The Generalized Weibull Uniform Log Logistic Distribution

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Abstract

In this article the T-X family is introduced by givening the cumulative distribution function (CDF) R{W(F(x))}, where R is the CDF of a random variable T, F is the CDF of X and W is an increasing function defined on [0, 1] having the support of T as its range. This family provides a new method of generating univariate distributions. Different choices of the R, F and W functions give different families of distributions. We used the quintile functions to define the W function. Some general properties of this T-X system of distributions are studied. Three new distributions of the T-X family are derived, namely, the Generalized Weibull Uniform Log Logistic Distribution (GRU {LL}). Different methods of estimation are used to estimate the unknown parameters in complete and censored random samples.

Key wards

T-X families, Weibull Distribution, Log logistic distribution , Generalized Weibull Uniform Log Logistic Distribution, quintile function.

1-Introduction

Many existing well-known distributions have been extensively used for modeling several areas of applications, such as medical, biological, economics studies and lifetime analysis. Despite the large number of Distribution, it is not enough to meet the various real applications. The recent literature has suggested many ways of extending will known distributions, for generating new generalized families. The generalized families are Pearson system [14],Burr system[2], Johnson system [7], Marshall Olkin(1997)[12],Lai [11], Tukey [15], Generalized Lambda distribution (G L D) Ramberg and Schiser [16], Freimer et al .[5], Karian and Dudewicz [9]; 2002,Jones,2004) [8] ,Kumaraswamy-G(Kw) Cordeiro and Castro , (2011)[3],; Alzatreh et al.(2013))[1].

Let r(t) be the probability density function (pdf) of a random Variable T, where $T \in [m,n]$ for $-\infty \le m \le n \le \infty$ and let W[F(x)δ] be a Continuous function of Cumulative distribution function (CDF) of a Continuous random Variable X, with Probability density function (PDF) depending on the Vector Parameter θ Satisfying the following conditions:

1- $W[F(X;\delta)] \in [m,n]$,

2- $W[F(X;\delta)]$ is differentiable and monotonically increasing,

3- $W[F(X;\delta)] \to m \quad as \to x \to \infty$

Alzaatreh et al.(2013)[1], defined the CDF of the T-X family of distribution by:

$$G(X;\delta) = \int_{m}^{W[F(X;\delta)]} r(t)dt \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots.(1)$$

If both Functions $W(.)$ and $F(.)$ are absolutely continuous, then $G(X;\delta)$ is absolutely continuous and has the PDF $v(y)$ and its quantile function is $Q_y(\theta) = V^{-1}(\theta)$ is a continuous and strictly increasing [2; Shorak and Wellner 1986]. Taking $W(.)$ in (1) to be the quantile function of . Then the CDF $G(X;\delta)$ of (1) is defined by :

$$G(X;\delta) = \int_{m}^{Q_y[F(X;\delta)]} r(t)dt \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots.(2)$$

$$G(X;\delta) = R\{Q_y[F(X;\delta)]\}, x \in [-\infty,\infty]$$

and the corresponding PDF of (2) is:

$$g(X;\delta) = \frac{f(X;\delta)}{v\{Q_y[F(X;\delta)]\}} r\{Q_y[F(X;\delta)]\} \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots.(3)$$

1.2 The Generalized Weibull_T –Uniform_X { Log – Logistic_y} Distribution GW_T-U_x\{ LL_y\}.

Let the variable y has the log – logestic {LL} distribution with Parameter (c) then the PDF$v(y)$ and the quantile function$Q_y(\theta)$ are:

$$v(y) = \frac{\left(\frac{1}{c}\right)}{\left[1+\left(\frac{y}{c}\right)\right]^2} \quad ,c > 0 \ ,y \geq 0 \ldots\ldots\ldots\ldots\ldots..(4)$$

$$Q_y(\theta) = c\left(\frac{\theta}{1-\theta}\right) \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots.(5)$$

Then ,From (4) and (5) ,the definition (3) defines the PDF $g(X;\delta)$of the T-X {LL} family as : $g(X;\delta) = \frac{\left(\frac{1}{c}\right)f(X;\delta).r\left\{c\left(\frac{F(X;\delta)}{1-F(X;\delta)}\right)\right\}}{\left(1-F(X;\delta)\right)^2} \ldots\ldots\ldots.(6)$

and from (3)and (5) the CDF $G(X;\delta)$ as :

$$G(X;\delta) = R\{Q_y[F(X;\delta)]\} \ldots\ldots\ldots.\ldots\ldots\ldots.(7)$$

Then from (5) and (7) the CDF $G(X;\delta)$ is defined as :

$$G(X;\delta) = R\left\{c\left(\frac{F(X;\delta)}{1-F(X;\delta)}\right)\right\} \ldots\ldots\ldots\ldots\ldots\ldots\ldots.(8)$$

when $c = 1$ the PDF $g(X;\delta)$ and the CDF $G(X;\delta)$ of the T-X {LL} family reduced to :

$$g(X;\delta) = \frac{F(X;\delta)}{\left(1-F(X;\delta)\right)^2}.r\left\{\frac{F(X;\delta)}{1-F(X;\delta)}\right\} \ldots\ldots\ldots.(9)$$

$$G(X;\delta) = R\left\{\frac{F(X;\delta)}{1-F(X;\delta)}\right\} \ldots\ldots\ldots\ldots\ldots\ldots\ldots.(10)$$

Suppose that T is a random variable has the standard Weibull distribution with the PDF $r(t)$ and CDF $R(t)$ with Parameter c, $f(X;\delta)$ and $F(X;\delta)$ are the PDF and CDF of a uniform distribution with two parameter (a,b) , $-\infty \leq a \leq b \leq \infty$, then

$$r(t) = c.t^{c-1}e^{-t^c} \dots\dots\dots\dots\dots\dots\dots\dots\dots.(11)$$

Where $c > 0$ is a shape parameter , $t > 0$

$$R(t) = 1 - e^{-t^c} \dots\dots\dots\dots\dots\dots\dots\dots\dots.(12)$$

$$f(x) = \frac{1}{b-a} \;,\; a \le x \le b \quad \dots\dots\dots\dots\dots\dots..(13)$$

$$F(x) = \frac{x-a}{b-a} \quad a \le x \le b \quad \dots\dots\dots\dots\dots\dots..(14)$$

Then substituting in (9) and (10) from (11) , (12),(13) and (14) we obtain the PDF$g(x)$ and the CDF $G(x)$ of the Generalized Weibull-Uniform {Log-Logestic} GWU{LL} distribution as :

$$g(x) = \frac{b-a}{(b-x)^2}.c\left(\frac{x-a}{b-x}\right)^{c-1} e^{\left\{-\left(\frac{x-a}{b-x}\right)^c\right\}} \dots\dots\dots\dots\dots..(15)$$

,andc $> 0 , -\infty \le a \le x \le b \le \infty$

$$G(x) = 1 - e^{\left\{-\left(\frac{x-a}{b-x}\right)^c\right\}} \dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots(16)$$

c $> 0 , -\infty \le a \le x \le b \le \infty$

Note : $G(a) = 1 - 1 = 0 , G(b) = 1 - 0 = 1$

then $G(x)$ Satisfies the required Conditions to be a CDF function.

