ODDS BURR XII-GENRALIZED FAMILY OF DISTRIBUTIONS WITH SIMULAION AND APPLICATION

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Abstract

Statistical distributions can be grouped into families or systems. The T-X family is the generalized family of distributions, suggested by Alzaatreh, *et al*.(2013). which consists of several sub-families of distributions, which can be used to develop new distributions. In this paper, Odds Burr XII generalized family of distributions is introduced. The special cases based on the Odds Burr XII-generalized family of distributions are proposed. The two suggested OBXII-distributions referred to as Odds Burr XII-logistic and Odds Burr XII -beta II distributions are constructed. Some properties of the proposed distributions are obtained. The maximum likelihood method is used under Type-I censored samples for estimating the parameters. A Monte Carlo simulation study is given to investigate the precision of the maximum likelihood estimates of the two suggested distributions. An application using COVID-19 data sets is introduced to show how the theoretical results can be used in practice.

Keywords: *T-X family; Odds Burr XII-generalized family of distributions; Type-I censored samples; Maximum likelihood method.*

1. Introduction

Statistical distributions are very useful in describing and predicting real-world phenomena. From the past several years, there is a growing trend of generating new families of distributions from existing distributions by adding one or more additional parameters to the baseline distribution to study the behavior of the shapes of density and hazard rate, and for checking the goodness-of-fit of proposed distributions. The T-X family suggested by Alzaatreh, *et al*. (2013) is one of the most important of these families. Ortega *et al*. (2014) introduced the log-odd Weibull regression model. Abdelall (2016) proposed the odd generalized exponential modified Weibull distribution. The odd-Burr family is presented by Alizadeh *et al*. (2017). Gomes *et al*. (2017) introduced the odd Lindley- G family. Afify *et al*.(2018) proposed the Odd Lindley Burr XII distribution. Aisha (2019) proposed the odd Frechet inverse Weibull distribution. Al-Marzouki *et al*. (2020) introduced an extension of the odd Fréchet family called the “Topp-Leone strategy” Eliwa *et al*. (2021) introduced Exponential Odd Chin-G family of distributions. Mohammed *et al*. (2022) proposed the power Odds generalized exponential Lomax distributions.

Burr (1942) suggested a number of cumulative distributions where the most popular one is called Burr XII distribution, whose two-parameter *probability density function* (pdf) is given by:

$$f(y;c,k) = cky^{c-1}[1+y^c]^{-k-1}, \qquad y > 0. \tag{1.1}$$

where:

$c > 0$ and $k > 0$ are both shape parameters.

Based in the transformer T-X which is introduced by Alzaatreh, *et al*. (2013), the class of continuous distributions called the odds Burr XII generalized family of distributions referred to as OBXII-G is proposed by taking the integrating of the Burr XII density function having *cumulative distribution function* (cdf) given by:

$$F(y;c,k,\xi) = \int_0^{\frac{G(y,\xi)}{1-(y,\xi)}} cky^{c-1}[1+y^c]^{-k-1} \quad dy.$$

$$= 1 - \left[1 + \left(\frac{G(y,\xi)}{1-G(y,\xi)}\right)^c\right]^{-k} \quad . \tag{1.2}$$

- The corresponding density function

$$f(y;c,k,\xi) = ck\; g(y,\xi)\frac{(G(y,\xi))^{c-1}}{(1-G(y,\xi))^{c+1}}\left[1+\left(\frac{G(y,\xi)}{1-G(y,\xi)}\right)^{c}\right]^{-k-1} \quad (1.3)$$

where:

$c > 0$ and $k > 0$ are both shape parameters, $G(y,\xi)$and $g(y,\xi)$ is the cdf and pdf respectively of any baseline distribution, and ξ is the vector of parameters in a baseline distribution.

- The *survival function* (sf) of the random variable Y corresponding to (1.3) is given by:

$$S(y;c,k,\xi) = \left[1+\left(\frac{G(y,\xi)}{1-G(y,\xi)}\right)^{c}\right]^{-k} \quad (1.4)$$

- The *hazard rate function* (hrf) of the random variable Y corresponding to (1.3) is given by:

$$h(y;c,k,\xi) = ck\; g(y,\xi)\frac{(G(y,\xi))^{c-1}}{(1-G(y,\xi))^{c+1}}\left[1+\left(\frac{G(y,\xi)}{1-G(y,\xi)}\right)^{c}\right]^{-1} \quad (1.5)$$

- The quantile function of the OBXII-G family of distributions is given by:

$$y_q = \left[1+\left[(1-q)^{\frac{-1}{k}}-1\right]^{\frac{-1}{c}}\right]^{-1}, \qquad 0 < q < 1. \quad (1.6)$$

This paper aims to use the new family of distributions called OBXII-G family of distributions to derive two new distributions. The two suggested distributions are referred to as the *odds Burr XII-logistic* (OBXII-L) and *odds Burr XII-beta II* (OBXII-beta II).

The rest of the paper is outlined as follows: in Section 2, some special models of the OBXII-G family of distributions is introduced and some statistical properties of the two suggested distributions are studied. *Maximum likelihood* (ML) estimation of the parameters based on the Type-I censoring scheme is derived in Section 3. In Section 4 the simulation study investigates the precision of the ML estimates. An application using COVID-19 data set is given in Section 5 to demonstrate how the results can be used in practice. Finally, The general conclusion is discussed in Section 6.

2. Some special models of the OBXII-G family

The importance of the OBXII-G family of distributions is that it has several new distributions. The two new distributions such as *odds Burr XII-Logistic* (OBXII-L) and *odds Burr XII-beta II* (OBXII-beta II) will be discussed below.

Remark: It is noticed that there are many distributions that can be obtained from OBXII-G family of distributions such as; OBXII-uniform distribution, OBXII-exponential distribution and OBXII-Lomax distribution and many other of distributions.

2.1 OBXII-L distribution

- The OBXII-L distribution is a special case of OBXII-G family of distributions when $G(y;\xi) = \left(1+e^{-\lambda y}\right)^{-1}$ and $g(y;\xi) = \lambda e^{-\lambda y}\left(1+e^{-\lambda y}\right)^{-2}$, respectively in (1.3) with the following pdf:

$$f(y;\theta) = ck\lambda e^{c\lambda y}\left[1+e^{c\lambda y}\right]^{-k-1},$$

$$-\infty < y < \infty;\ c,k,\lambda > 0. \qquad (2.1)$$

where:

$\underline{\theta} = (c,k,\lambda)$, c and λ are scale parameters, k is the shape parameter.

- The cdf of the random variable Y corresponding to (2.1) is given by:

$$F(y;\theta) = 1-\left[1+\frac{\left(1+e^{-\lambda y}\right)^{-c}}{\left(1-\left(1+e^{-\lambda y}\right)^{-1}\right)^{c}}\right]^{-k}$$

$$= 1-\left[1+e^{c\lambda y}\right]^{-k},\ -\infty < y < \infty;\ \ c,k,\lambda > 0. \qquad (2.2)$$

- The sf of the random variable Y corresponding to (2.1) is given by:

$$S(y;\theta)=\left[1+e^{c\lambda y}\right]^{-k}\ ,\ -\infty< y<\infty\ ;\quad c,k,\lambda>0. \qquad (2.3)$$

- The hrf of the random variable Y corresponding to (2.1) is given by:

$$h(y;\theta)=ck\lambda e^{c\lambda y}\left[1+e^{c\lambda y}\right]^{-1},\ -\infty< y<\infty\ ;\quad c,k,\lambda>0. \quad (2.4)$$

- The quantile function of the OBXII-L distribution is given by:

$$y_q=\ln\left[(1-q)^{\frac{-1}{k}}-1\right]^{\frac{1}{c\lambda}}\ ,\qquad 0<q<1. \qquad (2.5)$$

Graphical description of the OBXII-L distribution

The pdf curves of the OBXII-L distribution are plotted in Figure 1 for some different values of the parameters. $(c=3,k=1,\lambda=0.5)$, $(c=1,k=1.5,\lambda=0.8\,)$, $(c=2,k=0.1,\lambda=1.2\,)$ and $(c=1.1,\ k=2.4,\lambda=0.38\,)$

