Progressive Type-II Censoring Scheme of Alpha Power Transformed Weibull - Rayleigh Distribution with Application in Medicine Field

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Progressive Type-II Censoring Scheme of Alpha Power Transformed Weibull - Rayleigh Distribution with Application in Medicine Field

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Abstract

In this paper, the classical and Bayesian estimation of the parameters, reliability and hazard rate functions are considered for alpha power transformed Weibull-Rayleigh distribution under progressive Type-II censored scheme. The maximum likelihood and the Bayes estimators for the unknown parameters of the alpha power transformed Weibull-Rayleigh distribution are derived. Also the 95% asymptotic confidence intervals based on maximum likelihood and highest posterior density credible intervals based on Monte Carlo Markov chain samples are constructed. A simulation study is provided to illustrate the theoretical results and an application using real data set is applied to demonstrate how the results can be used in practice.

Keywords: *Alpha power transformed Weibull-Rayleigh distribution; Progressive Type-II censoring; Maximum likelihood estimation; Bayesian estimation; Confidence intervals; credible intervals.*

1. Introduction

In life-testing and reliability studies censoring occurs when the failure times are known only for a portion of the individuals or items under study. The complete failure times may not have been observed by the experimenter, for example, individuals in a clinical trial may drop out of the study, or the study may have to be terminated early for lack of funds. In an industrial experiment, units may break accidently. There are also situations wherein the removal of items prior to failure is pre-planned in order to reduce the cost and time associated with testing. Data obtained from such experiments are called censored data.

Type-I and Type-II censoring schemes are the two most popular censoring schemes which have been used in practice. In Type-I censoring scheme, the experiment continues up to a pre-specified time T but the number of items during the experiment are random. Where, in Type-II censoring scheme the experiment is continued (i.e time varies) until a pre-specified number of failures $r < n$ occur. Unfortunately, none of these censoring schemes allows the removal of any experimental units during the experiment. For this reason, a more general censoring scheme called progressive Type-II censoring is considered. Balakrishnan and Aggarwala (2000) introduced a progressively Type-II censored sample as follows. Suppose that n experimental units are placed on a life test, and suppose that, the experimenter decides to have only m of these n units fail. At the time of the first failure, it is decided R_1 of the $n-1$ that of the surviving units are randomly removed from the life testing experiment. Continuing on, at the time of the second failure, R_2 of the $n-R_1-2$ of the surviving units are randomly removed. Finally, at the time of the m^{th}failure, all of the remaining $n-m-R_1-\cdots-R_{m-1}=R_m$surviving units are withdrawn from the experiment. The m ordered observed failure times denoted by $X_{1:m:n}, X_{2:m:n}, \dots X_{m:m:n}$ are called progressively Type-II censored order statistics of size m from a sample of size n with progressive censoring scheme$(R_1, R_2, \dots R_m)$. It is clear that $n=m+\sum_{i=1}^{m} R_i$. The special case when $R_1=R_2=\cdots, R_{m-1}=0$ so that $R_m = n - m$ is the case of Type-II censored sampling. Also when $R_1, R_2, \dots, R_m = 0$, so that $m=n$, the progressively Type-II censoring scheme reduces to the case of no censoring (complete sample).

The statistical inference for life time distributions under progressive Type-II censoring scheme has been studied by several authors see for example [Almetwally and Almongy (2018), Karakoca and Pekgor (2019), Li and Gui (2020), Salah (2020), Alshenawy *et al*. (2020), AL-Sayed *et al*. (2022) and AL-Sayed and Behairy (2023)].

Statistical distributions are important for parametric inferences and applications. In recent years, it is a common practice in the statistical distribution theory to add an extra parameter to an existing family of distribution functions. Such a technique (adding an extra parameter) is adopted to bring in more flexibility to a class of distribution functions. Besides, it can be very useful for data analysis purposes.

Many methods have been developed to generate statistical distributions in the literature. Lee *et al*. (2013) introduced a brief overview of some general methods developed prior to 1980 and give more detailed review of methods developed since 1980s. They indicated that the majority of methods developed after 1980s shifted to adding parameters to an existing distribution or combining existing distributions. Thus, they categorized as ‘methods of combination’. For more details see Lee *et al*. (2013).

Mahdavi and Kundu (2017) introduced a family for generating univariate distributions called the *alpha power transformation* (APT). They studied *alpha power exponential* (APE) distribution as a special case. Mead *et al* (2019) introduced *alpha power exponentiated Weibull* (APEW) and studied some of its structural properties. Elbatal *et al.* (2019) used a new technique of APT to propose a new class of lifetime distributions called as *new* APT (NAPT). EL-Sherpieny and Hwas (2021) introduced a generalization to APT method which called *generalized* APT (GAPT) method.