The Survival function $S(x)$ and the hazard function $h(x)$ of the GWU{LL} distribution are defined as :

$$S(x) = 1 - G(x) \dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots.(17)$$

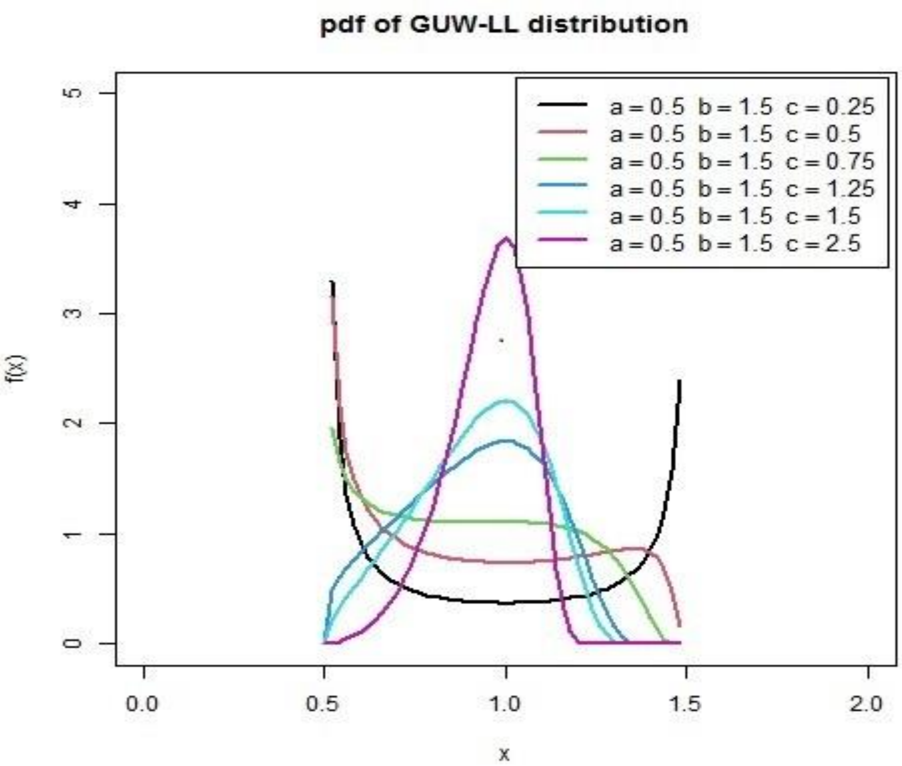
$$h(x) = \frac{g(x)}{1-G(x)} = \frac{g(x)}{S(x)} \dots\dots\dots\dots\dots\dots\dots\dots\dots..(18)$$

where $g(x)$ and $G(x)$ as given 1n (15) and (16)

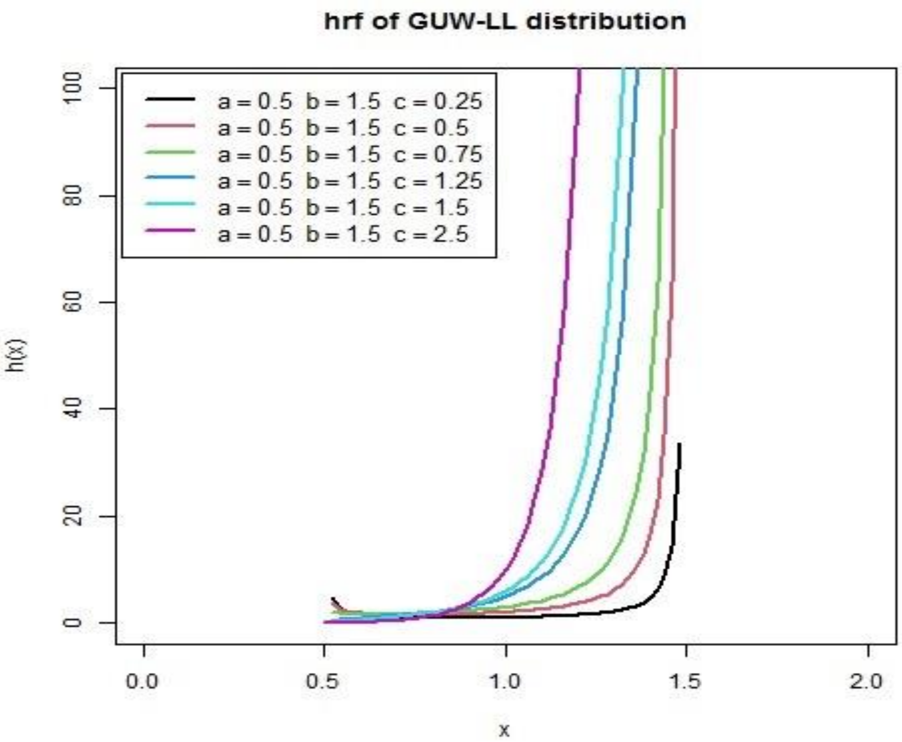
The random variable with PDF in (15) is said to follow the generalized Weibull – Uniform {Log-Logistic} (GWU{LL}) distribution. Plots of the GWU{LL} density and hazard functions are given in Figures (1,2). The grabhs in Figure (1) show that the GWU{LL} distribution can be wright skewed , bathtub shape , or Unimodal

.The graphs in Figure (2) show increasing failure which can be useful in analyzing various data sets.

Graph(1)



Graph(2)



The CDF of the GWU{LL} distribution is given in (16), and hence the quantile function of the GWU{LL} can be written as :

$$Q(p) = \frac{bz + a}{1 + z} \dots\dots\dots\dots\dots\dots\dots\dots . (19)$$

$$Z = \left[ln\left(\frac{1}{1-p}\right)\right]^{\frac{1}{c}}$$

where p is a vector of percentiles .

If we replaced p by a vector of Uniform (0,1) random variables we can use $Q(u)$ to generate random samples of GWU{LL} distribution .

The Bowley skewnes measure Bsk [10], and the Moor's kurtosis measure Mkur[13] are defined by :

$$Bsk = \frac{Q_{0.75} - 2Q_{0.5} + Q_{0.25}}{Q_{0.75} - Q_{0.25}}$$

$$Mkur = \frac{Q_{0.875} - Q_{0.625} - Q_{0.375} + Q_{0.125}}{Q_{0.75} - Q_{0.25}}$$

The above measures are less Senstive to outlires. Table(1) given below the Bsk and Mkur of the GWU{LL}distribution for the different values of the parameters.