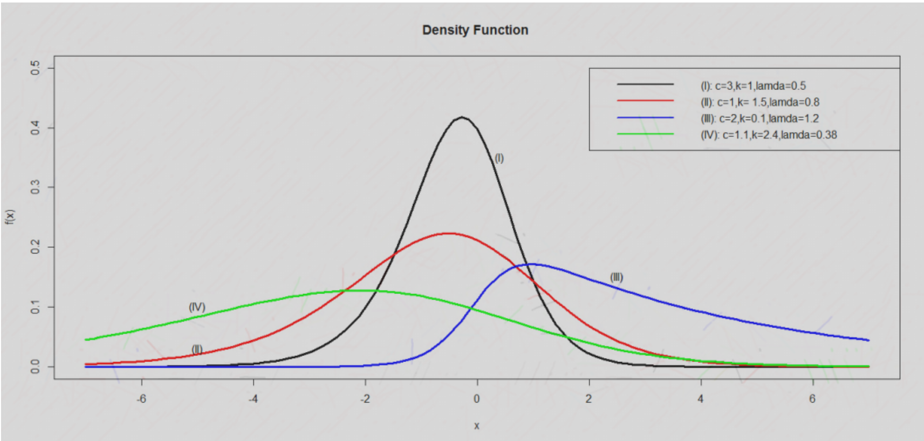


Figure 1 the pdf of OBXII-L distribution for some values of c, k and λ.

Figure 1 shows that:

- The pdf curves of the OBXII-L are more flexible for changing the values of the parameters.

- when $k = 1, c > 1$ and $\lambda < 1$ the pdf curve is bell shape.
- when $c = 1, k > 1$ and $\lambda < 1$ the pdf curve is approximately symmetric.

The hrf curves of OBXII-L distribution are plotted in Figure 2 for some selected values of the parameters. $(c = 1.3,\ k = 2,\ \lambda = 1.8)$, $(c = 1.3,\ k = 2,\ \lambda = 1)$, $(c = 0.3,\ k = 4.3,\ \lambda = 1.5)$ and $(c = 2,\ k = 0.4,\ \lambda = 0.1)$.

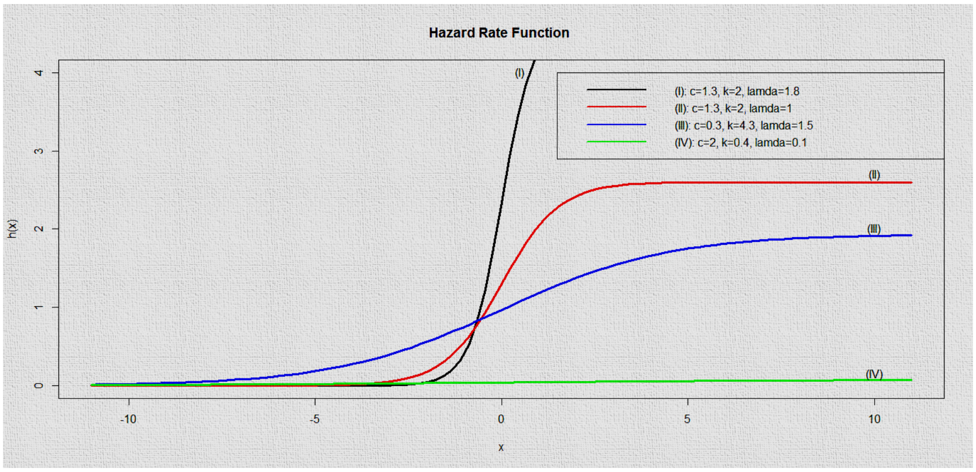


Figure 2 hrf of OBXII-L distribution for some values of c, k and λ.

Figure 2 shows that:

- The hrf curves of the OBXII-L are more flexible for changing the values of the parameters.
- when $c, k, \lambda > 1$ the curve of the hrf is monotone increasing.
- when $c, k > 1$ and $\lambda = 1$ the hrf curve is monotone increasing and then approximately constant.
- when $c < 1$ and $k, \lambda > 1$ the hrf curve is monotone increasing.

- when $c > 1$ and $k, \lambda < 1$ the hrf curve is approximately constant.

Remark: It is noticed that logistic distribution introduced by Verhulst (1838) is a special case of OBXII-L when $k = 1$ in (2.1) and Burr XII distribution introduced by Burr (1942) can be derived from (2.1) when $y = e^{\lambda x}$ in. In addition scale-shape OBXII-L(I) distribution, scale-shape OBXII-L(II) distribution, shape OBXII-L distribution, shape OBXII are new special cases of OBXII-L distribution when $c = 1, \lambda = 1, c\lambda = 1$ and $y = e^{c\lambda x}$in (2.1) respectively.

2.2 Odds Burr XII-beta II distribution

- The OBXII-beta II distribution is a special case of OBXII-G family of distributions when $G(y;\beta) = y^{\beta}(1+y)^{-\beta}$ and $g(y;\beta) = \beta y^{\beta-1}(1+y)^{-\beta-1}$, respectively in (1.3) with the following pdf:

$$f(y;\vartheta) = ck\beta y^{-2}(y^{-1}+1)^{\beta-1}\left[(y^{-1}+1)^{\beta}-1\right]^{-c-1} \times$$

$$\left[1+\left[(y^{-1}+1)^{\beta}-1\right]^{-c}\right]^{-k-1}, \quad y > 0; \quad c,k,\beta > 0. \quad (2.6)$$

where:

$\underline{\vartheta} = (c,k,\beta)$, c, k and β are scale parameters.

- The cdf of the random variable Y corresponding to (2.6) is given by:

$$F(y;\vartheta) = 1 - \left[1 + \left(\frac{y^{\beta}(1+y)^{-\beta}}{1-y^{\beta}(1+y)^{-\beta}}\right)^{c}\right]^{-k}$$

$$= 1 - \left[1+\left[(y^{-1}+1)^{\beta}-1\right]^{-c}\right]^{-k}, y > 0;\ c,k,\beta > 0. \quad (2.7)$$

- The sf of the random variable Y corresponding to (2.6) is given by:

$$S(y;\vartheta) = \left[1+\left[(y^{-1}+1)^{\beta}-1\right]^{-c}\right]^{-k}, y > 0;\quad c,k,\beta > 0. \quad (2.8)$$

- The hrf of the random variable Y corresponding to (2.6) is given by:

$$h(y;\vartheta) = ck\beta y^{-2}(y^{-1}+1)^{\beta-1}\left[(y^{-1}+1)^{\beta}-1\right]^{-c-1} \times$$

$$\left[1+\left[(y^{-1}+1)^{\beta}-1\right]^{-c}\right]^{-1}, y>0; \quad c,k,\beta>0. \qquad (2.9)$$

- The quantile function of the OBXII-beta II distribution is given by:

$$y_q = \left[\left(1+\left[(1-q)^{-1/k}-1\right]^{-1/c}\right)^{1/\beta}-1\right]^{-1}, \quad 0<q<1. \qquad (2.10)$$

