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On the oscillation of bounded solutions of higher order nonlinear neutral delay differential equations with oscillating coefficients

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Abstract

In this paper, new sufficient conditions are established for the oscillation of all bounded solutions of higher order dynamic equations

$$[r(t)(z^{\Delta n-1}(t))^{\alpha}]^{\Delta}+q(t)f(x(\delta(t)))=0 \quad \text{for } t\in[t_0,\infty)_{\mathbb{T}},$$

where $z(t):=x(t)+p(t)x(\tau(t))$ and $\alpha\geq 0$ is a constant. The obtained results extend and supplement certain known results in literature. .

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1. Introduction

In this paper, we introduce new sufficient conditions for the oscillation of solutions of the neutral differential equation

$$[r(t)(z^{\Delta n-1}(t))^{\alpha}]^{\Delta}+q(t)f(x(\delta(t)))=0 \quad \text{for } t\in[t_0,\infty)_{\mathbb{T}} \qquad (1.1)$$

where $z(t):=x(t)+p(t)x(\tau(t))$ and $\alpha\geq 0$ is constant. We assume the following conditions .

(H_1) $r\in C_{rd}([t_0,\infty)_{\mathbb{T}},\mathbb{R})$, $r(t)>0$, $r^{\Delta}(t)>0$, $\int_{t_0}^{\infty}r^{-1/\alpha}\Delta s=\infty$.

(H_2) $\tau,\delta\in C^1_{rd}([t_0,\infty)_{\mathbb{T}},\mathbb{T})$, $\tau\circ\delta=\delta\circ\tau$, $\lim_{t\to\infty}\tau(t)=\infty$, $\delta(t)\leq t$ $\lim_{t\to\infty}\delta(t)=\infty$.

(H_3) $p,q\in C_{rd}([t_0,\infty)_{\mathbb{T}},\mathbb{R})$, p is an oscillating function, $\lim_{t\to\infty}p(t)=0$, and $q(t)>0$.

(H_4) $f\in C(\mathbb{T},\mathbb{T})$, and there exists a positive constant k such that $\frac{f(x)}{x}\geq k$ for all $x\neq 0$

The theory of time scales was introduced by Hilger (see [9]) in 1988 in order to unify continuous and discrete analysis. A *time scale,* which inherits the standard topology on $\mathbb{R}$, is a nonempty closed subset of reals. Here, and later throughout this paper, a time scale will be denoted by the symbol $\mathbb{T}$, and the intervals with a subscript $\mathbb{T}$ are used to denote the intersection of the usual interval with $\mathbb{T}$. For $t\in\mathbb{T}$, the *forward jump operator* $\sigma:\mathbb{T}\to\mathbb{T}$ is defined by $\sigma(t):=\inf(t,\infty)_{\mathbb{T}}$, while the *backward jump operator* $\rho:\mathbb{T}\to\mathbb{T}$ is defined by $\rho(t):=\sup(-\infty,t)_{\mathbb{T}}$, and the *graininess function* $\mu:\mathbb{T}\to\mathbb{R}^{+}$ is defined to be $\mu(t):=\sigma(t)-t$. A point $t\in\mathbb{T}$ is called *right-dense* if $\sigma(t)=t$ and/or equivalently $\mu(t)=0$ holds; otherwise, it is called *right-scattered*, and similarly *left-dense* and *left-scattered points* are defined with respect to the backward jump operator.