Table (1): Quantiles, Skewness, and Kurtosis

Quantiles	parm1	parm2	parm3	parm4	parm5	parm6	parm7
Q.25	0.229305	0.910555	1.047333	1.133779	1.192926	1.235806	1.268264
Q.5	0.973593	1.31765	1.362941	1.390243	1.408486	1.421533	1.431327
Q.75	1.97324	1.662675	1.622216	1.597851	1.581578	1.569942	1.56121
Q.12	0.052555	0.621407	0.802873	0.926636	1.014725	1.08007	1.130258
Q.375	0.542803	1.130266	1.220185	1.275202	1.312236	1.338836	1.358854
Q.625	1.470968	1.490322	1.492741	1.494193	1.495161	1.495852	1.496371
Q.875	2.436522	1.858952	1.771512	1.718074	1.682122	1.656309	1.636887
Bskewness	0.146427	-0.08253	-0.09799	-0.10528	-0.10927	-0.11169	-0.11326
Mkurtosis	0.272548	-0.18644	-0.24099	-0.26868	-0.28444	-0.29422	-0.30067

2-The Raw Moments.

The r^{th} row moment of a random variable X having The GWU{LL} distribution is given as

$$:\mu_r^{\backslash} = E(x^r) = c(b-a)\int_a^b \frac{1}{(b-x)^2}\left(\frac{x-a}{b-x}\right)^{c-1} e^{\left\{-\left(\frac{x-a}{b-x}\right)^c\right\}} dx$$

The above integration can be easily calculated using the integer (.) function of the R software. Table (2) given below presents the first four raw moments $(\mu_1, \mu_2, \mu_3, \mu_4)$ the variance (μ_2) , the coefficient of variation (cv), the skewness (sk) and the kurtosis (kur) of the GWU{LL} distribution for different parameter values.

Table (2): Moments

moments		parm1	parm2	parm3	parm4
1	mean	0.506096	0.613194	0.680954	0.726962
2	m2	0.521234	0.680793	0.796611	0.883194
3	m3	0.595985	0.809942	0.977979	1.111464
4	m4	0.722384	1.005721	1.239914	1.433932
5	M2	0.265101	0.304787	0.332913	0.354721
6	CV	1.017356	0.900328	0.84732	0.824471
7	sk	0.467841	0.11112	-0.09305	-0.21926
8	kur	-1.29108	-1.59122	-1.67044	-1.692

3-Mean Residual life (MRL) For a random lifetime X,

The mean residual life (MRL) or the life expectancy at age t is the expected additional life length for a unit which is a life at age t . The MRL has many important applications in fuzzy set engineering , modeling ,insurance assessment of human life expectancy, demography, and economic etc .The MRL is the conditional expectation $E(x-t|x>t)$ where $t>0$. The MRL function cab be simply represented with the survival function $S(x)$. For a random lifetime X, the MRL is :

$$MRL = \frac{1}{S(x)} \int_t^{\infty} S(x)dx \quad , \qquad S(x) > 0$$

When $S(0)=1$ and $=0$, the MRL equal the average lifetime. When the MRL is represented with $S(x)$ we denote it by the theoretical MRL (TMRL), and when we calculate it from a random sample $x_1, x_2, .., x_n$ of size n of a GWU{LL}| distribution using the following expression is called the empirical MRL (EMRL) which can be calculated as :

$$EMRL = \frac{1}{(n-k)} \sum_{k=1}^{n-1} (x_{k+1} - x_k)$$

Where $x_{(k)}$ is the k^{th} orders statistic of the sample . Table(3) given below present the first and last 10 values of the TMRL and EMRL when the values of the age t is the order statistics of a random sample of size 50 from a GWU{LL}distribution with parameters $a=0, b=3, and\ c=4$

Table(3): MRL

	Death.Time	EMRL	TMRL
1	0.409027	0.95816	0.929195
2	0.515098	0.86984	0.839025
3	0.532758	0.870312	0.824408
4	0.743629	0.673777	0.658915
5	0.836386	0.593931	0.591614
6	0.845656	0.597949	0.585078
7	0.855473	0.60181	0.578196
8	0.867339	0.60399	0.56993
9	0.991738	0.491289	0.486777
10	1.017579	0.477083	0.470318

4 Some Methods of Estimation

Here five well-known methods of point estimation are presented: maximum likelihood method, maximum product spacing method, ordinary and weighted least squares methods and finally percentile method. Let $x = (x_1, \dots, x_n)^T$ be a random sample of size n from GWU{LL} distribution with $pdf\ g(x_i; a, b, c)$ given in (15) and cdf G(x;a,b,c) given in (16) with unknown parameters $\theta = (a, b, c)$ then we can introduce the five methods of estimation as given below.

4.1 Maximum Likelihood Method

The idea behind maximum likelihood parameter estimation is to determine the values of the parameters $\theta = (a, b, c)$ that maximize the probability (likelihood) of the sample data. The likelihood function is given as follows:

$$\ell(\theta) \equiv \ell(\theta; x_1, \dots, x_n) = g(x_1, \dots, x_n;\ \theta)$$

$$\ell(\theta) = \prod_{i=1}^{n} g(x_i; \theta) \qquad (2.3)$$

then the log-likelihood function is

$$LL = \sum_{i=1}^{n} \ell n g(x_i; a, b, s, c)$$

We use the R software to get the best estimates of the parameters a, b and c which maximize the above function LL.

4.2 Maximum Product Spacing Method

let $D_i\ (a, b, c\ ,)$ is the uniform spacings of a random sample from the GWU{LL} distribution defined by:

$D_i\ (a, b,\ c) = G\left(x_{(i)}; a, b, c\right) - G\left(x_{(i-1)}; a, b, c\right), i = 1, 2, \dots, n$

Where G(.) as given by (16), $G\left(x_{(0)}; a, b, c\right) = 0$ and $G\left(x_{(n+1)}; a, b, c\right) = 1$

The MPS estimators of a, b and c, are obtained by maximizing he following quantity.

$$\frac{1}{n+1} \sum_{i=1}^{n+1} \ell n\, D_i\ (a, b, c)$$

With respect to a, b and c .

4.3 Ordinary Least Squares Method

The best estimates minimize the difference between the observed values of the $cdf\ G(x; a, b, c)$ given in (16) for the order statistics $X_{(1)} < X_{(2)} < \cdots < X_{(n)}$ of a random sample $X_1, X_2, \ldots, X_n$ drawn from GWU{LL} population, and the corresponding expected values of $G(x; a, b, c)$. Let $\frac{i}{n+1}$ be an estimate of $G(x_{(i)}; a, b, c)$ then the LS estimators of a, b and c can be obtained by minimizing the following quantity:

$$Q_1 = \sum_{i=1}^{n} \left(G\left(x_{(i);a,b,c}\right) - \frac{i}{n+1} \right)^2$$

With respect to a, b and c.