Graphical description of the OBXII-beta II distribution

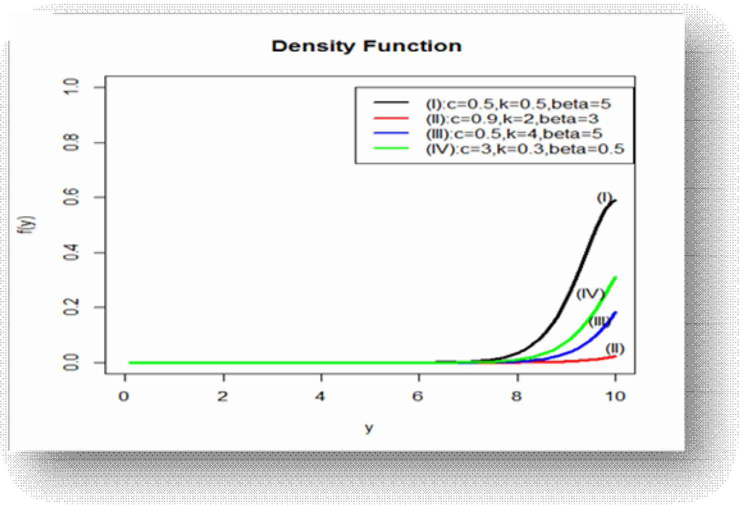


Figure 3.a

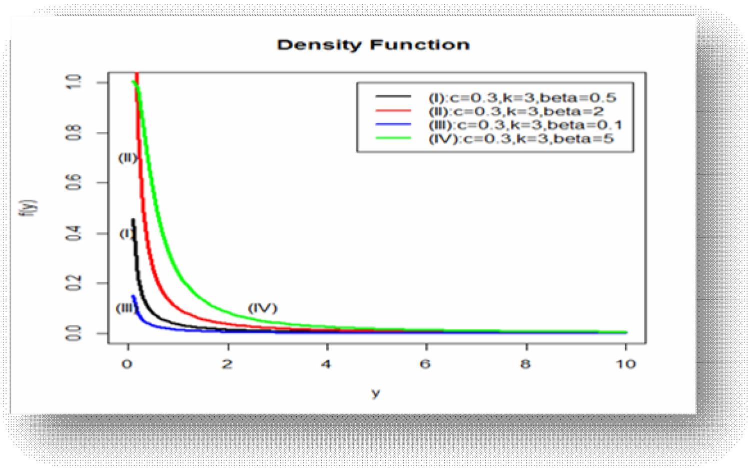


Figure 3.b

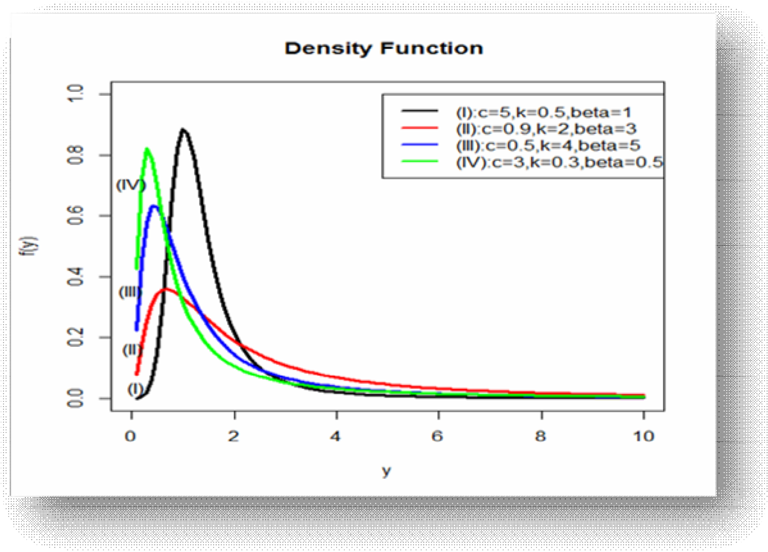


Figure 3.c

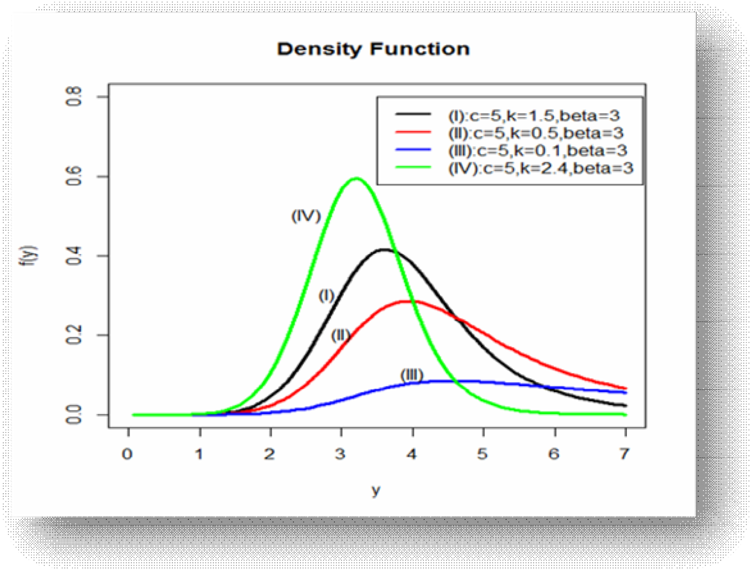


Figure 3.d

Figure 3: Plots of the pdf of OBXII-beta II for some values of the parameters

The plots of the pdf are provided for some values of the parameters in Figure 3.In Figure 3.a, the OBXII-beta II density is monotonically increasing at $c = 0.5, 0.9, 0.5, 0.3,\ k = 0.5, 4, 2, 0.3,$ $\beta = 5, 3, 5, 0.5,$ while in Figure 3.b, the curves are monotonically decreasing at $c = 0.3, k = 3$ and $\beta = 0.5,\ 2, 0.1, 5$. In Figure 3.c, the curves are monotone increasing and then monotone decreasing at $c = 5, 0.9, 0.5, 3, k = 0.5, 2, 4, 0.3$ and $\beta = 1,\ 3, 5, 0.5,$while in Figure 3.d, the curves are approximately symmetric and bell shape at $c = 5, k = 1.5, 0.5, 0.1, 2.4$ and $\beta = 3$.