The set of all such *rd*-continuous functions is denoted by $C_{rd}(\mathbb{T},\mathbb{R})$. The set of functions $f:\mathbb{T}\to\mathbb{R}$ which are differentiable and whose derivative is an *rd*-continuous function is denoted by $C^1_{rd}(\mathbb{T},\mathbb{R})$.The Delta derivative of a function $f:\mathbb{T}\to\mathbb{R}$ is defined by

$$f^{\Delta}(t)=\begin{cases}\dfrac{f^{\sigma}(t)-f(t)}{\mu(t)}, & \mu(t)>0\\[2ex] \lim\limits_{s\to t}\dfrac{f(t)-f(s)}{t-s}, & \mu(t)=0\end{cases}$$

The Δ derivative of the product of two differentiable functions f and g :

$$(fg)^{\Delta}(t)=f^{\Delta}(t)g(t)+f(\sigma(t))g^{\Delta}(t)$$

and Δ derivative of the quotient of two differentiable functions f and $g\neq 0$: is given by

$$(\frac{f}{g})^{\Delta}(t)=\frac{g(t)f^{\Delta}(t)-f(t)g^{\Delta}(t)}{g(\sigma(t))g(t)}$$

F is called an antiderivative of a function f defined on $\mathbb{T}$ if $F^{\Delta}=f$ holds on $\mathbb{T}^k$. In this case integration of f is defined by

$$\int_s^t f(\tau)\Delta\tau=F(t)-F(s)\quad where\ s,t\in\mathbb{T}$$

An antiderivative of 0 is 1, an antiderivative of 1 is t; but it is not possible to find a polynomial which is an antiderivative of t .The role of t^2 is therefore played in the time scales calculus by

$$\int_0^t\sigma(\tau)\Delta\tau\quad and\quad \int_0^t\tau\Delta\tau$$

In general, the functions

$$g_0(t,s)\equiv 1,\ and\ \ g_{k+1}(t,s)=\int_s^t g_k(\sigma(\tau),s)\Delta\tau,\quad k\geq 0$$

and

$$h_0(t,s)\equiv 1,\ and\ \ h_{k+1}(t,s)=\int_s^t h_k(\tau,s)\Delta\tau,\quad k\geq 0$$

may be considered as the polynomials on $\mathbb{T}$. The relationship between g_k and h_k is

$$g_k(t,s)=(-1)^k h_k(t,s)\quad for\ all\ k\in\mathbb{N}$$

The following is the dynamic generalization of the well-known Taylor's formula

Lemma 1.1 *(Taylorâ€™s formula [3]) Let* $n\in\mathbb{N}$ *and* $s\in\mathbb{T}$*, and let* $f\in C^n_{rd}(\mathbb{T},\mathbb{R})$. *Then,*

$$f(t)=\sum_{k=0}^{n-1}h_k(t,s)f^{\Delta k}(s)+\int_s^t h_{n-1}(t,\sigma(\eta))f^{\Delta n}(\eta)\Delta\eta\quad for\ \ t\in\mathbb{T}.$$

By a solution of (1.1) we mean a nontrivial function $x\in C_{rd}([T_x,\infty)_{\mathbb{T}},\mathbb{R})$, where $T_x\in[t_0,\infty)_{\mathbb{T}}$, which has the property that $[r(t)(z^{\Delta n-1}(t))^{\alpha}]\in C^1_{rd}([T_x,\infty)_{\mathbb{T}},\mathbb{R})$ and satisfies (1.1) identically on $[T_x,\infty)_{\mathbb{T}}$. A solution x of (1.1) is said to be oscillatory if it is neither eventually positive nor eventually negative; otherwise, it is nonoscillatory. Equation (1.1) is called oscillatory if all its solutions oscillate.

In recent years there has been much research activities concerning the oscillation of solutions of several classes of neutral dynamic equations, see [1, 2, 7, 10-14, 16-19]

Several papers are devoted to study the cases in which $0<p(t)<1$ and $0<p(t)\leq p_0<\infty$, for instance, In 2015 Karpuz [11] studied the qualitative behavior of solutions of the higher order delay dynamic equations of the form

$$[x(t)+A(t)x(\alpha(t))]^{\Delta n}+B(t)x(\beta(t))=0 \quad \text{for } t\in[t_0,\infty)_{\mathbb{T}},$$

where $n\in\mathbb{N}$, $A\in C_{rd}([t_0,\infty)_{\mathbb{T}},\mathbb{R})$, and $\alpha(t),\beta(t)\le t$ for all $t\in[t_0,\infty)_{\mathbb{T}}$.

