Bayesian estimation and prediction for Chen distribution based on upper record values. *Journal of Mathematics and Statistical Science.* Vol. 1, No. 6, pp. 235-243.

Gupta, I. (2017). Bayesian and E-Bayesian method of estimation of parameter of Rayleigh distribution-A Bayesian approach under LINEX loss function. *International Journal of Statistics and Systems***.** Vol. 12, No. 4, pp. 791-796.

Kundu, D.and Howlader, H. (2010). Bayesian inference and prediction of the inverse Weibull distribution for Type-II censored data. *Computational Statistics and Data Analysis.* Vol. 6, No.54, pp. 1547-1558.

Han, M. (2007). E-Bayesian estimation of failure probability and its application. *Mathematical and Computer Modelling,* Vol.45, No. (9-10), pp. 1272-1279.

Han, M. (2019). E-Bayesian estimation of the exponentiated distribution family parameter under LINEX loss function**.** *Communications in Statistics-Theory and Methods*. Vol. 48, No. 3, pp. 648-659. https://doi.org/10.1080/03610926.2017.1417432**.**

Han, M. (2020). E-Bayesian estimation and its E-posterior risk of the exponential distribution parameter based on complete and Type-I censored samples. *Communications in Statistics - Theory and Methods.* Vol. 49, No. 8, pp. 1858-1872. https://doi.org/10.1080/03610926.2019.1565837.

Hassan, A. S., Elsherpieny, E. A., and Mohamed, R. E. (2018). Odds Generalized Exponential-Inverse Weibull Distribution: Properties and Estimation. *Pakistan Journal of Statistics and Operation Research*, No. 1, pp. 1-22. https:// doi.org /10.1016/j.mcm.2006.11.007.

Liu, X., Ahmad, Z., Gemeay, A. M., Abdulrahman, A.T., Hafez, E. H. and Khalil, N. (2021). Modeling the survival times of the COVID-19 patients with a new statistical model: A case study from China. *Plos One*. Vol. 16, No.7.

Murthy, D. N. P., Xie, M. and Jiang, R. (2004). *Weibull Models*. John Wiley and Sons, *Inc.*, Hoboken, New Jersey.

Okasha, H. M., Wang, C. and Wang, J. (2020). E-Bayesian prediction for the Burr XII model based on Type-II censored data with two samples. *Advances in Mathematical Physics.* Vol. 5, pp. 1-13. https://doi.org/10.1155/2020/3510673.

Rabie, A. and Li, J. (2020). E-Bayesian estimation for Burr-X distribution based on generalized Type-I hybrid censoring scheme. *American Journal of Mathematical and Management Sciences.* Vol. 39, No. 1, pp. 41-55. https://doi.org/10.1080/01966324.2019.1579123.

Reyad, H. M. and Ahmed, S. O. (2015). E-Bayesian analysis of the Gumbel Type-II distribution under Type-II censored scheme. *International Journal of Advanced Mathematical Sciences.* Vol. 3, No. 2, pp.108-120. https://doi.org/10.14419/ijams.v3i2.5093.

Singh, S. K. and Maddala, G. S. (1976). A function for the size distribution of incomes. *Econometrica,* Vol. 44, pp. 963-970.

Valiollahi, R., Asgharzadeh, A., and Kundu, D. (2017). Prediction of future failures for generalized exponential distribution under Type-I or Type-II hybrid censoring. *Brazilian Journal of Probability and Statistics*, pp. 41-61.

Zellner, A. (1994). Bayesian and non-Bayesian estimation using balanced loss functions. In: S. S. Gupta and J. O. Burger, Eds. *Statistical Decision Theory and Related Topics,* Springer, New York. pp. 377-390.

Table 1

Conditional predictors and bounds of the future observation based on Type II censoring under two-sample prediction (N=10000, n=100, r=80% , m=25,in sample size, $\alpha = .6, \beta = .3, \zeta = .1.4$)

s	$\hat{y}_{(s)}(ML)$	LL	UL	Length
1	0.7274	0.9611	0.0100	0.9511
13	4414.١	7184.١	0.0100	7084.١
21	2.0056	2.4427	0.0100	2.4327

Table 2

Conditional predictors and bounds of the future observation based on Type II censoring under two-sample prediction (N=10000, n=100, r=80% , m=25,in sample size, $\alpha = 1.63, \beta = .37, \zeta = 2.48$)

s	$\hat{y}_{(s)}(ML)$	LL	UL	Length
1	0.2255	2.0000	10.0000	8.0000
13	0.4343	0.3403	20.0000	19.6597
21	0.5960	2.0000	30.0000	28.0000