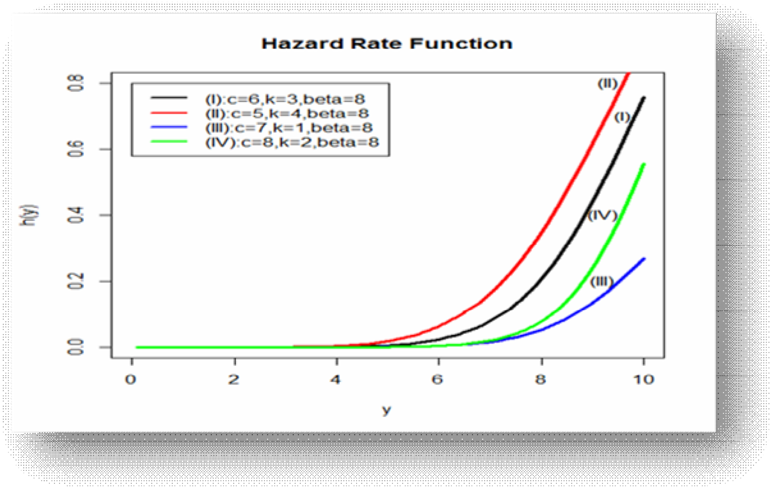


Figure 4.a

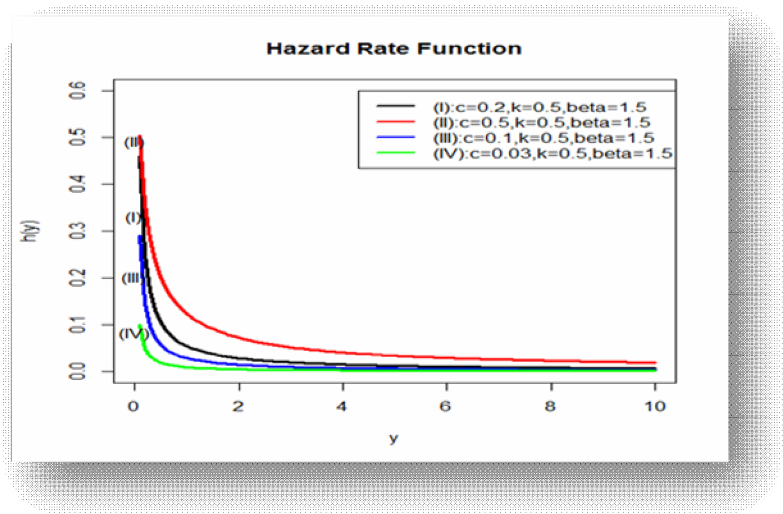


Figure 4.b

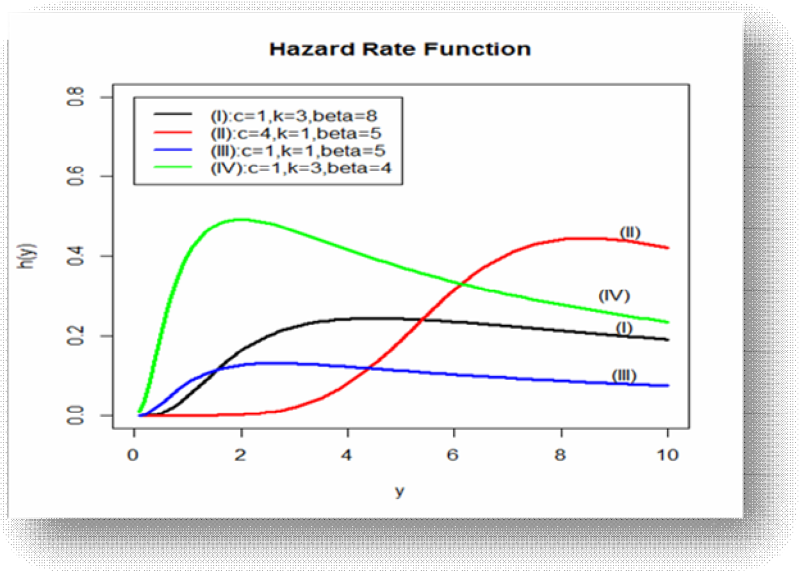


Figure 4.c

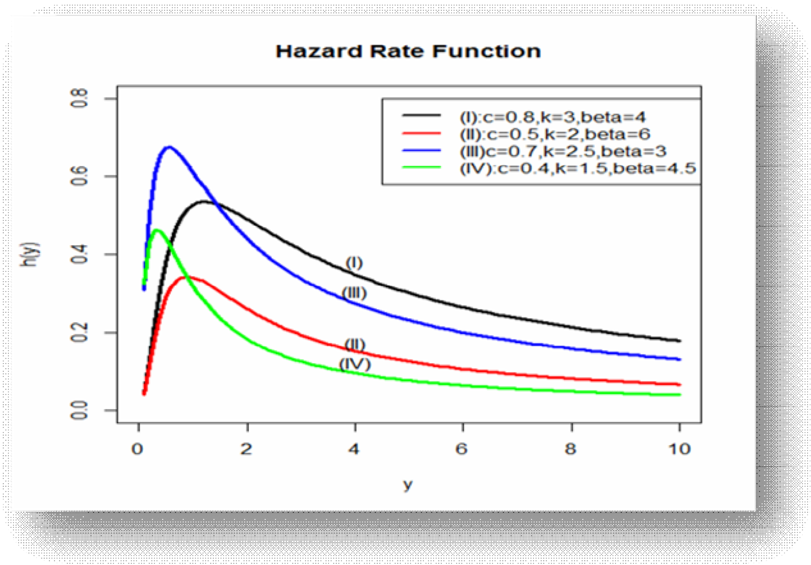


Figure 4.d

Figure 4: Plots of the hazard rate function of OBXII-beta II for some values of the parameters

The plots of the hrf are provided for some values of the parameters in Figure 4. The hrf curves of the OBXII-beta II are more flexible for changing the values of the parameters. In Figure 4.a, the OBXII-beta II hrf is monotonically increasing at $c = 6, 5, 7, 8$, $k = 3, 4, 1, 2, \beta = 8,$ while in Figure 4.b, the curves are monotonically decreasing at $c = 0.2, 0.5, 0.1, 0.03,$ $k = 0.5$ and $\beta = 1.5$. In Figure 4.c, the curves are monotone increasing and then monotone decreasing at $c = 1, 4, 1, 1, k = 3, 1, 1, 3$ and $\beta = 8, 5, 5, 4$,while in Figure 4.d, the curves are unimodal at $c = 0.8, 0.5, 0.7, 0.4, k = 3, 2, 2.5, 1.5$ and $\beta = 4, 6, 3, 4$.

Remark: It is noticed that Beta II distribution introduced by Thomas Bays (1763) is a special case of OBXII-beta II when $c = k = 1$ in (2.6) and Burr XII can be derived when $\beta = 1$ in (2.6). In addition shape OBXII, shape BXII, Two shape OBXII-beta II, OBXII-beta II distribution with two shape parameters, standard OBXII-beta II distribution are new special cases of OBXII-beta II when $c = \beta = 1, k = \beta = 1, c = 1, k = 1,$ and $c = k = \beta = 1$ in (2.6) respectively.

3.Maximum Likelihood Estimation Based on Type I Censored Data

In this section the ML estimation of the parameters of two suggested distributions OBXII-L and OBXII-beta II based on the Type-I censoring scheme is derived.

Consider a sample $(y_1, y_2, \ldots, y_n)$ of n independent observations. The ith individual is assumed to have a lifetime T_i and a censoring time C_i. The ith response is defined by

$$Y_i = \min\{T_i, C_i\}.$$

Under Type I censoring each individual has a fixed censoring time $C_i > 0$ such that T_i is observed if $T_i \leq C_i$; otherwise, the only known is that $T_i > C_i$. The likelihood function based on Type I censored sample is given by:

$$L\left(\underline{\omega}\middle|\underline{y}\right) = \Pi_{i=1}^{n}[f(y_i|\underline{\omega})]^{\delta_i}[s(y_i|\underline{\omega})]^{1-\delta_i} \tag{3.1}$$

where:

$\underline{\omega}$ is the vector of parameters.

$f(y_i|\underline{\omega})$ and $s(y_i|\underline{\omega})$ are the pdf and sf of the models respectively, δ_i is an indicator of whether or not the value Y_i is censored.

$$\delta_i = \begin{cases} 0 & when \text{ the value } Y_i \text{ is censor} \\ 1 & when \text{ the value } Y_i \text{ is uncensore} \end{cases}$$

[for more details see Lawless (2003)].

3.1 The maximum likelihood estimation of the parameters of OBXII-L distribution

The likelihood function of OBXII-L distribution based on Type I censored sample can be obtained by substituting $f(y_i|\underline{\theta})$ and $s(y_i|\underline{\theta})$ given by (2.1) and (2.3) respectively in (3.1), as follows:

$$L\left(\underline{\theta}\middle|\underline{y}\right) = \Pi_{i=1}^{n}\left[ck\lambda e^{c\lambda y}\left[1+e^{c\lambda y}\right]^{-k-1}\right]^{\delta_i}\left[\left[1+e^{c\lambda y}\right]^{-k}\right]^{1-\delta_i}. \tag{3.2}$$

The natural logarithm of the likelihood function is given by:

$$\ell = r\ln(c) + r\ln(k) + r\ln(\lambda) + \textstyle\sum_{i=1}^{n}\delta_i c\lambda y_i - (k+1)\times$$

$$\textstyle\sum_{i=1}^{n}\delta_i\ln\left[1+e^{c\lambda y_i}\right] - k\sum_{i=1}^{n}(1-\delta_i)\ln\left[1+e^{c\lambda y_i}\right]. \tag{3.3}$$

The first partial derivatives of (3.3) with respect to $\underline{\theta} = (c,k,\lambda)^T$ after equating to zero are given as follows:

$$\frac{\partial\ell}{\partial c} = \frac{r}{\hat{c}} + \sum_{i=1}^{n}\delta_i\hat{\lambda}y_i - (\hat{k}+1)\sum_{i=1}^{n}\delta_i\frac{\hat{\lambda}y_i e^{\hat{c}\hat{\lambda}y_i}}{\left[1+e^{\hat{c}\hat{\lambda}y_i}\right]} - \hat{k}\times$$

$$\sum_{i=1}^{n}(1-\delta_i)\frac{\hat{\lambda}y_i e^{\hat{c}\hat{\lambda}y_i}}{\left[1+e^{\hat{c}\hat{\lambda}y_i}\right]}=0. \quad (3.4)$$

$$\frac{\partial \ell}{\partial k}=\frac{r}{\hat{k}}-\sum_{i=1}^{n}\delta_i \log\left[1+e^{\hat{c}\hat{\lambda}y_i}\right]=0. \quad (3.5)$$

$$\frac{\partial \ell}{\partial \lambda}=\frac{r}{\hat{\lambda}}-\sum_{i=1}^{n}\delta_i\frac{\hat{c}y_i}{\hat{c}\hat{\lambda}y_i}-(\hat{k}+1)\sum_{i=1}^{n}\delta_i\frac{\hat{c}y_i e^{\hat{c}\hat{\lambda}y_i}}{\left[1+e^{\hat{c}\hat{\lambda}y_i}\right]}-\hat{k}\times$$

$$\sum_{i=1}^{n}(1-\delta_i)\frac{\hat{c}y_i e^{\hat{c}\hat{\lambda}y_i}}{\left[1+e^{\hat{c}\hat{\lambda}y_i}\right]}=0. \quad (3.6)$$

The system of non-linear $(3.4-3.6)$ have no closed form solution, in which case an iterative algorithm can be used to find point estimators of $\hat{c}$, $\hat{k}$ and $\hat{\lambda}$.