Chen . [5] established sufficient conditions for the oscillation and asymptotic behavior of solutions of the nth-order nonlinear neutral delay dynamic equation

$$\{a(t)\psi(x(t))[|(x(t)+p(t)x(\tau(t)))^{\Delta n-1}|^{\alpha-1}|(x(t)+p(t)x(\tau(t)))^{\Delta n-1}|]^{\gamma}\}^{\Delta}+\lambda f(t,x(\delta(t)))=0,$$

where $\alpha>0$ is a constant, $\gamma>0$ is a quotient of odd positive integers and $\lambda=\pm 1$;

$p(t)\in C_r d(\mathbb{T},\mathbb{R})$ and $0\le p(t)\le 1$.

On the other hand, few papers discussed the case when $p(t)$ oscillates . In [15] Mustafa studied the oscillatory behavior of certain higher order differential equations of the form.

$$[x(t)+p(t)x(\tau(t))]^{(n)}+q(t)[x(\sigma(t))]^{\alpha}=0,\ t\ge t_0,$$

where $\alpha>0$, $p(t)$ is oscillatory and $\lim_{t\to\infty}p(t)=0$.

On the particular case $\mathbb{T}=\mathbb{Z}$, Bolat et al,[4] introduced new oscillation criteria for bounded solutions of the higher order difference equation

$$\Delta^n(y(k)+p(k)y(k-\tau)+q(k)f(y(\sigma(k)))=0, \quad n\ge 2\in\mathbb{N},\ k\in\mathbb{N}$$

where $P(k)$ is oscillatory.

Recently, by employing a generalized Riccati technique and an integral averaging technique, Chen [6] established several oscillation criteria for all bounded solutions of the equation

$$[r(t)([y(t)+p(t)y(\tau(t))]^{\Delta})^{\alpha}]^{\Delta}+f(t,y(\delta(t)))=0 \quad \text{for } t\in[t_0,\infty)_{\mathbb{T}},$$

where $\alpha>0$, $p(t)$ is oscillatory and $\lim_{t\to\infty}p(t)=0$.

In what follows, we present some known results, which will be useful in the proof of our main results.

Theorem 1.1 *[3] Assume that* $v:\mathbb{T}\to\mathbb{R}$ *is strictly increasing and* $\tilde{\mathbb{T}}:=v(\mathbb{T})$ *is a time scale. Let* $y:\tilde{\mathbb{T}}\to\mathbb{R}$ *. If* $y^{\tilde{\Delta}}[v(t)]$ *and* $v^{\Delta}(t)$ *exist for* $t\in\mathbb{T}_k$ *, then*

$$\left(y[v(t)]\right)^{\Delta}=y^{\tilde{\Delta}}[v(t)]v^{\Delta}(t).$$

Lemma 1.2 *[3] Let* $n\in N$, $f\in C_{rd}^{n}(\mathbb{T},\mathbb{R})$ *and* $\sup\mathbb{T}=\infty$. *Suppose that* f *is either positive or negative,* $f^{\Delta n}$ *is not identically zero and is either nonnegative or nonpositive on* $[t_0,\infty)_{\mathbb{T}}$ *for some* $t_0\in\mathbb{T}$. *Then, there exist* $t_1\in[t_0,\infty)_{\mathbb{T}}$, $m\in[0,n)_{\mathbb{Z}}$ *such that* $(-1)^{n-m}f(t)f^{\Delta n}(t)\ge 0$ *for all* $t\in[t_0,\infty)_{\mathbb{T}}$ *with*

- $f(t)f^{\Delta j}(t)>0$ for all $t\in[t_0,\infty)_{\mathbb{T}}$ and all $j\in[0,m)_{\mathbb{Z}}$,
- $(-1)^{m+j}f(t)f^{\Delta j}(t)\ge 0$ for all $t\in[t_0,\infty)_{\mathbb{T}}$ and all $j\in[m,n)_{\mathbb{Z}}$,