Table 3

Two-Sample conditional prediction and the relevant 95% conditional interval bounds with their lengths for COVID-19 data of China

n,m	*r*	s	$\hat{y}_{(s)}(ML)$	LL	UL	Length
53,25	42	1	0.1935	1.0977	7.0000	5.9023
		13	0.5110	6.2585	29.1000	22.8415
		25	1.3707	32.7065	90.1200	57.4135

Table 4

Two-Sample conditional prediction and the relevant 95% conditional interval bounds with their lengths for two applications

n,m	***r***	**s**	$\hat{y}_{(s)}(ML)$	**LL**	**UL**	**Length**
30,25	24	1	0.2248	0.0938	0.4445	0.3507
		13	1.3449	0.8156	2.0473	1.2316
		25	6.2817	3.3480	11.2805	7.9324

Table 5

Two-Sample conditional prediction and the relevant 95% conditional interval bounds with their lengths for three applications

n,m	***r***	**s**	$\hat{y}_{(s)}(ML)$	**LL**	**UL**	**Length**
60, 25	48	1	0.4932	0.3453	0.6477	0.3024
		13	0.9478	0.7977	14.9874	14.1897
		25	1.8244	1.3698	16.9223	15.5524

Table 6

Bayes, E-Bayes predictor and bounds of the future observation under balanced squared error loss function based on Type II censoring under two-sample prediction

(N=10000, n=100, r=80%,m=25,in sample size, ($\boldsymbol{\alpha=.6,\beta=.3,\zeta=1.4,\omega=.3}$)

s	Bayesian				E-Bayesian			
	(BBS)$\hat{y}_{(s)}$	LL	UL	Length	(EBBS)$\hat{y}_{(s)}$	LL	UL	Length
1	1.2050	1.2030	1.2059	0.0029	1.2034 1.2056 1.2040	1.2022 1.2044 1.2007	1.2047 1.2069 1.2057	0.0025 0.0025 0.0050
18	1.5041	1.5022	1.5055	0.0033	1.5047 1.5035 1.5047	1.5033 1.5022 1.5040	1.5057 1.5045 1.5054	0.0024 0.0023 0.0014
25	1.9889	1.9861	1.9920	0.0059	1.9898 1.9877 1.9879	1.9882 1.9865 1.9871	1.9911 1.9884 1.9889	0.0029 0.0019 0.0018

Table 7

Bayes, E-Bayes predictor and bounds of the future observation under balanced linear exponential loss function based on Type II censoring under two-sample prediction

(N=10000, n=100, r=80%,m=25,in sample size,($\boldsymbol{\alpha=.6,\beta=.3,\zeta=1.4,\nu=-1.8,\omega=.3}$)

s	Bayesian				E-Bayesian			
	(BBL)$\hat{y}_{(s)}$	LL	UL	Length	(EBBL)$\hat{y}_{(s)}$	LL	UL	Length
1	1.2015	1.2004	1.2024	0.0020	1.2017 1.2004 1.2011	1.2002 1.2024 1.1999	1.2031 0.0020 1.2017	0.0029 0.0030 0.0018
18	1.5054	1.5035	1.5067	0.0032	1.5047 1.5045 1.5048	1.5034 1.5028 1.5031	1.5054 1.5063 1.5062	0.0020 0.0035 0.0031
25	1.9901	1.9879	1.9916	0.0037	1.9895 1.9916 1.9906	1.9881 1.9898 1.9894	1.9910 1.9924 1.9915	0.0029 0.0026 0.0021

Table 8

Bayes, E-Bayes predictor and bounds of the future observation on Type II censoring under two-sample under balanced prediction
(N=10000, n=100, r=80%, m=25,in sample size, $(\alpha = 1.63, \beta = .37, \zeta = 2.48, \omega = .3)$

s	Bayesian				E-Bayesian			
	$(BBS)\hat{y}_{(s)}$	LL	UL	Length	$(EBBS)\hat{y}_{(s)}$	LL	UL	Length
1	0.2931	0.2914	0.2944	0.0030	0.2933 0.2911 0.2917	0.2922 0.2895 0.2900	0.2941 0.2931 0.2931	0.0019 0.0036 0.0031
18	0.5884	0.5865	0.5901	0.0036	0.5890 0.5899 0.5885	0.5871 0.5879 0.5870	0.5906 0.5910 0.5892	0.0035 0.0031 0.0022
25	1.6284	1.6258	1.6303	0.0045	1.6310 1.6279 1.6271	1.6280 1.6265 1.6257	1.6323 1.6293 1.6286	0.0043 0.0028 0.0029