3.2 The maximum likelihood estimation of the parameters of OBXII-beta II distribution

By substituting $f(y_i|\underline{\vartheta})$ and $s(y_i|\underline{\vartheta})$ given by (2.6) and (2.8) respectively in (3.1), the likelihood function of OBXII-L distribution can be obtained in the form:

$$L\left(\underline{\vartheta}\middle|\underline{y}\right)=\prod_{i=1}^{n}\left[ck\beta y^{-2}(y^{-1}+1)^{\beta-1}\left[(y^{-1}+1)^{\beta}-1\right]^{-c-1}\right]^{\delta_i}\times$$
$$\left[1+\left[(y^{-1}+1)^{\beta}-1\right]^{-c}\right]^{-k-\delta_i} \quad (3.7)$$

The natural logarithm of the likelihood function, in (3.7) is given by:

$$\ell=r\ln(c)+r\ln(k)+r\ln(\beta)-2\sum_{i=1}^{n}\delta_i\ln y_i+(\beta-1)\sum_{i=1}^{n}\delta_i\times$$
$$\ln[y_i^{-1}+1]-(c+1)\sum_{i=1}^{n}\delta_i\ln\left[(y_i^{-1}+1)^{\beta}-1\right]-$$
$$\sum_{i=1}^{n}(k+\delta_i)\ln\left[1+\left[(y_i^{-1}+1)^{\beta}-1\right]^{-c}\right]. \quad (3.8)$$

where:

r $=\sum_{i=1}^{n}\delta_i$ is the observed number of uncensored lifetimes.

Taking the partial derivatives of (3.8) with respect to $\theta=(c,k,\beta)^T$ and equaling to zero.

$$\frac{\partial\ell}{\partial c}=\frac{r}{\hat{c}}+\sum_{i=1}^{n}\delta_i\ln\left[(y_i{}^{-1}+1)^{\hat{\beta}}-1\right]+$$

$$\sum_{i=1}^{n}(\hat{k}+\delta_i)\frac{\left[(y_i{}^{-1}+1)^{\hat{\beta}}-1\right]^{-\hat{c}}\ln\left[(y_i{}^{-1}+1)^{\hat{\beta}}-1\right]}{\left[1+\left[(y_i{}^{-1}+1)^{\hat{\beta}}-1\right]^{-\hat{c}}\right]}=0. \qquad (3.9)$$

$$\frac{\partial\ell}{\partial k}=\frac{r}{\hat{k}}-\sum_{i=1}^{n}\ln\left[(y_i{}^{-1}+1)^{\hat{\beta}}-1\right]=0. \qquad (3.10)$$

$$\frac{\partial\ell}{\partial\beta}=\frac{r}{\hat{\beta}}+\sum_{i=1}^{n}\delta_i\ln[y_i{}^{-1}+1]-(\hat{c}+1)\sum_{i=1}^{n}\delta_i\frac{(y_i{}^{-1}+1)^{\hat{\beta}}\ln(y_i{}^{-1}+1)}{\left[(y_i{}^{-1}+1)^{\hat{\beta}}-1\right]}+$$

$$\hat{c}\sum_{i=1}^{n}(\hat{k}+\delta_i)\frac{\left[(y_i{}^{-1}+1)^{\hat{\beta}}-1\right]^{-\hat{c}-1}(y_i{}^{-1}+1)^{\hat{\beta}}\ln(y_i{}^{-1}+1)}{\left[1+\left[(y_i{}^{-1}+1)^{\hat{\beta}}-1\right]^{-\hat{c}}\right]}=0. \qquad (3.11)$$

The system of non-linear (3.9-3.11) have no closed form solution, in which case an iterative algorithm can be used to find point estimators $\hat{c}$, $\hat{k}$ and $\hat{\beta}$.

4. Simulation Study

In this section, the Monte Carlo simulation study is performed to investigate the precision of the theoretical results on the basis of simulated data generated from the OBXII-L and OBXII-beta II density functions (2.1) and (2.6) respectively.

For each combination of the parameter values, censoring percentages and sample sizes, the model is fitted and the variance, absolute bias, *mean square error* (MSE), *relative mean square error* (RMSE), *asymptotic confidence interval* (A.C.I) are calculated using the following formulae:

1) $|bias| = \left| \begin{array}{c} average\ estimate\ of\ the\ parameter\ - \\ true\ value\ of\ the\ parameter \end{array} \right|$ (4.1)

2) $MSE = variance(estimate) + bias^2(estimate)$ (4.2)

3) $RMSE = \frac{MSE\ of\ the\ parameter}{true\ value\ of\ the\ parameter}.$ (4.3)

4) The asymptotic normality of ML estimation can be used to compute the asymptotic 100 (1- $\tau)\%$ A.C.I for ω as follows

$$\hat{\omega} \pm Z_{(1-\frac{\tau}{2})}\sqrt{\widetilde{var}(\hat{\omega})} \tag{4.4}$$

4.1 The steps of OBXII-L simulation study

The following steps are used to compute the ML estimates for the suggested OBXII-L distribution for different sample sizes [n=20,30,50 and 100] and different censoring percentages 0% (complete sample), 10%,30% and 50%.

1) For given values of the parameters c, k and λ the inverse cdf, can be used to generate the random variable of lifetimes (T_i) from OBXII-L whose cdf is given by (2.2), Thus, by solving the nonlinear equation

$$T_i = \ln\left[(1-u_i)^{\frac{-1}{k}} - 1\right]^{\frac{1}{c\lambda}}, \quad i = 1,2,\dots,n.$$

where: $u_i \sim$ standarded uniform distribution(0,1).

2) Generate the number of censoring time observations C_i using the binomial distribution with parameters (n, d), where d is the censoring percentages $[d = 0\%, 10\%, 30\%$ and $50\%]$.

3) $Y_i = \min\{T_i, C_i\}$.

where:

T_i The generated lifetimes and C_i, $i = 1,2,\dots,n$ are the censoring times.

4) Obtain the ML estimates by solving Equations (3.4) – (3.6).

5) Compute the absolute bias, RMSE and A.C.I for each estimate using Equations (4.1), (4.3) and (4.4) respectively.

6) Repeat the above steps for all censoring percentages and all sample sizes 1000 times using the R program (virgin, 3.6.2 and package maxLik).

Table 1 display the ML average estimates, absolute bias, variance, RMSE and A.C.I of the unknown parameters based on Type-I censoring for all percentages under different sample sizes for the OBXII-L density function.