Lemma 1.3 *[11] Let* $\sup\mathbb{T}=\infty$, $n\in\mathbb{N}$ *and* $f\in C_{rd}^{n}([t_0,\infty),\mathbb{R}_0^{+})$ *with* $f^{\Delta n}\le 0$ *on* $[t_0,\infty)_{\mathbb{T}}$. *Let Lemma 1.2 hold with* $m\in[0,n)_{\mathbb{Z}}$ *and* $s\in[t_0,\infty)_{\mathbb{T}}$. *Then*

$$f(t)\ge h_m(t,s)f^{\Delta m}(t) \quad \textit{for all } t\in[s,\infty)_{\mathbb{T}} \tag{1.2}$$

Lemma 1.4 *[8] Let* $\sup\mathbb{T}=\infty$ *and* $f\in C_{rd}^{n}(\mathbb{T},\mathbb{R}^{+})$, $(n\ge 2)$. *Suppose that Kneser's theorem holds with* $m\in[1,n)_{\mathbb{N}}$ *and* $f^{\Delta n}(t)\le 0$ *on* $\mathbb{T}$. *Then there exists a sufficiently large* $t_1\in\mathbb{T}$ *such that*

$$f^{\Delta}(t)\ge h_{m-1}(t,t_1)f^{\Delta m}(t) \quad \textit{for} \quad \textit{all} \ \ t\in[t_1,\infty)_{\mathbb{T}}.$$

Our aim in this paper is to obtain some sufficient conditions for the oscillation of all bounded solutions of (1.1) when

$p \in C_{rd}(\mathbb{T};\mathbb{R})$ is an oscillating function using a generalized Riccati technique.

2. Main results

In this section, we state the main results which guarantee that every bounded solution of (1.1) is oscillatory.

Lemma 2.1 *Let the conditions (H₁)-(H₃) be satisfied . If $x(t)$ is an eventually positive solution of (1.1), then there exists $t_1 \in [t_0,\infty)_{\mathbb{T}}$ such that*

$$z^{\Delta n-1}(t) > 0,\ \ z^{\Delta n}(t) \le 0,\ \ z^{\Delta}(t) > 0,\quad t > t_1. \tag{2.1}$$

Proof. Suppose that (1.1) has a nonoscillatory solution $x(t)$ on $[t_0,\infty)$, such that $x(t)>0$, $x(\tau(t))>0$, $x(\delta(t))>0$ on $[T_0,\infty)$, the assumptions 3 and 4. Then (1.1) implies

$$[r(t)(z^{\Delta n-1}(t))^{\alpha}]^{\Delta} \le -kq(t)x^{\alpha}(\delta(t)) < 0,\quad t \ge t_1. \tag{2.2}$$

Therefore, $r(t)(z^{\Delta n-1}(t))^{\alpha}$ is decreasing and either $z^{\Delta n-1}(t)>0$ or $z^{\Delta n-1}(t)<0$ eventually for $t \ge t_1$. We claim that $z^{\Delta n-1}(t)>0$ for $t\in[t_1,\infty)_{\mathbb{T}}$. If this is false, then there exists $t_2>t_1$ such that $z^{\Delta n-1}(t_2)<0$, and so $r(t_2)(z^{\Delta n-1}(t_2))^{\alpha} \le 0$. Since $r(t)(z^{\Delta n-1}(t))^{\alpha}$ is decreasing, it is clear that

$$r(t)(z^{\Delta n-1}(t))^{\alpha} \le r(t_3)(z^{\Delta n-1}(t_3))^{\alpha} := c < 0,\quad t_3 > t_2$$

Integrating from t_3 to t, we get

$$z^{\Delta n-2}(t) \le c^{1/\alpha}\int_{t_3}^{t} \frac{1}{r^{\frac{-1}{\alpha}}(s)}\Delta s.$$