4.4 Weighted Least Squares Method

The WLS estimators of $a, b\ and\ c$ of GWU{LL} distribution can be obtained by minimizing the following quantity:

$$Q_2 = \sum_{i=1}^{n} \frac{(n+1)^2\,(n+2)}{n-i+1} \left(G\left(x_{(i)}; a, b, c\right) - \frac{i}{n+1} \right)^2$$

With respect to a, b, s and c.

4.5 Percentile Method

If p_i denotes some estimate of $G(x_{(i)}; \theta)$, then the estimate of $\theta = (a, b, c)$ can be obtained by minimizing

$$\sum_{i=1}^{n} [\ln(p_i) - \ln[G(x_{(i)}; \theta)]]^2,$$

with respect to θ.

It is possible to use several p_i as estimators of $G(x_{(i)};\theta)$. For example $p_i = \left(\frac{i}{n+1}\right)$ is the most used estimator of $G(x_{(i)};\theta)$, as $(\frac{i}{n+1})$ is an unbiased estimator of $G(x_{(i)};\theta)$.

4.6 Simulation Study and Data Analysis

The aim of this section is to compare the performance of the methods of estimation, namely: MLE, MPS, LS, WLS, and PE for the GWU{LL} distribution which discussed in the previous section. A Monte Carlo study is employed to check the behavior of the proposed methods of estimation. Also, a real data set is analyzed for illustrative purpose. *R*-statistical programming language will be used for calculation.

4.6.1 Simulation Study

A simulation study is employed to compare the performance of proposed methods of estimation using Monte Carlo. The Monte Carlo process is carried by generating 5000 random data from the GWU_LL distribution with the following assumptions:

1. Sample sizes are $n = 20, 50, 100.$
2. Assume the following values of parameters a, b and c of the GWU_LL distribution:
   a. $\alpha = 0.25, 0.75, 1.25$
   b. $a = 0.75,\ 1.5,\ 2.5$
   c. $c = 0.25, 0.50, 0.75, 1.25, 1.5, 2.5$

Based on the generated data and applying different methods of estimation. All the means square error (MSE) and relative biases (RB) are reported from Table (1) to Table (9) for six different methods of estimation.

Table (1.a): The MSE and RB for different estimates of the GWU_LL distribution with parameters $a = 0.25, b = 0.75$ and different values of c at sample size $n = 25$.

Parameters	MLE		MPS		LS		WLS		PE	
	MSE	RB	MSE	RB	MSE	RB	MSE	RB	MSE	RB
Case I:										
$a = 0.25$	2.5E-07	0.0004	2.5E-07	0.0004	0.0003	0.0118	0.0001	0.0057	0.0003	0.0332
$b = 0.75$	0.0003	0.0162	0.0003	0.0171	0.0085	0.0251	0.0050	0.0091	0.0081	0.0067
$c = 0.25$	0.0028	0.1041	0.0064	0.2415	0.0462	0.2425	0.0281	0.2122	0.0317	0.3268
Case II:										
$a = 0.25$	4.5E-05	0.0186	4.5E-05	0.0185	0.0010	0.0341	0.0005	0.0149	0.0006	0.0097
$b = 0.75$	0.0019	0.0470	0.0027	0.0559	0.0507	0.0139	0.0464	0.0014	0.0067	0.0145
$c = 0.50$	0.0075	0.0619	0.0214	0.1704	0.1684	0.2651	0.1033	0.2269	0.0688	0.1700
Case III:										
$a = 0.25$	0.0005	0.0774	0.0005	0.0769	0.0017	0.0170	0.0014	0.0074	0.0013	0.0142
$b = 0.75$	0.0035	0.0561	0.0043	0.0724	0.0592	0.0104	0.0244	0.0035	0.0383	0.0163
$c = 0.75$	0.0217	0.0205	0.0382	0.0898	0.4959	0.2455	0.3402	0.2268	0.1887	0.1324
Case IV:										
$a = 0.25$	0.0039	0.2357	0.0040	0.2408	0.0041	0.0916	0.0038	0.0797	0.0036	0.0970
$b = 0.75$	0.0124	0.1340	0.0093	0.1178	0.2896	0.0189	0.1119	0.0010	0.0210	0.0262
$c = 1.25$	0.3756	0.3571	0.1523	0.1341	1.3125	0.1011	1.3866	0.1538	1.2066	0.0941
Case V:										
$a = 0.25$	0.0065	0.3049	0.0067	0.3152	0.0057	0.1513	0.0057	0.1370	0.0052	0.1457
$b = 0.75$	0.0185	0.1459	0.0400	0.1323	0.4115	0.0160	0.0919	0.0167	0.1037	0.0265
$c = 1.50$	0.6374	0.3817	0.2669	0.2153	1.7290	0.0181	2.2958	0.0947	1.8100	0.0395
Case VI:										
$a = 0.25$	0.0172	0.5087	0.0178	0.5239	0.0133	0.3782	0.0123	0.3545	0.0122	0.3507
$b = 0.75$	0.1387	0.1785	0.2037	0.1663	0.6072	0.0462	0.3904	0.0611	0.2095	0.0777
$c = 2.50$	2.5603	0.5226	2.0128	0.4425	3.7260	0.2781	4.9501	0.2158	4.3907	0.2218

Table (1.b): The MSE and RB for different estimates of the GWU_LL distribution with parameters $a = 0.25, b = 0.75$ and different values of c at sample size $n = 50$.

