

(Article)

## Critical Launch Angle for Projectile Motion

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### ABSTRACT

In this research article, we have successfully demonstrated an interesting characteristic of parabolic motion when the launch angle is at a critical value. Specifically, we have discovered an unexpected property associated with the distance between the launched object and the launcher. This property exhibits both a maximum and a minimum distance. We will illustrate that this phenomenon occurs only when the launch angle exceeds a certain threshold, specifically when  $\cos(\alpha) = \frac{1}{3}$ . It is essential to highlight that our investigation of the launch scenario assumes a ground-level launch and does not account for the effects of air resistance. To emphasize the angular variable, we used dimensionless coordinates in our analysis. Critical launch angle is a key parameter determining the optimal angle for maximizing range and accuracy in projectile trajectories. This study offers a refined method to optimize trajectories for maximum efficiency and precision across various applications, enhancing performance and control. It can be used in various fields, such as sports and civil engineering.

**Keywords:** *parabolic motion, motion in two dimensions, projectile, launcher.*

### 1- Introduction

In a gravitational field, such as on the Earth's surface, when an object or particle is thrown or projected, it undergoes a type of motion known as projectile motion. This particular motion occurs due to the influence of gravity, causing the object to follow a curved path. Specifically, the

trajectory of an object in projectile motion is symmetrical on both sides and can be described as a parabolic path.

The critical launch angle in projectile motion has significant implications across diverse fields. In sports, it optimizes throws and kicks for maximum distance and accuracy. In the military, it ensures precise targeting and effective use of artillery and missiles. Aerospace engineering uses it to design efficient launch trajectories for rockets and satellites. Civil engineering applies this concept to safely manage debris trajectories in demolitions and avoid structural collisions. Projectile motion is a fundamental concept in physics that is frequently encountered. It involves the motion of an object when it is launched or thrown into the air, experiencing only the influence of gravity as the sole acceleration. This type of motion has captivated the curiosity of scientists throughout history, as they have sought to understand and analyze the movement of objects propelled through the air. It reveals the interesting properties of objects propelled into the air by only gravity. In the realm of physics, both baseballs and tennis balls exhibit what is known as projectile behavior when they traverse through the air during a home run hit or a field goal kick ((Wójcicki, 2011) - (Dittrich, 2023)). These balls follow a distinct curved trajectory as they move through the air. Parabolic motion, which describes this curved path, has been extensively studied over the years ((Krist, 1993) - (Yabushita, 2007)). We believed we comprehensively understood its characteristics, but a recent discovery has taken us by surprise. When we launch a projectile, the object seems to be moving away from the launcher continually, from both the launcher's perspective and the particle itself. This observation holds even when viewed from a vantage point situated away from the launcher. It is essential to provide a clear explanation of the physical mechanism underlying the critical launch angle for projectile motion and the principles for mitigating undesirable effects.

The critical launch angle is the specific angle at which a projectile achieves optimal performance, typically in terms of maximum range or accuracy. Understanding this mechanism involves analyzing the forces acting on the projectile, including gravity, air resistance, and initial velocity. By comprehensively examining these factors, one can determine how they influence the projectile's trajectory. Additionally, strategies for reducing undesirable effects, such as drag and wind interference, are crucial. These can include optimizing the projectile's shape, surface texture, and launch conditions. By thoroughly explaining these principles, the study aims to provide a robust framework for improving projectile performance in various practical applications.

Parabolic motion from a critical launch angle has an expected property related to the distance between the object and the launcher (Escobar, 2022). A mathematically regular approximation of the pair interaction between the particles combined with the power expansion to the magnetic interaction parameter is used in (Abu-Bakr, 2019). The stability analysis of this body was converted from a two-dimensional phase plane to a three-phase space (Amer, 2024). The basic equation of the body motion used to get the regulating motion's system as well as the three available independent first integrals, are studied in (Amer, 2022). In (He, 2023), the effects of the various body parameter values on the motion's behavior, which can be used to optimize the charged rigid body, are represented.

In this study, we aim to demonstrate a rather unexpected phenomenon. Contrary to common belief, it is not always true that the projectile moves further away from its launch location. Surprisingly, we have observed instances where the projectile actually approaches the launching point. These results challenge the intuition of both students and teachers. We have identified a specific range of angles where the distance between the launch location and the projectile decreases. This goes against the conventional understanding that the projectile always moves away.

## **2- Mathematical mode**

The Critical Launch Angle for Projectile Motion addresses the optimal angle for maximizing projectile range or accuracy. A schematic problem depicts a projectile launched at an angle above the ground with its path. This problem's resolution is crucial for achieving the desired projectile performance.