Letting $t\to\infty$,then it follows from 1, that $\lim_{t\to\infty} z^{\Delta n-2}(t) = -\infty$. Therefore, $\lim_{t\to\infty} z(t) = -\infty$ and this is a contradiction. Consequently,

$$z^{\Delta n-1}(t) > 0\quad \mathit{for}\quad t \ge t_1. \tag{2.3}$$

Now, we show that $z^{\Delta n}(t) \le 0$. Since

$$[r(t)(z^{\Delta n-1}(t))^{\alpha}]^{\Delta} = r^{\Delta}(t)(z^{\Delta n-1}(t))^{\alpha} + r^{\sigma}(t)[(z^{\Delta n-1}(t))^{\alpha}]^{\Delta} \le 0, \tag{2.4}$$

Using Pötzche chain rule [3], we obtain

$$[(z^{\Delta n-1}(t))^{\alpha}]^{\Delta} = \{\alpha\int_0^1 \left[z^{\Delta n-1}(t)+\mu h z^{\Delta n}(t)\right]^{\alpha-1} dh\} z^{\Delta n}$$

$$\ge \begin{cases} \alpha(z^{\Delta n-1}(t))^{\alpha-1} z^{\Delta n}(t), & \alpha > 1 \\ \alpha(z^{\Delta n-1}(\sigma(t)))^{\alpha-1} z^{\Delta n}(t), & 0<\alpha\le 1 \end{cases} \tag{2.5}$$

Substituting into (2.4) using (2.2),(2.5) , we get

$$r^{\Delta}(t)(z^{\Delta n-1}(t))^{\alpha} + \alpha r^{\sigma}(t)(z^{\Delta n-1}(t))^{\alpha-1} z^{\Delta n}(t) \le 0,\quad \text{when}\ \alpha > 1$$

and

$$r^{\Delta}(t)(z^{\Delta n-1}(t))^{\alpha} + \alpha r^{\sigma}(t)(z^{\Delta n-1}(\sigma(t)))^{\alpha-1} z^{\Delta n}(t) \le 0,\quad \text{when}\ 0<\alpha\le 1$$

i.e.

$$z^{\Delta n}(t) \le 0.$$

Applying Lemma1.2 and Lemma1.4, we obtain

$$z^{\Delta n-1}(t) > 0,\ \ z^{\Delta n}(t) \le 0,\ \ z^{\Delta}(t) > 0,\quad t > t_1.$$

Theorem 2.1 *Suppose that (H₁)-(H₄) hold. Furthermore, assume that there exists a constant $M \in (0,1)$ and a positive* *function* $\rho \in C_{rd}^{1}(\mathbb{T},\mathbb{R})$ *such that for all sufficiently large* $T \geq t_0$,

$$\limsup_{t\to\infty} \int_{T_1}^{t} [kM^{\alpha} q(s)\rho(t)\psi^{\alpha}(s,T) - \frac{1}{(\alpha+1)^{\alpha+1}} \frac{(\rho_{+}^{\Delta}(s))^{\alpha+1} r(s)}{\rho^{\alpha}(s) h^{\alpha}(s,T)}]\Delta s = \infty. \quad (2.6)$$

where $T_1 > T$, and $\psi^{\alpha}(s,T) = \frac{h_{n-2}(\delta(t),T)}{h_{n-2}(t,T)} (\int_T^t r^{-1/\alpha}(s)\Delta s)^{-1} \int_T^{\delta(t)} r^{-1/\alpha}(s)\Delta$ Then every bounded solution of (1.1) is oscillatory.