Parameters	MLE		MPS		LS		WLS		PE	
	MSE	RB	MSE	RB	MSE	RB	MSE	RB	MSE	RB
Case I:										
$a = 0.25$	2.3E-09	8.1E-05	2.3E-09	8.1E-05	3.6E-05	0.0024	1.2E-05	0.0004	0.0001	0.0176
$b = 0.75$	8.0E-05	0.0100	8.8E-05	0.0102	0.0012	0.0086	0.0010	0.0010	0.0034	0.0130
$c = 0.25$	0.0025	0.1520	0.0044	0.2246	0.0084	0.1647	0.0142	0.1758	0.0116	0.2061
Case II:										
$a = 0.25$	9.2E-06	0.0104	9.2E-06	0.0104	0.0003	0.0109	5.3E-05	0.0023	0.0001	0.0065
$b = 0.75$	0.0011	0.0403	0.0014	0.0448	0.0037	0.0087	0.0017	0.0183	0.0031	0.0247
$c = 0.50$	0.0061	0.0999	0.0106	0.1436	0.0447	0.1696	0.0219	0.1484	0.0244	0.1061
Case III:										
$a = 0.25$	0.0002	0.0571	0.0002	0.0571	0.0010	0.0015	0.0003	0.0222	0.0004	0.0387
$b = 0.75$	0.0023	0.0569	0.0025	0.0593	0.0212	0.0062	0.0031	0.0248	0.0028	0.0314
$c = 0.75$	0.0108	0.0283	0.0163	0.0816	0.1865	0.1717	0.0739	0.1260	0.0434	0.0680
Case IV:										
$a = 0.25$	0.0028	0.2052	0.0028	0.2088	0.0032	0.0685	0.0023	0.0903	0.0021	0.1203
$b = 0.75$	0.0086	0.1155	0.0063	0.1012	0.0819	0.0037	0.0600	0.0283	0.0042	0.0517
$c = 1.25$	0.1831	0.2524	0.0642	0.1186	0.8057	0.1243	0.5243	0.0857	0.1663	0.0084
Case V:										
$a = 0.25$	0.0049	0.2753	0.0051	0.2816	0.0044	0.1261	0.0035	0.1383	0.0033	0.1707
$b = 0.75$	0.0106	0.1305	0.0088	0.1213	0.1069	0.0158	0.0598	0.0312	0.0106	0.0654
$c = 1.50$	0.3083	0.2942	0.1502	0.2009	1.5913	0.0729	1.4217	0.0617	0.3010	0.0646
Case VI:										
$a = 0.25$	0.0150	0.4846	0.0156	0.4967	0.0115	0.3445	0.0105	0.3287	0.0102	0.3584
$b = 0.75$	0.0192	0.1801	0.0199	0.1810	0.2697	0.0702	0.1775	0.0761	0.0491	0.1149
$c = 2.50$	1.5980	0.4557	1.6617	0.4303	4.4683	0.2055	3.4559	0.1652	2.2022	0.2517

Table (1.c): The MSE and RB for different estimates of the GWU_LL distribution with parameters $a = 0.25, b = 0.75$ and different values of c at sample size $n = 100$.

Parameters	MLE		MPS		LS		WLS		PE	
	MSE	RB	MSE	RB	MSE	RB	MSE	RB	MSE	RB
Case I:										
$a = 0.25$	1.2E-10	3.2E-05	1.2E-10	3.2E-05	4.1E-07	0.0002	8.5E-10	2.6E-05	0.0001	0.0130
$b = 0.75$	3.8E-05	0.0073	4.0E-05	0.0074	0.0003	0.0007	0.0001	0.0046	0.0017	0.0177
$c = 0.25$	0.0027	0.1787	0.0036	0.2148	0.0032	0.1375	0.0025	0.1621	0.0050	0.1424
Case II:										
$a = 0.25$	4.0E-06	0.0075	4.0E-06	0.0075	2.9E-05	0.0021	4.3E-06	0.0061	0.0001	0.0135
$b = 0.75$	0.0009	0.0365	0.0010	0.0396	0.0012	0.0170	0.0008	0.0221	0.0016	0.0260
$c = 0.50$	0.0060	0.1237	0.0073	0.1369	0.0105	0.1238	0.0094	0.1335	0.0117	0.0863
Case III:										
$a = 0.25$	0.0001	0.0467	0.0001	0.0467	0.0003	0.0168	0.0001	0.0340	0.0002	0.0439
$b = 0.75$	0.0018	0.0518	0.0018	0.0528	0.0019	0.0216	0.0012	0.0316	0.0016	0.0337
$c = 0.75$	0.0077	0.0580	0.0102	0.0828	0.0374	0.1095	0.0181	0.0955	0.0208	0.0616
Case IV:										
$a = 0.25$	0.0022	0.1883	0.0023	0.1904	0.0022	0.0839	0.0015	0.1280	0.0016	0.1447
$b = 0.75$	0.0060	0.0994	0.0050	0.0921	0.0641	0.0201	0.0032	0.0559	0.0030	0.0617
$c = 1.25$	0.0797	0.1714	0.0360	0.1079	0.7153	0.0831	0.0864	0.0069	0.0479	0.0494
Case V:										
$a = 0.25$	0.0042	0.2591	0.0044	0.2633	0.0034	0.1398	0.0028	0.1781	0.0029	0.2026
$b = 0.75$	0.0078	0.1147	0.0073	0.1122	0.0318	0.0360	0.0049	0.0698	0.0043	0.0793
$c = 1.50$	0.1494	0.2213	0.1032	0.1877	1.0190	0.0278	0.1944	0.0572	0.0922	0.1136
Case VI:										
$a = 0.25$	0.0140	0.4726	0.0145	0.4803	0.0101	0.3268	0.0099	0.3551	0.0105	0.3998
$b = 0.75$	0.0170	0.1727	0.0174	0.1748	0.2171	0.0696	0.1008	0.1101	0.0120	0.1405
$c = 2.50$	1.2121	0.4292	1.1788	0.4251	3.4488	0.1813	1.4266	0.2418	0.8973	0.3322

Table (2.a): The MSE and RB for different estimates of the GWU_LL distribution with parameters $a = 0.75, b = 1.50$ and different values of c at sample size $n = 25$.

Parameters	MLE		MPS		LS		WLS		PE	
	MSE	RB	MSE	RB	MSE	RB	MSE	RB	MSE	RB
Case I:										
$a = 0.75$	3.5E-07	0.0002	3.5E-07	0.0002	0.0017	0.0085	0.0005	0.0031	0.0007	0.0163
$b = 1.50$	0.0006	0.0113	0.0007	0.0117	0.0606	0.0213	0.0086	0.0067	0.0181	0.0028
$c = 0.25$	0.0043	0.1249	0.0070	0.2532	0.0410	0.2511	0.0257	0.2145	0.0317	0.3312
Case II:										
$a = 0.75$	0.0001	0.0091	0.0001	0.0090	0.0045	0.0248	0.0022	0.0102	0.0015	0.0057
$b = 1.50$	0.0044	0.0341	0.0064	0.0409	0.0862	0.0097	0.1618	0.0006	0.0175	0.0110
$c = 0.50$	0.0153	0.0682	0.0283	0.1658	0.1986	0.2887	0.1312	0.2307	0.0875	0.1735
Case III:										
$a = 0.75$	0.0011	0.0391	0.0011	0.0391	0.0086	0.0209	0.0063	0.0097	0.0039	0.0051
$b = 1.50$	0.0089	0.0494	0.0103	0.0556	0.4724	0.0169	0.2151	0.0006	0.0643	0.0128
$c = 0.75$	0.0318	0.0385	0.0406	0.0814	0.4259	0.2357	0.3675	0.2188	0.1974	0.1316
Case IV:										
$a = 0.75$	0.0089	0.1163	0.0091	0.1189	0.0166	0.0237	0.0151	0.0198	0.0129	0.0325
$b = 1.50$	0.2484	0.0905	0.0206	0.0873	1.3601	0.0339	0.7486	0.0157	0.0661	0.0183
$c = 1.25$	0.3595	0.3159	0.1504	0.1283	1.6371	0.1366	1.7831	0.1752	0.9366	0.0827
Case V:										
$a = 0.75$	0.0146	0.1510	0.0151	0.1564	0.0193	0.0529	0.0169	0.0500	0.0164	0.0568
$b = 1.50$	0.0321	0.1081	0.2448	0.0951	1.1477	0.0134	1.1731	0.0137	0.3987	0.0139
$c = 1.50$	0.5834	0.3600	0.2771	0.2101	1.9484	0.0433	1.6850	0.0651	2.0371	0.0537
Case VI:										
$a = 0.75$	0.0385	0.2527	0.0402	0.2604	0.0324	0.1751	0.0305	0.1597	0.0298	0.1626
$b = 1.50$	0.1615	0.1403	0.6863	0.1197	4.6992	0.0591	3.2458	0.0226	0.5422	0.0619
$c = 2.50$	2.5685	0.5155	1.6731	0.4413	4.1491	0.2665	6.1525	0.1844	3.3080	0.2275

Table (2.b): The MSE and RB for different estimates of the GWU_LL distribution with parameters $a = 0.75, b = 1.50$ and different values of c at sample size $n = 50$.