Table 9

Bayes, E-Bayes predictor and bounds of the future observation linear exponential loss function based onType II under balanced censoring under two-sample prediction
(N=10000, n=100, r=80%,m=25,in sample size,$(\alpha = 1.63, \beta = .37, \zeta = 2.48, \nu = -2, \omega = .3)$
.

s	Bayesian				E-Bayesian			
	$(BBL)\hat{y}_{(s)}$	LL	UL	Length	$(EBBL)\hat{y}_{(s)}$	LL	UL	Length
1	0.2920	0.2904	0.2927	0.0023	0.2905 0.2919 0.2920	0.2890 0.2902 0.2911	0.2921 0.2934 0.2928	0.0031 0.0032 0.0017
18	0.5890	0.5877	0.5904	0.0027	0.5900 0.5896 0.5889	0.5886 0.5886 0.5879	0.5909 0.5903 0.5895	0.0023 0.0017 0.0016
25	1.6254	1.6230	1.6270	0.0040	1.6234 1.6249 1.6245	1.6220 1.6222 1.6233	1.6247 1.6263 1.6253	0.0027 0.0041 0.0020

Table 10

Bayes, E-Bayes predictor and bounds of the future observation squared error loss for COVID-19 data of China under balanced function based on Type II censoring under two-sample prediction (N=10000, n=53, r=42 , m=25,in sample size, $(\boldsymbol{\alpha} = .6, \boldsymbol{\beta} = .3, \boldsymbol{\zeta} = .6, \boldsymbol{\omega} = .3)$

n,m	r	s	Bayesian				E-Bayesian			
			(BBS)$\hat{y}_{(s)}$	LL	UL	Length	(EBBS)$\hat{y}_{(s)}$	LL	UL	Length
53,25	42	1	0.058	0.0560	0.0597	0.0037	0.0587 0.0585 0.0605	0.0573 0.0570 0.0579	0.0596 0.0600 0.0615	0.0023 0.0030 0.0037
		13	0.5963	0.5947	0.5989	0.0042	0.5966 0.5955 0.5965	0.5953 0.5937 0.5941	0.5978 0.5966 0.5976	0.0025 0.0029 0.0035
		21	2.5825	2.5807	2.5851	0.0044	2.5807 2.5839 2.5851	2.5791 2.5823 2.5825	2.5822 2.5849 2.5878	0.0031 0.0026 0.0053

Table 11

Bayes, E-Bayes predictor and bounds of the future observation linear exponential for COVID-19 data of China under balanced loss function based on Type II censoring under two-sample prediction (N=10000, n=53, r=42 , m=25,in sample size,$(\boldsymbol{\alpha} = .6, \boldsymbol{\beta} = .3, \boldsymbol{\zeta} = .6, \boldsymbol{\omega} = .3, \boldsymbol{\nu} = -1.8)$

n,m	r	s	Bayesian				E-Bayesian			
			(BBL)$\hat{y}_{(s)}$	LL	UL	Length	(EBBL)$\hat{y}_{(s)}$	LL	UL	Length
53,25	42	1	0.0554	0.0542	0.0566	0.0024	0.0555 0.0591 0.0565	0.0542 0.0573 0.0546	0.0564 0.0601 0.0575	0.0022 0.0028 0.0029
		13	0.5977	0.5963	0.5991	0.0028	0.5971 0.5970 0.5964	0.5958 0.5960 0.5944	0.5982 0.5976 0.5986	0.0024 0.0016 0.0042
		21	2.5794	2.5775	2.5809	0.0034	2.5783 2.5779 2.5806	2.5772 2.5763 2.5788	2.5795 2.5795 2.5822	0.0023 0.0032 0.0034