The trajectory of an object refers to the precise route it follows while in motion. For projectile motion to occur, an initial force must be exerted at the start of the trajectory. Subsequently, the only factor that affects the object's movement is gravity. Two-dimensional free-fall motion with only gravity acting on it is called projectile motion. Ignoring the impact of air resistance produces results that show the reality of comparatively large objects traveling over comparatively short distances at a relatively slow velocity. The motion of a projectile can be described as a trajectory that follows a parabolic path resulting from the combination of two independent motions (Biggs, 2020).

Given a particle launched with a velocity  $v_0$  and angle  $\alpha$  from the origin of the Cartesian coordinate system, we can mathematically describe its motion using.

$$x(t) = v_0 \cos(\alpha)t, \tag{1}$$

$$y(t) = v_0 \sin(\alpha)t - \frac{1}{2}gt^2. \tag{2}$$

The dimensionless position vector is delineated in Eqs. (3) and (4) denote the projectile's location at any particular instance.

$$\vec{r}(t) = x(t)\vec{i} + y(t)\vec{j}, \tag{3}$$

$$\vec{r}(t) = v_0 \cos(\alpha)t\vec{i} + [v_0 \sin(\alpha)t - \frac{1}{2}gt^2]\vec{j}. \tag{4}$$

The magnitude of the position vector represents the distance between the object and the launcher at any given point. Eq. (5) calculates the square root of a polynomial that represents the square root of a fourth-degree polynomial in the new dimensionless time variable, which determines this magnitude.

$$r(t) = \sqrt{x(t)^2 + y(t)^2}, \tag{5}$$

$$r(t) = \sqrt{(v_0 \cos(\alpha)t)^2 + \left(v_0 \sin(\alpha)t - \frac{1}{2}gt^2\right)^2}, \tag{6}$$

$$r(t) = \sqrt{v_0^2 t^2 - gv_0 \sin(\alpha)t^3 + \frac{1}{4}g^2 t^4}. \tag{7}$$

Let's examine this function in greater detail, looking at its critical points. To do this, we detect the first derivative and determine its roots.

$$\frac{dr}{dt} = \frac{2v_0^2 t - 3gv_0 \sin(\alpha)t^2 + g^2 t^3}{2\sqrt{v_0^2 t^2 - gv_0 \sin(\alpha)t^3 + \frac{1}{4}g^2 t^4}}. \tag{8}$$

To get the maximum and minimum points, put  $\frac{dr}{dt} = 0$

$$2v_0^2 t - 3gv_0 \sin(\alpha)t^2 + g^2 t^3 = 0. \tag{9}$$

The trivial solution  $t = 0$  will be disregarded as it corresponds to the start of the launch. The other two solutions

$$t_{1,2} = \frac{3gv_0 \sin(\alpha) \pm \sqrt{\frac{9}{2}g^2v_0^2 - \frac{9}{2}g^2v_0^2 \cos(2\alpha) - 8g^2v_0}}{2g^2} \tag{10}$$

$$\frac{9}{2}g^2v_0^2 - \frac{9}{2}g^2v_0^2 \cos 2\alpha - 8g^2v_0 \geq 0 . \tag{11}$$

For  $g = 9.8m/s^2$  and  $v_0=1m/s$ , we obtain

$$\alpha \geq 70.53 \tag{12}$$

Based on all the information we have reviewed so far if we commence a launch at an angle greater than the critical angle  $\alpha \geq 70.53$ .

### 3- Results and Discussion

In **Table 1**, the Critical Launch angle is verified with available literature. As can be concluded from this table, the results are in good agreement with the available literature.

**Table 1: Compression of the Critical Launch angle with the previous work**

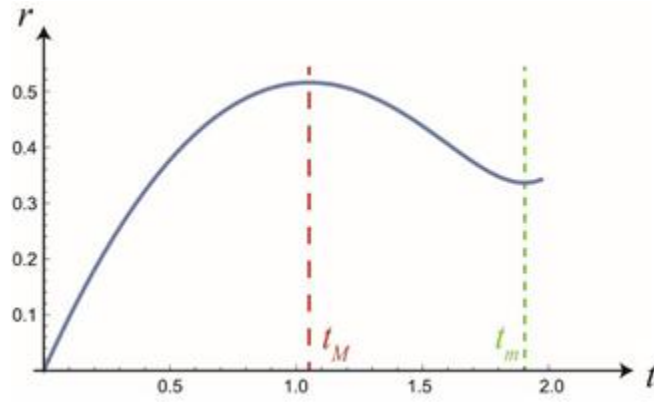
	Critical Launch angle
Present Critical Launch angle	$\alpha \geq 70.53$ .
Ribeiro & Sousa, 2021 ( Ribeiro, 2021)	$\alpha \geq 65$

In cases where we launch an object with an angle that exceeds the critical angle, we have thoroughly analyzed all relevant aspects when  $\alpha > \alpha_c$ .