Parameters	MLE		MPS		LS		WLS		PE	
	MSE	RB	MSE	RB	MSE	RB	MSE	RB	MSE	RB
Case I:										
$a = 0.75$	3.8E-09	3.9E-05	3.8E-09	3.9E-05	0.0002	0.0016	3.0E-05	0.0002	0.0003	0.0093
$b = 1.50$	0.0001	0.0064	0.0002	0.0067	0.0030	0.0063	0.0006	0.0016	0.0077	0.0102
$c = 0.25$	0.0039	0.1824	0.0050	0.2392	0.0090	0.1681	0.0049	0.1619	0.0116	0.2060
Case II:										
$a = 0.75$	2.2E-05	0.0052	2.2E-05	0.0052	0.0014	0.0077	0.0001	0.0012	0.0003	0.0039
$b = 1.50$	0.0027	0.0283	0.0034	0.0330	0.0071	0.0060	0.0060	0.0128	0.0067	0.0175
$c = 0.50$	0.0111	0.1049	0.0132	0.1411	0.0375	0.1683	0.0206	0.1479	0.0246	0.1044
Case III:										
$a = 0.75$	0.0005	0.0282	0.0005	0.0283	0.0038	0.0053	0.0011	0.0096	0.0009	0.0188
$b = 1.50$	0.0058	0.0441	0.0061	0.0456	0.0236	0.0059	0.0086	0.0186	0.0065	0.0240
$c = 0.75$	0.0166	0.0203	0.0196	0.0768	0.2391	0.1781	0.0934	0.1310	0.0443	0.0691
Case IV:										
$a = 0.75$	0.0062	0.1024	0.0064	0.1041	0.0129	0.0210	0.0077	0.0379	0.0047	0.0600
$b = 1.50$	0.0176	0.0830	0.0142	0.0754	0.4252	0.0066	0.1570	0.0178	0.0113	0.0385
$c = 1.25$	0.1483	0.2231	0.0645	0.1151	0.9381	0.1351	0.6844	0.0994	0.2167	0.0063
Case V:										
$a = 0.75$	0.0112	0.1384	0.0117	0.1414	0.0148	0.0490	0.0119	0.0591	0.0079	0.0849
$b = 1.50$	0.0229	0.0962	0.0200	0.0911	0.7280	0.0056	0.2892	0.0242	0.0140	0.0501
$c = 1.50$	0.2794	0.2819	0.1591	0.2025	2.0083	0.0902	0.9491	0.0435	0.3248	0.0640
Case VI:										
$a = 0.75$	0.0337	0.2420	0.0351	0.2480	0.0290	0.1554	0.0267	0.1533	0.0238	0.1758
$b = 1.50$	0.1164	0.1320	0.2539	0.1300	3.1840	0.0270	1.3826	0.0303	0.1697	0.0881
$c = 2.50$	1.5653	0.4585	1.4053	0.4331	4.0954	0.1892	3.9035	0.1708	1.5986	0.2634

Table (2.c): The MSE and RB for different estimates of the GWU_LL distribution with parameters $a = 0.75, b = 1.50$ and different values of c at sample size $n = 100$.

Parameters	MLE		MPS		LS		WLS		PE	
	MSE	RB	MSE	RB	MSE	RB	MSE	RB	MSE	RB
Case I:										
$a = 0.75$	2.7E-10	1.6E-05	2.6E-10	1.6E-05	5.6E-06	0.0001	2.2E-09	1.4E-05	0.0001	0.0065
$b = 1.50$	0.0001	0.0044	0.0001	0.0047	0.0006	4.7E-06	0.0001	0.0037	0.0038	0.0136
$c = 0.25$	0.0038	0.2105	0.0041	0.2300	0.0027	0.1322	0.0024	0.1569	0.0048	0.1368
Case II:										
$a = 0.75$	9.0E-06	0.0038	9.0E-06	0.0038	0.0001	0.0010	8.8E-06	0.0031	0.0001	0.0065
$b = 1.50$	0.0020	0.0254	0.0024	0.0289	0.0028	0.0144	0.0019	0.0178	0.0038	0.0212
$c = 0.50$	0.0094	0.1263	0.0089	0.1339	0.0108	0.1189	0.0091	0.1279	0.0116	0.0811
Case III:										
$a = 0.75$	0.0003	0.0234	0.0003	0.0234	0.0007	0.0090	0.0002	0.0171	0.0005	0.0221
$b = 1.50$	0.0045	0.0408	0.0042	0.0399	0.0042	0.0167	0.0028	0.0236	0.0037	0.0253
$c = 0.75$	0.0103	0.0479	0.0111	0.0803	0.0333	0.1079	0.0180	0.0972	0.0203	0.0629
Case IV:										
$a = 0.75$	0.0050	0.0940	0.0051	0.0951	0.0083	0.0331	0.0036	0.0620	0.0035	0.0714
$b = 1.50$	0.0129	0.0731	0.0114	0.0691	0.3497	0.0095	0.0072	0.0412	0.0066	0.0457
$c = 1.25$	0.0694	0.1597	0.0369	0.1075	0.4757	0.0791	0.0922	0.0017	0.0493	0.0455
Case V:										
$a = 0.75$	0.0095	0.1296	0.0099	0.1318	0.0121	0.0554	0.0067	0.0872	0.0065	0.1009
$b = 1.50$	0.0174	0.0858	0.0165	0.0842	0.4751	0.0116	0.0124	0.0521	0.0097	0.0593
$c = 1.50$	0.1443	0.2178	0.1040	0.1874	1.0703	0.0475	0.2167	0.0547	0.0915	0.1113
Case VI:										
$a = 0.75$	0.0317	0.2368	0.0327	0.2406	0.0258	0.1553	0.0231	0.1765	0.0237	0.2007
$b = 1.50$	0.0384	0.1299	0.0393	0.1315	1.9527	0.0159	0.0878	0.0893	0.0269	0.1061
$c = 2.50$	1.2217	0.4314	1.1878	0.4272	3.5619	0.1844	1.3099	0.2513	0.8651	0.3357

Table (3.a): The MSE and RB for different estimates of the GWU_LL distribution with parameters $a = 1.25, b = 1.75$ and different values of c at sample size $n = 25$.