Proof. Suppose that (1.1) has a nonoscillatory solution $x(t)$ on $[t_0,\infty)$, such that $x(t) > 0$, $x(\tau(t)) > 0$, $x(\delta(t)) > 0$ on $[T_0,\infty)$, Using the definition of z and Lemma 2.1, we have $z(t) > 0,$ $z^{\Delta}(t) > 0$, $z^{\Delta n-1}(t) > 0$, and $z^{\Delta n}(t) \leq 0$. Since $z(t) > 0$ and is bounded, we get $\lim_{t\to\infty} z(t) = L > 0$. But since $M \in (0,1)$, then by 3, we have

$$\lim_{t\to\infty} [x(t) - Mz(t)] = \lim_{t\to\infty} [(1-M)z(t) - p(t)x(\tau(t))] = (1-M)L > 0.$$

This means that there exists $t_2 > t$ such that $x(t) - Mz(t) > 0$, i.e. $x(t) > Mz(t)$ for all sufficiently large t and so $x(\delta(t)) > Mz(\delta(t))$. Then (2.2) takes the form

$$[r(t)(z^{\Delta n-1}(t))^{\alpha}]^{\Delta} \leq -kM^{\alpha} q(t) z^{\alpha}(\delta(t)) < 0, \quad t \geq t_1. \quad (2.7)$$

Define the generalized Riccati substitution

$$\omega(t) = \rho(t) \frac{r(t)(z^{\Delta n-1}(t))^{\alpha}}{z^{\alpha}(t)}, \quad t \in [T,\infty)_{\mathbb{T}}. \quad (2.8)$$

then clearly we have $\omega(t) > 0$, and

$$\omega^{\Delta}(t) = \frac{\rho(t)}{z^{\alpha}(t)} [r(t)(z^{\Delta n-1}(t))^{\alpha}]^{\Delta} + [r(t)(z^{\Delta n-1}(t))^{\alpha}]^{\sigma} [\frac{\rho(t)}{z^{\alpha}(t)}]^{\Delta}$$

$$= \rho(t) \frac{[r(t)(z^{\Delta n-1}(t))^{\alpha}]^{\Delta}}{z^{\alpha}(t)} + \frac{\rho^{\Delta}(t)}{\rho(\sigma(t))} \omega(\sigma(t)) - \frac{\rho(t)}{\rho(\sigma(t))} \frac{(z^{\alpha}(t))^{\Delta}}{z^{\alpha}(t)} \omega(\sigma(t))$$

This with (2.7) leads to

$$\omega^{\Delta}(t) \leq -kM^{\alpha} q(t) z^{\alpha}(\delta(t)) \frac{\rho(t)}{z^{\alpha}(t)} + \frac{\rho^{\Delta}(t)}{\rho(\sigma(t))} \omega(\sigma(t)) - \frac{\rho(t)}{\rho(\sigma(t))} \frac{(z^{\alpha}(t))^{\Delta}}{z^{\alpha}(t)} \omega(\sigma(t)) \quad (2.10)$$

Since $r(t)(z^{\Delta n-1}(t))^{\alpha}$ is strictly decreasing on $[t_1,\infty)$, for $t \in [T,\infty)$ we obtain

$$z^{\Delta n-2}(t) - z^{\Delta n-2}(\delta(t)) = \int_{\delta(t)}^{t} \frac{[r(s)(z^{\Delta n-1}(s))^{\alpha}]^{1/\alpha}}{r^{1/\alpha}(s)} \Delta s \leq [r(\delta(t))(z^{\Delta n-1}(\delta(t)))^{\alpha}]^{1/\alpha} \int_{\delta(t)}^{t} r^{-1/\alpha}(s)\Delta s$$

and

$$\frac{z^{\Delta n-2}(t)}{z^{\Delta n-2}(\delta(t))} \leq 1 + \frac{[r(\delta(t))(z^{\Delta n-1}(\delta(t)))^{\alpha}]^{1/\alpha}}{z^{\Delta n-2}} \int_{\delta(t)}^{t} r^{-1/\alpha}(s)\Delta s. \quad (2.11)$$