Table 1. The Average estimates, $|Bias|$, Variance, RMSE and A.C.I (LL and UL) for the Parameters c, k and λ for the OBXII-L Density Function for Different Censoring Percentages and Sample Sizes$(c = 1.1, k = 0.9\ and \lambda = 1.2)$.

n	censoring	parameters	averages estimates	$\lvert Bias\rvert$	variance	RMSE	LL	UL	averages length
20	0%	c	1.1389	0.0389	0.0468	0.0439	0.7148	1.5630	0.8481
		k	0.8486	0.0513	0.2750	0.3085	0.0000	2.0559	2.0559
		λ	1.2328	0.0328	0.0385	0.0329	0.8482	1.6175	0.7693
	10%	c	1.1002	0.0002	0.0693	0.0630	0.5839	1.6165	1.0325
		k	0.6624	0.2375	0.2565	0.3477	0.0000	1.6550	1.6550
		λ	1.1994	0.0005	0.0581	0.0484	0.7267	1.6721	0.9454
	30%	c	1.0175	0.0824	0.1005	0.0975	0.3961	1.6390	1.2429
		k	0.4145	0.4854	0.1866	0.4692	0.0000	1.2612	1.2612
		λ	1.1243	0.0756	0.0848	0.0754	0.5535	1.6950	1.1415
	50%	c	0.9874	0.1125	0.4020	0.3770	0.0000	2.2302	2.2302
		k	0.2406	0.6593	0.1228	0.6195	0.0000	0.9275	0.9275
		λ	1.0995	0.1004	0.3897	0.3331	0.0000	2.3231	2.3231
30	0%	c	1.1292	0.0292	0.0303	0.0283	0.7878	1.4707	0.6828
		k	0.9446	0.0446	0.1374	0.1549	0.2179	1.6713	1.4534
		λ	1.2258	0.0258	0.0257	0.0219	0.9115	1.5401	0.6285
	10%	c	1.0465	0.0535	0.0629	0.0597	0.5548	1.5381	0.9832
		k	0.6733	0.2266	0.2075	0.2876	0.0000	1.5662	1.5662
		λ	1.1501	0.0498	0.0524	0.0458	0.7010	1.5991	0.8980
	30%	c	0.9975	0.1024	0.0853	0.0870	0.4250	1.5700	1.1449
		k	0.6588	0.2411	0.2353	0.3260	0.0000	1.6096	1.6096
		λ	1.1065	0.0725	0.0934	0.0677	0.5787	1.6344	1.0556
	50%	c	0.9212	0.1787	0.1207	0.1388	0.2401	1.6023	1.3621
		k	0.6549	0.2450	0.2468	0.3409	0.0000	1.6287	1.6287
		λ	1.0371	0.1628	0.1023	0.1073	0.4101	1.6641	1.2539
50	0%	c	1.1263	0.0263	0.0255	0.0238	0.8128	1.4399	0.6271
		k	0.9103	0.0103	0.1334	0.1483	0.1944	1.6262	1.4317
		λ	1.2241	0.0241	0.0216	0.0185	0.9356	1.5126	0.5770

(1)continued

	10%	c	1.0555	0.0444	0.0423	0.0403	0.6521	1.4589	0.8068
		k	0.6813	0.2186	0.1675	0.2392	0.0000	1.4836	1.4836
		λ	1.1594	0.0405	0.0357	0.0311	0.7888	1.5301	0.7412
	30%	c	1.0087	0.0912	0.0719	0.0729	0.4831	1.5343	1.0511
		k	0.6639	0.2360	0.1755	0.2569	0.0000	1.4851	1.4851
		λ	1.1176	0.0823	0.0610	0.0565	0.6334	1.6019	0.9684
	50%	c	0.9365	0.1634	0.1078	0.1223	0.2927	1.5802	1.2874
		k	0.6533	0.2466	0.1841	0.2722	0.0000	1.4945	1.4945
		λ	1.0519	0.1480	0.0921	0.0950	0.4569	1.6469	1.1900
100	0%	c	1.1158	0.0158	0.0166	0.0153	0.8628	1.3689	0.5061
		k	0.9088	0.0088	0.1125	0.1251	0.2514	1.5663	1.3148
		λ	1.2143	0.0143	0.0140	0.0118	0.9820	1.4467	0.4647
	10%	c	1.0726	0.0273	0.0270	0.0252	0.7501	1.3950	0.6448
		k	0.7119	0.1880	0.1174	0.1698	0.0401	1.3837	1.3436
		λ	1.175٠	0.0250	0.0228	0.0195	0.8785	1.4714	0.5929
	30%	c	1.0363	0.0636	0.0350	0.0355	0.6695	1.4031	0.7336
		k	0.6713	0.2286	0.1387	0.2122	0.0000	1.4014	1.4014
		λ	1.1416	0.0583	0.0294	0.0273	0.8053	1.4778	0.6724
	50%	c	0.9909	0.1090	0.0786	0.0823	0.4412	1.5406	1.0993
		k	0.6607	0.2392	0.1597	0.2410	0.0000	1.4440	1.4440
		λ	1.1086	0.0913	0.0683	0.0639	0.5961	1.6210	1.0249

It is observed from Table 1 that:

- The ML averages estimates are very close to the population parameter values as the sample size increases.
- The absolute bias, variance, RMSEs and the approximated length of A.C.Is.are decreasing when the sample size is increasing.
- The estimates are consistent and approach the population parameter values as the sample size increases.
- As expected, the absolute bias, variance, RMSEs and the approximated length of A.C.Is. increased when censoring percentage increased.

4.2 The steps of OBXII-beta II simulation study

The following steps are used to compute the ML estimates for the suggested OBXII-beta II distribution for different sample sizes [n=20,30,50 and 100] and different censoring percentages 0% (complete sample), 10%,30% and 50%..

1) For given values of the parameters c, k and β the inverse cdf, can be used to generate the random variable of lifetimes (T_i) from OBXII-beta II whose cdf is given by (2.7), Thus, by solving the nonlinear equation

$$T_i = \left[\left(1 + \left[(1-u_i)^{-1/k} - 1\right]^{-1/c}\right)^{1/\beta} - 1\right]^{-1} \quad , \quad i = 1,2,\dots,n.$$

where: $u_i \sim$ standarded uniform distribution(0,1).

2) Generate the number of censoring time observations C_i using the binomial distribution with parameters (n, d), where d is the censoring percentages $[d = 0\%, 10\%, 30\%$ and $50\%]$.
3) $Y_i = \min\{T_i, C_i\}$

where: T_i The generated lifetimes and C_i , $i = 1,2,\dots,n$ are the censoring times.

4) Obtain the ML estimates by solving Equations (3.9) – (3.11).
5) Compute the absolute bias, MSE and C.I. for each estimate using Equations (4.1), (4.2) and (4.4) respectively.
6) Repeat the above steps for all censoring percentages and all sample sizes 1000 times using the R program (virgin, 3.6.2 and package maxLik).

Table 2 display the ML average estimates, absolute bias, variance, MSE and A.C.I of the unknown parameters based on Type-I censoring for all percentages under different sample sizes for the OBXII-beta II density function.

Table 2 The Average estimates, $|Bias|$, Variance, MSE and A.C.I (LL and UL) for the Parameters c, k and β for the OBXII-beta II Density Function for Different Censoring Percentages and Different Sample Sizes. $(c = 0.2, k = 0.5\ and \beta = 0.6)$