Table 12

Bayes, E-Bayes predictor and bounds of the future observation for for two applications under balanced squared error loss function based on Type II censoring under two-sample prediction (N=10000, n=30, r=24 , m=25,in sample size, $(\boldsymbol{\alpha=.9,\beta=.5,\zeta=.8,\omega=.3})$

n,m	r	s	Bayesian				E-Bayesian			
			(BBS)$\hat{y}_{(s)}$	LL	UL	Length	*(EBBS)*$\hat{y}_{(s)}$	LL	UL	Length
30,25	24	1	0.2212	0.2196	0.2225	0.0029	0.2211	0.2200	0.2219	0.0019
							0.2200	0.2187	0.2215	0.0028
							0.2223	0.2208	0.2234	0.0026
		13	1.1778	1.1756	1.1790	0.0034	1.1784	1.1766	1.1796	0.0030
							1.1769	1.1757	1.1779	0.0022
							1.1784	1.1761	1.1797	0.0036
		21	2.4993	2.4974	2.5014	0.0040	2.4992	2.4978	2.5003	0.0025
							2.4991	2.4979	2.5002	0.0023
							2.5003	2.4987	2.5013	0.0026

Table 13

Bayes, E-Bayes predictor and bounds of the future observation for two applications under balanced linear exponential loss function based onType II censoring under two-sample prediction (N=10000, n=30, r=24 , m=25,in sample size, $(\boldsymbol{\alpha=.9,\beta=.5,\zeta=.8,\nu=-.5,\omega=.3})$

n,m	r	s	Bayesian				E-Bayesian			
			(BBL)$\hat{y}_{(s)}$	LL	UL	Length	*(EBBL)*$\hat{y}_{(s)}$	LL	UL	Length
30,25	24	1	0.2200	0.2187	0.2207	0.0020	0.2202	0.2192	0.2210	0.0018
							0.2215	0.2199	0.2226	0.0027
							0.2179	0.2158	0.2193	0.0035
		13	1.1783	1.1770	1.1792	0.0022	1.1777	1.1763	1.1787	0.0024
							1.1787	1.1776	1.1797	0.0021
							1.1782	1.1768	1.1792	0.0024
		21	2.5011	2.4994	2.5020	0.0026	2.5004	2.4987	2.5015	0.0028
							2.4999	2.4984	2.5013	0.0029
							2.4980	2.4964	2.5002	0.0038

Table 14

Bayes, E-Bayes predictor and bounds of the future observation squared error loss for three applications under balanced function based on Type II censoring under two-sample prediction (N=10000, n=60, r=48 , m=35,in sample size, $(\boldsymbol{\alpha} = .9, \boldsymbol{\beta} = .6, \boldsymbol{\zeta} = .3, \boldsymbol{\omega} = .3)$

n,m	r	s	Bayesian				E-Bayesian			
			$(BBS)\hat{y}_{(s)}$	LL	UL	Length	$(EBBS)\hat{y}_{(s)}$	LL	UL	Length
60,35	48	1	0.4942	0.4930	0.4952	0.0022	0.4932	0.4921	0.4940	0.0019
							0.4935	0.4920	0.4951	0.0031
							0.4932	0.4910	0.4954	0.0044
		18	1.5079	1.5063	1.5095	0.0032	1.5079	1.5067	1.5087	0.0020
							1.5063	1.5045	1.5076	0.0031
							1.5065	1.5042	1.5081	0.0039
		35	4.0046	4.0030	4.0064	0.0034	4.0040	4.0028	4.0050	0.0022
							4.0060	4.0036	4.0071	0.0035
							4.0041	4.0026	4.0049	0.0023

Table 15

Bayes, E-Bayes predictor and bounds of the future observation linear exponential loss for three applications under balanced function based on Type II censoring under two-sample prediction (N=10000, n=60, r=48 , m=35,in sample size, $(\boldsymbol{\alpha} = .9, \boldsymbol{\beta} = .6, \boldsymbol{\zeta} = .3, \boldsymbol{\nu} = -1.8, \boldsymbol{\omega} = .3)$

n,m	r	s	Bayesian				E-Bayesian			
			$(BBL)\hat{y}_{(s)}$	LL	UL	Length	$(EBBL)\hat{y}_{(s)}$	LL	UL	Length
60,35	48	1	0.4937	0.4924	0.4945	0.0021	0.4932	0.4909	0.4944	0.0035
							0.4944	0.4920	0.4963	0.0043
							0.4946	0.4933	0.4959	0.0026
		18	1.5098	1.5082	1.5108	0.0026	1.5100	1.5087	1.5108	0.0021
							1.5114	1.5095	1.5127	0.0032
							1.5097	1.5082	1.5113	0.0031
		35	4.0066	4.0048	4.0078	0.0030	4.0066	4.0053	4.0075	0.0022
							4.0068	4.0060	4.0075	0.0015
							4.0061	4.0044	4.0071	0.0027