Elbatal *et al.* (2021) introduced a new extended generator called APT *Weibull-G* (APTW-G) family of distributions based on combining the APT family of distributions which is introduced by Mahdavi and Kundu (2017) with the Weibull-G family of distributions which is introduced by Bourguignon *et al.* (2014). The proposed distributions were more flexible and performed better than some existing probability distributions to model life testing data. They applied the family to a specific class of distribution such as APTW- *Rayleigh* (APTW-R) distribution. This distribution is a very new distribution in survival analysis having very small number of studies in both classical and Bayesian paradigm and no one has paid attention under progressive censoring. Therefore, in this paper, the statistical inferences of the APTW-R distribution under progressive Type-II censoring scheme will be studied.

It is observed that APTW-R distribution offers greater distributional flexibility; its *probability density function* (pdf) can be decreasing, symmetry, unimodal and skewed to the right; so it is suitable for fitting positively skewed data which may not be adequately modeled by many other distributions. Thus, it can be used to fit data related to public health, biomedical studies, industrial reliability, survival analysis and several other areas. Also, the *hazard rate function* (hrf) of APTW-R distribution can be increasing, j-shaped and U-shaped. Figure 1 displays some plots of pdf and hrf of the $DAPTW - R$ distribution for various values of the parameters.

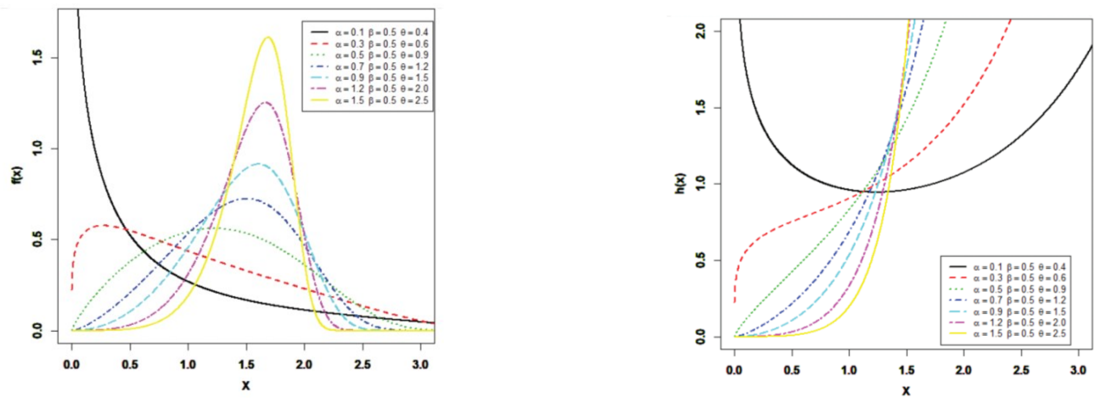

Figure1. Plots of the pdf and hrf for different values of the parameters of APTW-R distribution.

The *cumulative density function* (cdf) and pdf functions of APTW-R distribution are given, respectively, by

$$F_{APTW-R}(x;\alpha,\beta,\theta)=\begin{cases}\dfrac{\alpha^{1-e^{-\left(e^{\frac{\beta}{2}x^2}-1\right)^{\theta}}}-1}{\alpha-1}, & \alpha>0,\ \alpha\neq 1,\\[2ex] 1-e^{-\left(e^{\frac{\beta}{2}x^2}-1\right)^{\theta}}, & \alpha=1,\end{cases} \tag{1}$$

$$f_{APTW-R}(x;\alpha,\beta,\theta)=\begin{cases}\log(\alpha)\theta\beta x e^{-\frac{\beta}{2}x^2}\dfrac{\alpha^{1-e^{-\left(e^{\frac{\beta}{2}x^2}-1\right)^{\theta}}}\left(1-e^{-\frac{\beta}{2}x^2}\right)^{\theta-1}}{(\alpha-1)\left(e^{-\frac{\beta}{2}x^2}\right)^{\theta+1}}e^{-\left(e^{\frac{\beta}{2}x^2}-1\right)^{\theta}}, & \alpha>0,\ \alpha\neq 1,\\[2ex] \theta\beta x e^{-\frac{\beta}{2}x^2}\dfrac{\left(1-e^{-\frac{\beta}{2}x^2}\right)^{\theta-1}}{\left(e^{-\frac{\beta}{2}x^2}\right)^{\theta+1}}e^{-\left(e^{\frac{\beta}{2}x^2}-1\right)^{\theta}}, & \alpha=1,\end{cases} \tag{2}$$

where α,θ are shape parameters and β is a scale parameter. The *reliability function* (rf), hrf and *reversed* hrf (rhrf) of the APTW-R distribution are, respectively, given by