n	censoring	parameters	averages estimates	$\lvert Bias \rvert$	variance	MSE	LL	UL	averages length
20	0%	c	1.3785	1.1785	2.9755	4.3645	0.0000	4.7595	4.7595
		k	0.6293	0.1293	0.0368	0.0536	0.2530	1.0057	0.7527
		β	0.2665	0.3334	0.0893	0.2006	0.0000	0.8525	0.8525
	10%	c	1.6481	1.4481	4.4370	6.5341	0.0000	5.7767	5.7767
		k	0.6465	0.1465	0.2139	0.2354	0.0000	1.5531	1.5531
		β	0.3029	0.2970	0.0911	0.1794	0.0000	0.8947	0.8947
	30%	c	**2.5723**	**2.3723**	**31.3316**	**36.9595**	**0.0000**	**13.5433**	**13.5433**
		k	**0.5731**	**0.0731**	**0.2479**	**0.2533**	**0.0000**	**1.5491**	**1.5491**
		β	0.2195	0.3804	0.0827	0.2274	0.0000	0.7832	**0.7832**
	50%	c	2.8372	2.6372	47.4289	54.3843	0.0000	16.3355	16.3355
		k	0.5025	0.0025	0.2873	0.2873	0.0000	1.5531	1.5531
		β	0.3258	0.2741	0.0943	0.1695	0.0000	0.9278	0.9278
30	0%	c	0.8067	0.6067	0.8343	1.2025	0.0000	2.5970	2.5970
		k	0.5792	0.0792	0.0226	0.0289	0.2844	0.8740	0.5895
		β	0.4092	0.1907	0.0758	0.1122	0.0000	0.9488	0.9488
	10%	c	1.0752	0.8752	1.7776	2.5438	0.0000	3.6885	5.2265
		k	0.5804	0.0804	0.0797	0.0861	0.0269	1.1338	1.1068
		β	0.4012	0.1987	0.0791	0.1186	0.0000	0.9526	0.9526
	30%	c	1.2396	1.0396	4.2833	5.3642	0.0000	5.2961	5.2961
		k	0.5858	0.0858	0.1320	0.1398	0.0000	1.2991	1.2991
		β	0.3994	0.2005	0.0801	0.1203	0.0000	0.9542	0.9542
	50%	c	1.3317	1.1317	9.5074	10.7883	0.0000	7.3752	7.3752
		k	0.6850	0.1850	0.1481	0.1823	0.0000	1.4394	1.4394
		β	0.3633	0.2366	0.0974	0.1534	0.0000	0.9750	0.9750
٥0	0%	c	0.3843	0.1843	0.7066	0.7406	0.0000	2.0319	2.0319
		k	0.5532	0.0532	0.0181	0.0209	0.2895	0.8169	0.5274
		β	0.5625	0.0374	0.0191	0.0205	0.2912	0.8339	0.5426
	10%	c	0.7307	0.5307	1.1348	1.4164	0.0000	2.8186	2.8186
		k	0.5562	0.0562	0.0644	0.0676	0.0585	1.0538	0.9952
		β	0.4729	0.1270	0.0579	0.0740	0.0012	0.9446	0.9433

(2)continued

	30%	c	0.7920	0.5920	3.8344	4.1850	0.0000	4.6300	4.6300
		k	0.5572	0.0572	0.1089	0.1121	0.0000	1.2041	1.2041
		β	0.4699	0.1300	0.0606	0.0775	0.0000	0.9526	0.9526
	50%	c	1.8068	1.6068	25.0884	27.6703	0.0000	11.6241	11.6241
		k	0.5586	0.0586	0.1647	0.1681	0.0000	1.3541	1.3541
		β	0.4411	0.1588	0.0783	0.1035	0.0000	0.9896	0.9896
1٠0	0%	c	0.3352	0.1352	0.2610	0.2793	0.0000	1.3366	1.3366
		k	0.5190	0.0190	0.0056	0.0060	0.3712	0.6669	0.2956
		β	0.5698	0.0301	0.0172	0.0181	0.3124	0.8272	0.5148
	10%	c	0.3768	0.1768	0.3991	0.4304	0.0000	1.6152	1.6152
		k	0.5220	0.0220	0.0101	0.0106	0.3246	0.7194	0.3947
		β	0.5679	0.0320	0.0176	0.0187	0.3073	0.8284	0.5211
	30%	c	0.5934	0.3934	2.5270	2.6818	0.0000	3.7091	3.7091
		k	0.5228	0.0228	0.0350	0.0356	0.1556	0.8899	0.7343
		β	0.5638	0.0361	0.0189	0.0202	0.2943	0.8334	0.5390
	50%	c	0.6498	0.4498	6.4327	6.6351	0.0000	5.6209	5.6209
		k	0.5249	0.0249	0.0472	0.0478	0.0988	0.9510	0.8522
		β	0.5595	0.0404	0.0214	0.0230	0.2726	0.8463	0.5736

It is observed from Table 2 that:

- The ML estimates are very close to the population parameter values as the sample size increases.
- The absolute bias, variance, MSEs and the approximated length of A.C.Is. are decreasing when the sample size is increasing.
- The estimates are consistent and approach the population parameter values as the sample size increases.
- As expected, the absolute bias, variance, MSEs and the approximated length of A.C.Is. increased when censoring percentage increased.

5. An Application

In this section, the COVID-19 data set is investigated to illustrate the flexibility of the two suggested distributions. The data set is taken from a secondary source Information on confirmed COVID-19 individual cases in India is obtained from the web portal https://www.kaggle.com/. These data are refered to the recovery time of all the cases that are hospitalized with positive COVID-19 results in

India from March 1, 2020, to March 31, 2020. The total number of cases during this month used in the study is 130 alive individuals and 21 deaths. The recovery time (in days) is calculated from the day of hospitalization until recovery. The patients are considerable as censored if he/she died (in this case the recovery time is zero). The data is given as.

13, 25, 25, 25, 25, 25, 25, 25, 25, 25, 25, 25, 25, 11, 11, 11, 11, 9, 19, 21, 21, 21, 21, 18, 17, 16, 16, 11, 24, 14, 17, 14, 16, 16, 21, 21, 18, 18, 20, 0, 8, 11, 12, 0, 0, 11, 15, 13, 18, 15, 17, 14, 16, 15, 16, 19, 12, 18, 9, 0, 10, 18, 8, 8, 18, 7, 0, 0, 22, 10, 10, 14, 13, 17, 15, 9, 0, 20, 15, 20, 0, 11, 11, 11, 15, 14, 14, 20, 0, 0, 13, 16, 18, 18, 14, 0, 10, 20, 14, 0, 9, 12, 19, 19, 19, 0, 8, 8, 13, 8, 10, 8, 13, 0, 0, 16, 0, 16, 7, 7, 15, 14, 15, 15, 15, 0, 9, 0, 14, 14, 14, 6, 6, 7, 6, 6, 0, 8, 8, 8, 8, 0, 13, 13, 13, 5, 6, 6, 9, 0, 13.

Table 3: Descriptive Statistics for the COVID-19 Data Set

Mean	Median	St.D	Variance	Skewness	Kurtosis
12.56	13	7.14	50.94	-0.20	2.39

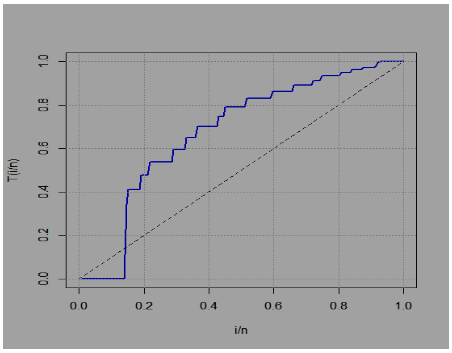


Figure 5. the TTT-plot for the recovery times of COVID-19.

Figure 5. presents the sequence of the recovery times of COVID-19 of the patient who recovered during the month of March.

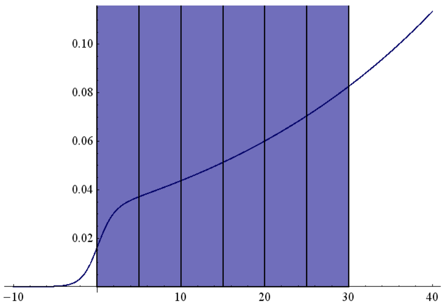
The goodness of fit is introduced in 5.1, the estimation of the parameters is given in 5.2 and the performance of the models is given in 5.3 for the OBXII-L distribution and its sub-models logistic, BXII and shape OBXII and for the OBXII-beta II distribution and its sub-models shape OBXII-beta II, BXII and Beta II.

5.1 The goodness of fit

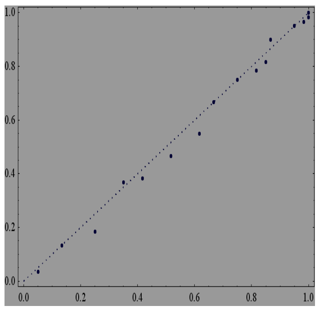
The Kolmogorov-Smirnov (K-S) goodness of fit test is used to check the validity of the fitted distributions.