Parameters	MLE		MPS		LS		WLS		PE	
	MSE	RB	MSE	RB	MSE	RB	MSE	RB	MSE	RB
Case I:										
$a = 1.25$	8.8E-08	0.0001	8.8E-08	0.0001	0.0020	0.0044	0.0008	0.0019	0.0003	0.0065
$b = 1.75$	0.0003	0.0062	0.0003	0.0062	0.2187	0.0169	0.1654	0.0071	0.0080	0.0010
$c = 0.25$	0.0038	0.0851	0.0074	0.2586	0.0457	0.2501	0.0305	0.2173	0.0323	0.3319
Case II:										
$a = 1.25$	4.6E-05	0.0037	4.6E-05	0.0037	0.0045	0.0122	0.0027	0.0063	0.0007	0.0021
$b = 1.75$	0.0025	0.0227	0.0029	0.0237	0.2705	0.0111	0.0397	0.0002	0.0085	0.0067
$c = 0.50$	0.0162	0.0046	0.0272	0.1652	0.1689	0.2831	0.1338	0.2465	0.0690	0.1674
Case III:										
$a = 1.25$	0.0005	0.0153	0.0006	0.0153	0.0087	0.0135	0.0054	0.0068	0.0021	0.0016
$b = 1.75$	0.0050	0.0316	0.0047	0.0318	0.5406	0.0136	0.1639	0.0042	0.0081	0.0085
$c = 0.75$	0.0543	0.1196	0.0535	0.0822	0.4169	0.2476	0.3121	0.2220	0.1543	0.1270
Case IV:										
$a = 1.25$	0.0040	0.0461	0.0041	0.0473	0.0148	0.0010	0.0130	0.0010	0.0092	0.0096
$b = 1.75$	0.0961	0.0554	0.2394	0.0448	2.1007	0.0499	1.5353	0.0344	0.4459	0.0006
$c = 1.25$	0.3953	0.3530	0.1762	0.1264	1.2826	0.1142	1.8578	0.1764	1.1555	0.0926
Case V:										
$a = 1.25$	0.0069	0.0603	0.0071	0.0622	0.0173	0.0152	0.0151	0.0137	0.0121	0.0188
$b = 1.75$	0.0152	0.0626	0.7628	0.0387	5.0627	0.0995	3.0632	0.0595	0.7889	0.0030
$c = 1.50$	0.6958	0.3745	0.3367	0.2038	1.5441	0.0145	2.0167	0.0745	1.4180	0.0259
Case VI:										
$a = 1.25$	0.0173	0.1004	0.0178	0.1040	0.0206	0.0628	0.0197	0.0587	0.0185	0.0595
$b = 1.75$	1.3412	0.0495	1.1943	0.0483	10.2103	0.2073	10.9462	0.2001	1.8159	0.0055
$c = 2.50$	2.5058	0.5194	1.6950	0.4454	4.5360	0.2570	3.6610	0.2269	4.6065	0.2127

Table (3.b): The MSE and RB for different estimates of the GWU_LL distribution with parameters $a = 1.25, b = 1.75$ and different values of c at sample size $n = 50$.

Parameters	MLE		MPS		LS		WLS		PE	
	MSE	RB	MSE	RB	MSE	RB	MSE	RB	MSE	RB
Case I:										
$a = 1.25$	2.7E-09	1.6E-05	2.7E-09	1.6E-05	0.0006	0.0011	9.4E-06	0.0001	0.0001	0.0036
$b = 1.75$	0.0001	0.0034	0.0001	0.0035	0.0020	0.0043	0.0003	0.0005	0.0034	0.0054
$c = 0.25$	0.0033	0.1593	0.0052	0.2403	0.0122	0.1766	0.0051	0.1695	0.0118	0.2063
Case II:										
$a = 1.25$	9.4E-06	0.0021	9.4E-06	0.0021	0.0013	0.0036	0.0001	0.0004	0.0001	0.0012
$b = 1.75$	0.0014	0.0182	0.0015	0.0193	0.0044	0.0033	0.0017	0.0077	0.0030	0.0100
$c = 0.50$	0.0091	0.0666	0.0118	0.1351	0.0529	0.1749	0.0215	0.1493	0.0240	0.1096
Case III:										
$a = 1.25$	0.0002	0.0114	0.0002	0.0114	0.0029	0.0035	0.0006	0.0036	0.0004	0.0076
$b = 1.75$	0.0032	0.0278	0.0027	0.0254	0.1853	0.0006	0.0032	0.0103	0.0028	0.0131
$c = 0.75$	0.0224	0.0360	0.0206	0.0807	0.1539	0.1677	0.0824	0.1330	0.0456	0.0741
Case IV:										
$a = 1.25$	0.0028	0.0411	0.0028	0.0417	0.0119	0.0026	0.0062	0.0124	0.0022	0.0246
$b = 1.75$	0.0087	0.0500	0.0064	0.0436	1.7049	0.0308	0.0526	0.0124	0.0042	0.0235
$c = 1.25$	0.1799	0.2537	0.0629	0.1190	1.2719	0.1401	0.5642	0.0900	0.1465	0.0197
Case V:										
$a = 1.25$	0.0049	0.0547	0.0051	0.0560	0.0127	0.0137	0.0101	0.0183	0.0052	0.0312
$b = 1.75$	0.1324	0.0523	0.2102	0.0478	1.8847	0.0365	0.9182	0.0039	0.1848	0.0236
$c = 1.50$	0.3115	0.2878	0.1543	0.1943	1.2296	0.0564	1.0928	0.0564	0.4922	0.0458
Case VI:										
$a = 1.25$	0.0152	0.0968	0.0158	0.0991	0.0173	0.0594	0.0159	0.0576	0.0121	0.0694
$b = 1.75$	0.1663	0.0733	0.0863	0.0750	7.1675	0.1255	3.6700	0.0407	0.5326	0.0383
$c = 2.50$	1.5971	0.4617	1.3761	0.4340	4.0050	0.2059	4.0184	0.1685	1.7378	0.2656

Table (3.c): The MSE and RB for different estimates of the GWU_LL distribution with parameters $a = 1.25, b = 1.75$ and different values of c at sample size $n = 100$.