Choosing $T_1 \in [T,\infty)$ such that $\delta(t) > T$ for $T_1 \in [T,\infty)$. Then for $t \in [T,\infty)$, we get

$$\begin{aligned} z^{\Delta n-2}(\delta(t)) &> z^{\Delta n-2}(\delta(t)) - z^{\Delta n-2}(T) = \int_T^{\delta(t)} \frac{[r(s)(z^{\Delta n-1}(s))^{\alpha}]^{1/\alpha}}{r^{1/\alpha}(s)} \Delta s \\ &\geq [r(\delta(t))(z^{\Delta n-1}(\delta(t)))^{\alpha}]^{1/\alpha} \int_T^{\delta(t)} r^{-1/\alpha}(s)\Delta s \end{aligned} \quad (2.12)$$

Hence

$$\frac{[r(\delta(t))(z^{\Delta n-1}(\delta(t)))^{\alpha}]^{1/\alpha}}{z^{\Delta n-2}(\delta(t))} < (\int_T^{\delta(t)} r^{-1/\alpha}(s)\Delta s)^{-1} \quad (2.13)$$

From (2.11) and (2.13), we get

$$\frac{z^{\Delta n-2}(t)}{z^{\Delta n-2}(\delta(t))} \le 1+(\int_T^{\delta(t)} r^{-1/\alpha}(s)\Delta s)^{-1}\int_{\delta(t)}^t r^{-1/\alpha}(s)\Delta s = (\int_T^{\delta(t)} r^{-1/\alpha}(s)\Delta s)^{-1}\int_T^t r^{-1/\alpha}(s)\Delta s.$$

i.e

$$\frac{z^{\Delta n-2}(\delta(t))}{z^{\Delta n-2}(t)} \ge (\int_T^t r^{-1/\alpha}(s)\Delta s)^{-1}\int_T^{\delta(t)} r^{-1/\alpha}(s)\Delta. \tag{2.14}$$

Using Lemma 1.2, we get

$$\frac{z(\delta(t))}{z(t)} \ge \frac{h_{n-2}(\delta(t),T)}{h_{n-2}(t,T)}(\int_T^t r^{-1/\alpha}(s)\Delta s)^{-1}\int_T^{\delta(t)} r^{-1/\alpha}(s)\Delta. \tag{2.15}$$

This with (2.10),and Lemma 1.4 leads to

$$\omega^{\Delta}(t) \le -kM^{\alpha}q(t)\rho(t)\psi^{\alpha}(t,T) + \frac{\rho_+^{\Delta}(t)}{\rho(\sigma(t))}\omega(\sigma(t)) - \alpha\frac{\rho(t)r^{-1/\alpha}(\sigma(t))}{\rho^{\frac{\alpha+1}{\alpha}}(\sigma(t))}h_{n-2}(t,T)\omega^{\frac{\alpha+1}{\alpha}}(\sigma(t)) \tag{2.16}$$

Applying the inequality

$$B\omega - A\omega^{\frac{\alpha+1}{\alpha}} \le \frac{\alpha^{\alpha}}{(1+\alpha)^{1+\alpha}}\frac{B^{1+\alpha}}{A^{\alpha}} \tag{2.17}$$

with $B = \frac{\rho^{\Delta}(t)}{\rho(\sigma((t))}$ and $A = \alpha\frac{\rho(t)r^{-1/\alpha}(\sigma(t))}{\rho^{\frac{\alpha+1}{\alpha}}(\sigma(t))}h_{n-2}(t,T) > 0$

This with (2.16), leads to

$$\omega^{\Delta}(t) \le -kM^{\alpha}q(t)\rho(t)\psi^{\alpha}(t,T) + \frac{1}{(\alpha+1)^{\alpha+1}}\frac{(\rho_+^{\Delta}(t))^{\alpha+1}r(t)}{\rho^{\alpha}(t)h^{\alpha}(t,T)} \tag{2.18}$$

Integrating from T_1 to t, we get

$$\int_{T_1}^t [kM^{\alpha}q(s)\rho(t)\psi^{\alpha}(s,T) - \frac{1}{(\alpha+1)^{\alpha+1}}\frac{(\rho_+^{\Delta}(s))^{\alpha+1}r(s)}{\rho^{\alpha}(s)h^{\alpha}(s,T)}]\Delta s \le \omega(T_1) - \omega(t) < \omega(T_1). \tag{2.19}$$

This contradicts (2.6).