- **The fitted of OBXII-L density function**

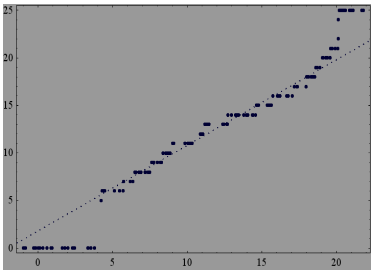
The OBXII-L density function is fitted to the COVID-19 data set. The p-value was declared to be = 0.7494. Figures 5 and 6 show the plots of the TTT-plot, fitted pdf, PP-plot, QQ-plot and for the OBXII-L distribution for the COVID-19 data set. Figures 5 and 6 indicate that the OBXII-L distribution provides better fits to these data sets.



Plot of the fitted pdf



PP-plot for the COVID-19 data

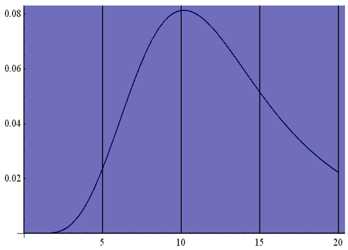


QQ-plot for the COVID-19 data

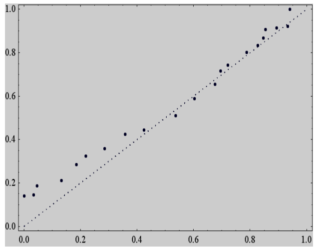
Figure 6. The fitted pdf of OBXII-L, PP and QQ for the COVID-19 data

- **The fitted OBXII-beta II density function**

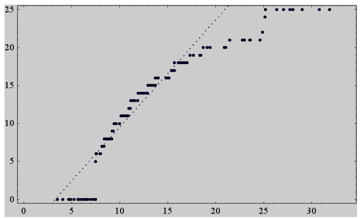
The OBXII-beta II density function is fitted to the COVID-19 data set. The p-value was decluced to be = 0.7166. Figures 5 and 7 show the plots of the TTT-plot, pdf, PP-plot, QQ-plot and for the OBXII-beta II distribution for the COVID-19 data set. Figures 5 and 7 indicate that the OBXII-beta II distribution provides better fits to these data sets.



Plot of the fitted pdf



PP-plot for the COVID-19 data



QQ-plot for the COVID-19 data

Figure 7. The fitted pdf of OBXII-beta II, PP and QQ for the COVID-19 data

5.2 The ML estimates of two suggested distributions

- **OBXII-L density function**

By using ML method of estimation the parameters of OBXII-L distribution and its special cases are estimated and the SEs and A.I.Cs are represented for all parameters.

Table 4 display the ML estimates, SEs and A.C.Is of the parameters using COVID-19 data set

Table 4 The ML estimates and SEs, A.C.I (LL and UL) of the Parameters of the OBXII-L distribution, and its sub-models (Logistic, BXII and Shape OBXII).

Density functions	parameters	estimates	SEs	A.C.I		
				LL	UL	Length
OBXII-L	c k λ	3.47842 3.48540 3.48817	0.00077 0.00076 0.00076	3.47830 3.48528 3.48805	3.47855 3.48552 3.48829	0.0002444 0.0002432 0.0002443
Logistic	c λ	3.242101 3.237399	0.000836 0.000835	3.241968 3.237266	3.242235 3.237533	0.000267 0.000266
BXII	c k	3.512069 3.414641	0.000815 0.000795	3.51193 3.414515	3.512199 3.414768	0.000260 0.000253
Shape OBXII	k	2.563603	0.000924	2.563456	2.563751	0.000295

It is observed from Table 4 that:

- It's important to note that the values of SEs and A.C.Is are considerably small and relatively suitable in the statistical analysis.

OBXII-beta II density function

By using ML method of estimation the parameters of OBXII-beta density function and its special cases are estimated and the SEs and A.I.Cs are represented for all parameters.

Table 5 displays the ML estimates, SEs and A.C.Is of the parameters using COVID-19 data set.

Table 5 The ML estimates and SEs, A.C.I (LL and UL) of the Parameters of the OBXII-beta II non-mixture cure rate model, and its sub-models (Shape OBXII-beta II, BXII and Beta II).

Distributions	parameters	estimates	SEs	A.C.I		
				LL	UL	Length
OBXII-beta II	c k β	0.810040 9.115632 0.003638	0.003924 0.073043 0.000033	0.809414 9.103981 0.003633	0.810666 9.127282 0.003644	0.0012520 0.0233008 0.0000107
Shape OBXII-beta II	c β	3.153828 1.151937	0.007887 0.002360	3.152570 1.151561	3.155086 1.152314	0.0025161 0.0007529
BXII	c k	0.381141 9.156002	0.003794 0.113982	0.380536 9.137822	0.381746 9.174182	0.0012105 0.0363602
Beta II	β	0.000060	0.000002	0.000060	0.000060	0.0000007

It is observed from Table 5 that:

- It's important to note that the values of SEs and A.C.Is are considerably small and relatively suitable in the statistical analysis.

5.3 The performance of two suggested distributions

Table 6 Numerical values of $-2\ln L$, AIC, BIC and $AICc$ for OBXII-L and its sub-models (Logistic, BXII and Shape OBXII).

criteria's / models	$-2\ln L$	AIC	BIC	AICc
OBXII-L	-20629459	-20629453	-20629444	-20629453
Logistic	-14972907	-14972901	-14972897	-14972901
BXII	-18454281	-18454275	-18454271	-18454275
Shape OBXII	-7678543	-7678537	-7678538	-7678537

Table 6 lists the numerical values of the criterion $-2\ln L$, AIC, BIC and $AICc$ for the OBXII-L distribution and its special cases Logistic, BXII and Shape OBXII. These results show that the OBXII-L distribution can be used quite effectively to provide better fits than the other models and hence it can be adequate for COVID-19 data set.

Table 7 Numerical values of $-2\ln L$, AIC, BIC and $AICc$ for OBXII-beta II and its sub-models Shape OBXII-beta II, BXII and Beta II

criteria's / models	$-2\ln L$	AIC	BIC	$AICc$
OBXII-beta II	**36474.67**	**36480.67**	**36489.72**	36480.83
Shape OBXII-beta II	**52804.04**	**52810.04**	**52814.07**	52810.12
BXII	**243197.3**	**243191.3**	**243207.3**	243191.4
Beta II	**263779**	**263773**	**263784**	263773

Table 7 lists the numerical values of the criterion $-2\ln L$, AIC, BIC and $AICc$ for the OBXII-beta II distribution and its sub-models. These results show that the OBXII-beta II distribution can be used quite effectively to provide better fits than the other models and hence it can be adequate for COVID-19 data set.

6. General Conclusions

1. In this paper OBXII-G family of distributions is introduced and the descriptive properties and some special cases of this suggested family are studied.

2. Two suggested distributions special cases of this family OBXII-L and OBXII-beta II are introduced.

3. The descriptive properties and some special cases of two suggested distributions are studied.

4. The parameters of two suggested distributions are estimated by ML method of estimation.

5. Some numerical results of OBXII-L and OBXII-beta II distributions are obtained using a simulation study to illustrate the theoretical work. It has been found that:

   - As expected, the absolute bias, variance, RMSE (for OBXII-L), MSE (for OBXII-beta II) and the length of A.C.I. decreased when the sample size increased.

- As expected, the absolute bias, variance, RMSE (for OBXII-L), MSE (for OBXII-beta II) and the approximated length of A.C.I. increased when censoring percentage increased.

- The ML averages are very close to the population parameter values as the sample size increases.

- This is indicative of the fact that the estimates are consistent and approach the population parameter values as the sample size increases.

- The approximated lengths of the A.C.Is of the parameters become narrower as the sample size increases.

6. An application using COVID-19 data set is used to demonstrate how the two proposed distributions based on Type-I censoring can be used in practice

- The K–S goodness of fit test is applied to check the validity of the introduced fitted models and the p-value given in each case showed that the two proposed models fit the data very well.

- The ML estimates and SEs of the unknown parameters of OBXII-L and OBXII-beta II for the real data sets are suitable in statistical analysis.

- The performance of the two suggested distributions are studied using the AIC, BIC and AICc criterion of model selection to choose among the competing models are used. It is found that the OBXII-L and OBXII-beta II models can be used quite effectively to provide better fits than its sub-models.

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