The behavior exhibited by the distance to the launching point is comparable to what is depicted in Figure 1.

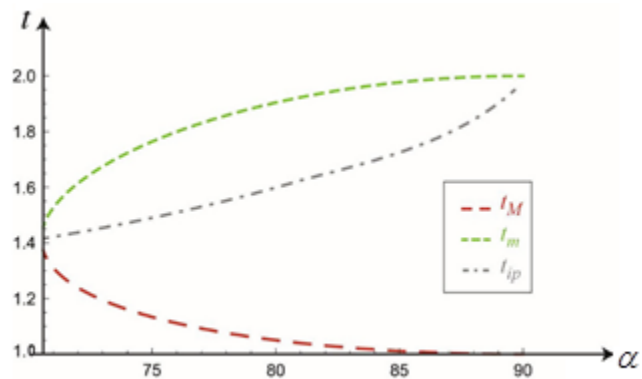
This observation indicates that during the time interval between Time Max ( $t_M$ ) and Time Min ( $t_m$ ), the distance between the point of launch and the projectile diminishes, implying that the object is getting closer to the launcher. This finding is intriguing and unexpected, as one would intuitively assume that the projectile would always move away from the launch point when considering only its horizontal movement along the x-axis. It is essential to conduct a

comprehensive analysis of the position vector to realize that there exists a range of launch angles within which, during a specific time interval, the projectile approaches the launcher.



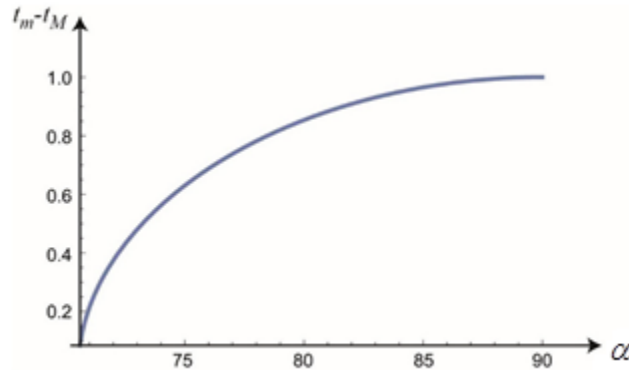
**Figure 1** The relation between time and distance

In Figure 1, the relationship between time and distance fluctuates as time progresses, demonstrating distinct patterns. Initially, as time increases, the distance traveled increases steadily, depicting a positive correlation. However, at a certain point, the distance reaches a peak or maximum value, after which it decreases, indicating a temporary reversal in the trend. This decline continues until reaching a minimum distance. Subsequently, as time further advances beyond this minimum point, the distance increases again, showcasing a renewed positive correlation between time and distance.



**Figure 2** The relation between time and angle  $\alpha$

Figure 2 illustrates the progression of critical points derived from examining the magnitude of the position vector, which signifies the distance between the projectile and the launch point. This evolution is contingent upon the launch angle, emphasizing angles surpassing the previously calculated critical angle.



**Figure 3** The relation between (Time max & Time min) and  $\alpha$

Figure 3 illustrates the correlation between the disparity in time intervals (Time max - Time min) and an angle, as shown by the increase in the difference in the angle.

#### 4- Conclusion

We have observed an interesting phenomenon in the context of projectile motion without air friction. Contrary to the common expectation that the distance between the launcher and the launched object would continuously increase, we have found a specific launch angle that decreases the distance for a certain period. After this interval, the object starts moving away from the launcher again, if such a trajectory is possible. It is important to note that this phenomenon occurs only when the launch angle surpasses a critical angle, which is approximately  $70.53^\circ$ . The initial velocity does not influence the critical angle. The described effect does not occur at angles below this, suggesting that it is unique to larger launch angles. The developed analytical analysis is directly related to the problem of determining the critical launch angle in projectile motion. This analysis helps in understanding the optimal angle at which a projectile should be launched to achieve a specific goal, such as maximizing range or height; the advantage of the proposed work

are precision, where the analytical approach provides precise mathematical formulas that describe the projectile's motion, allowing for accurate predictions and adjustments. It helps in predicting the projectile's behavior under ideal conditions, which is crucial for applications in various fields like sports, engineering, and physics, optimization where the analysis aids in identifying the optimal launch angle, improving efficiency in practical applications, the limitation of this model is the model typically assumes ideal conditions, such as the absence of air resistance, which may not reflect real-world scenarios. Future work could incorporate more realistic conditions, such as air resistance and varying launch elevations, to make the model more applicable to real-world situations.

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