Corollary 2.1 *Let* $\rho(t) = 1$, *suppose that 1-2 hold and that for sufficiently large* $T \ge t_0$

$$\int_{T_1}^{\infty}\psi^{\alpha}(s,T)q(s)\Delta s = \infty, \tag{2.20}$$

where $\psi^{\alpha}(t,T)$ defined in Theorem 2.1, then all bounded solutions of (1.1) are oscillatory.

Example 2.1 *Consider the following second-order nonlinear neutral delay dynamic equation with an oscillating coefficient*

$$[(t+1)^{\beta}((x(t)+(\frac{-1}{2})^t x(\tau(t)))^{\Delta})^{\beta}]^{\Delta} + \frac{1}{\psi^{\alpha}(t,t^*)}\frac{1}{t}x^{\beta}(\delta(t)) = 0 \tag{2.21}$$

Here $r(t) = (t+1)^{\beta}$, $p(t) = (\frac{-1}{2})^t$, $\tau(t) \le t$, $\delta(t) \le t$, $\beta > 0$ and $q(t) = \frac{1}{\psi^{\alpha}(t,t^*)}\frac{1}{t}$.

It is clear that $\int_0^{\infty} r^{-1/\beta}(s)\Delta s = \infty$, and $\lim_{t\to\infty} p(t) = 0$

According to corollary 2.1 we have

$$\int_{T_1}^{\infty}\psi^{\alpha}(s,T)q(s)\Delta s = \int_{T_1}^{\infty}(\frac{\psi^{\alpha}(s,T)}{\psi^{\alpha}(s,t^*)})^{\beta}\frac{1}{s}\Delta s = \int_{T_1}^{\infty}\frac{1}{s}\Delta s = \infty, \tag{2.22}$$

This mean that all bounded solutions of (2.21) are oscillatory.

Example 2.2 *Consider the equation*

$$[x(t)+4e^{\frac{-t+\pi}{2}}sin(t/2)x(\frac{t-\pi}{2})]''+\frac{4e^{\pi}-2}{te^{\frac{-\pi}{2}}}x(t-\pi/2)=0,\ t>0. \tag{2.23}$$

Here $n=2$, $r(t)=1$, $p(t)=4e^{\frac{-t+\pi}{2}}sin(t/2)$, $\tau(t)\le t$, $\delta(t)\le t$, $\alpha=1$ and $q(t)=\frac{4e^{\pi}-2}{te^{\frac{-\pi}{2}}}$.

In this example $\mathrm{T}=\mathrm{R}$, $\int_0^{\infty}r^{-1/\alpha}(s)\Delta s=\int_0^{\infty}r^{-1/\alpha}(s)ds=\infty$, and $\lim_{t\to\infty}p(t)=0$.

According to Theorem 2.1, $\mathrm{T}=\mathrm{R}$ and we have $h_n(t,s)=\frac{(t-s)^n}{n!}$,

$$\psi^{\alpha}(t,T)=\frac{t-\pi/2-T}{t-T},$$

Choosing $\rho(t)=1$, and $M\in(0,1)$, we obtain

$$\limsup_{t\to\infty}\int_{T_1}^{t}[kM^{\alpha}q(s)\rho(t)\psi^{\alpha}(s,T)-\frac{1}{(\alpha+1)^{\alpha+1}}\frac{(\rho_+^{\Delta}(s))^{\alpha+1}r(s)}{\rho^{\alpha}(s)h^{\alpha}(s,T)}]\Delta s$$
$$=\limsup_{t\to\infty}\int_{T_1}^{t}[kM\frac{4e^{\pi}-2}{se^{\frac{-\pi}{2}}}\frac{(s-\pi/2-T)}{(s-T)}]ds=\infty$$

This mean that all bounded solutions of (2.23) are oscillatory. In fact, $x(t)=e^{t}\sin t$ is such a solution.

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