Parameters	MLE		MPS		LS		WLS		PE	
	MSE	RB	MSE	RB	MSE	RB	MSE	RB	MSE	RB
Case I:										
$a = 1.25$	1.4E-10	6.5E-06	1.4E-10	6.5E-06	1.2E-06	4.9E-05	2.5E-10	5.8E-06	0.0001	0.0024
$b = 1.75$	2.4E-05	0.0024	2.7E-05	0.0025	0.0003	0.0002	3.9E-05	0.0015	0.0017	0.0074
$c = 0.25$	0.0034	0.1984	0.0042	0.2332	0.0028	0.1367	0.0026	0.1707	0.0050	0.1379
Case II:										
$a = 1.25$	4.0E-06	0.0015	4.0E-06	0.0015	2.8E-05	0.0003	3.7E-06	0.0012	0.0001	0.0027
$b = 1.75$	0.0010	0.0161	0.0010	0.0169	0.0013	0.0073	0.0008	0.0096	0.0017	0.0113
$c = 0.50$	0.0070	0.1035	0.0080	0.1323	0.0110	0.1252	0.0095	0.1339	0.0121	0.0858
Case III:										
$a = 1.25$	0.0001	0.0094	0.0001	0.0094	0.0004	0.0034	0.0001	0.0068	0.0002	0.0089
$b = 1.75$	0.0023	0.0247	0.0019	0.0228	0.0022	0.0097	0.0013	0.0138	0.0017	0.0148
$c = 0.75$	0.0111	0.0233	0.0115	0.0812	0.0421	0.1071	0.0183	0.0937	0.0202	0.0587
Case IV:										
$a = 1.25$	0.0022	0.0377	0.0023	0.0381	0.0049	0.0133	0.0017	0.0256	0.0016	0.0292
$b = 1.75$	0.0061	0.0428	0.0051	0.0397	0.2920	0.0061	0.0031	0.0244	0.0030	0.0268
$c = 1.25$	0.0808	0.1735	0.0367	0.1097	0.4735	0.0625	0.0769	0.0114	0.0479	0.0529
Case V:										
$a = 1.25$	0.0042	0.0518	0.0044	0.0526	0.0082	0.0205	0.0035	0.0343	0.0029	0.0404
$b = 1.75$	0.0078	0.0493	0.0073	0.0482	0.5814	0.0015	0.1338	0.0269	0.0043	0.0340
$c = 1.50$	0.1512	0.2234	0.1042	0.1894	1.0971	0.0317	0.1955	0.0565	0.0905	0.1141
Case VI:										
$a = 1.25$	0.0141	0.0947	0.0145	0.0963	0.0160	0.0557	0.0114	0.0690	0.0105	0.0803
$b = 1.75$	0.0170	0.0741	0.0174	0.0750	4.1820	0.0605	1.1837	0.0226	0.0119	0.0606

$c = 2.50$	1.2144	0.4307	1.1803	0.4265	3.1750	0.1788	1.7677	0.2410	0.8536	0.3361

4.6.2 Real Data Analysis

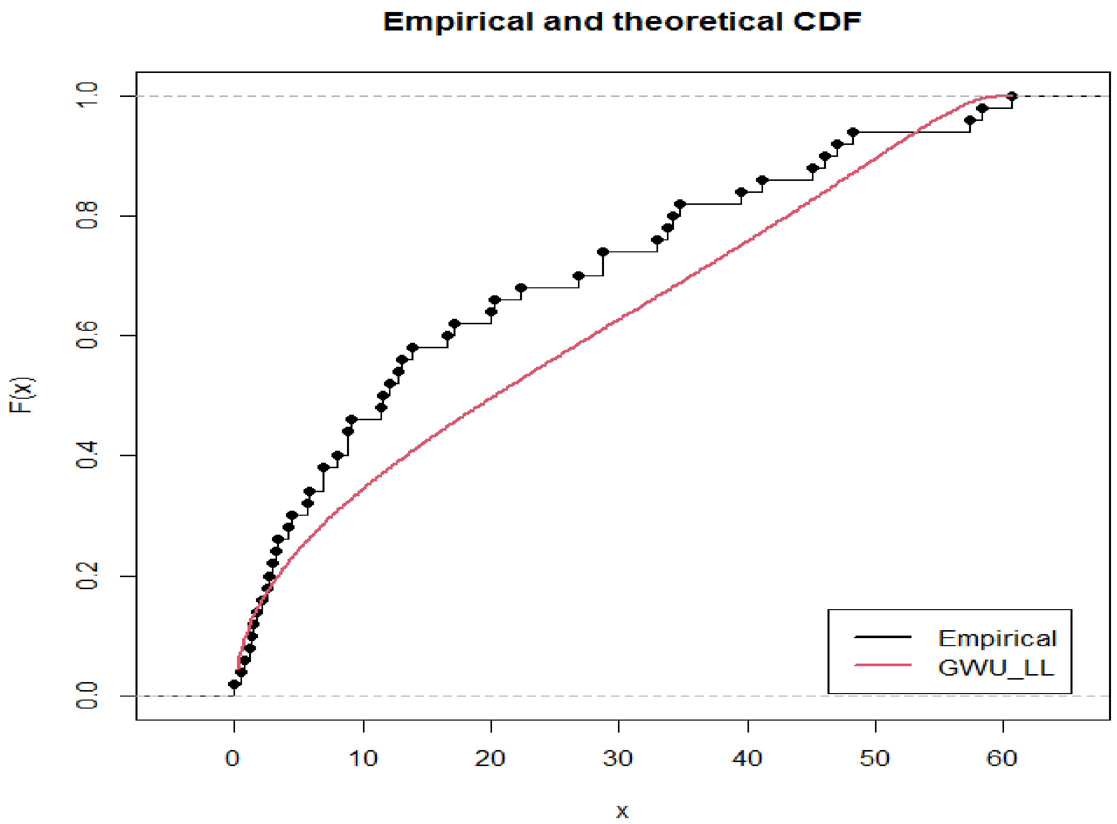
In this section, we analyze a real data set for illustrative purpose. The following data set represents leukemia-free survival times of 50 patients with Autologous transplant (Eugene, et. al (2002))[4]. The data are as follows:

0.030, 0.493, 0.855, 1.184, 1.283, 1.480, 1.776, 2.138, 2.500, 2.763, 2.993, 3.224, 3.421, 4.178, 4.441, 5.691, 5.855, 6.941, 6.941, 7.993, 8.882, 8.882, 9.145, 11.480, 11.513, 12.105, 12.796, 12.993, 13.849, 16.612, 17.138, 20.066, 20.329, 22.368, 26.776, 28.717, 28.717, 32.928, 33.783, 34.221, 34.770, 39.539, 41.118, 45.033, 46.053, 46.941, 48.289, 57.401, 58.322, 60.625.

We first check whether the GWU_LL distribution is suitable for analyzing this data set. The calculated Kolmogorov-Smirnov (K-S) distance between the empirical and the fitted extended for the GWU_LL distribution was 0.1778 and its p-value is 0.0847 where the MLE's are:

$\hat{a} = 0.0299, \quad \hat{b} = 60.625, \quad \hat{c} = 0.5304$

which indicate that this distribution can be considered as an adequate model for the given data set.



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