$$R_{APTW-R}(x;\alpha,\beta,\theta)=\begin{cases}\dfrac{\alpha-\alpha^{1-e^{-\left(e^{\frac{\beta}{2}x^2}-1\right)^{\theta}}}}{\alpha-1}, & \alpha>0,\ \alpha\neq 1,\\[2ex] e^{-\left(e^{\frac{\beta}{2}x^2}-1\right)^{\theta}}, & \alpha=1,\end{cases} \tag{3}$$

$$h_{APTW-R}(x;\alpha,\beta,\theta)=\begin{cases}\log(\alpha)\theta\beta x e^{-\frac{\beta}{2}x^2}\dfrac{\left(1-e^{-\frac{\beta}{2}x^2}\right)^{\theta-1}}{\left(\alpha-\alpha^{1-e^{-\left(e^{\frac{\beta}{2}x^2}-1\right)^{\theta}}}\right)\left(e^{-\frac{\beta}{2}x^2}\right)^{\theta+1}}e^{-\left(e^{\frac{\beta}{2}x^2}-1\right)^{\theta}}\alpha^{1-e^{-\left(e^{\frac{\beta}{2}x^2}-1\right)^{\theta}}}, & \alpha>0,\ \alpha\neq 1,\\[2ex] \theta\beta x e^{-\frac{\beta}{2}x^2}\dfrac{\left(1-e^{-\frac{\beta}{2}x^2}\right)^{\theta-1}}{\left(e^{-\frac{\beta}{2}x^2}\right)^{\theta+1}}, & \alpha=1\end{cases} \tag{4}$$

and

$$rh_{APTW-R}(x;\alpha,\beta,\theta) = \begin{cases} \log(\alpha)\theta\beta x e^{-\frac{\beta}{2}x^2} \dfrac{\alpha^{1-e^{-\left(e^{\frac{\beta}{2}x^2}-1\right)^{\theta}}}\left(1-e^{-\frac{\beta}{2}x^2}\right)^{\theta-1}}{\left(\alpha^{1-e^{-\left(e^{\frac{\beta}{2}x^2}-1\right)^{\theta}}}-1\right)\left(e^{-\frac{\beta}{2}x^2}\right)^{\theta+1}} e^{-\left(e^{\frac{\beta}{2}x^2}-1\right)^{\theta}}, & \alpha>0,\ \alpha\neq 1, \\ \theta\beta x e^{-\frac{\beta}{2}x^2} \dfrac{\left(1-e^{-\frac{\beta}{2}x^2}\right)^{\theta-1}}{\left(1-e^{-\left(e^{\frac{\beta}{2}x^2}-1\right)^{\theta}}\right)\left(e^{-\frac{\beta}{2}x^2}\right)^{\theta+1}} e^{-\left(e^{\frac{\beta}{2}x^2}-1\right)^{\theta}}, & \alpha=1. \end{cases} \quad (5)$$

The rest of this paper is organized as follows. In Section 2, the *maximum likelihood* (ML) estimators of the unknown parameters and approximate confidence intervals of the parameters are presented based on progressive Type-II censoring scheme. In Section 3, the Bayes estimators of the parameters under the *squared error* (SE) and *linear exponential* (LINEX) loss functions are derived and construction of credible intervals using the *Monte Carlo Markov chain* (MCMC) techniques are obtained. In Section 4, for illustrative purposes, a numerical example is given to illustrate the theoretical results and a real data set is analyzed to demonstrate how the results can be used in practice. Finally, conclusions appear in Section 5.

2. Maximum Likelihood Estimation under Progressive Type-II Censoring Scheme

This section is devoted to reflect the classical estimation of the model parameters. Using ML principle of estimation. Let n units are put on a test with corresponding lifetimes being identically distributed with cdf (1) and pdf (2). If $X_{1:m:n}, X_{2:m:n}, \ldots, X_{m:m:n}$ are denoted as progressively Type-II ordered censored samples of size m observed from (2). Then, the likelihood function based on all m progressively Type-II censored sample is,

$$L\left(\underline{\varphi};\underline{x}\right) = C(n,m-1)\prod_{i=1}^{m} f\left(x_i;\ \underline{\varphi}\right)\left[1-F\left(x_i;\ \underline{\varphi}\right)\right]^{R_i}, \quad (6)$$

where x_i is used instead of $x_{i:m:n}$, $\underline{\varphi} = (\alpha,\beta,\theta)$ and

$C(n,m-1) = n(n-R_1-1)(n-R_1-R_2-2)$

$\times (n-R_1-\cdots-R_{m-1}-m+1)$ with $C(n,0)=n$.

Then substituting (1) and (2) in (6) yields

$$L\left(\underline{\varphi}\,;\underline{x}\right)=C(n,m-1)\ \times\prod_{i=1}^{m}\log\,(\alpha)\theta\beta x_i e^{-\frac{\beta}{2}x_i^2}$$

$$\times\frac{\left(1-e^{-\frac{\beta}{2}x_i^2}\right)^{\theta-1}}{(\alpha-1)\left(e^{-\frac{\beta}{2}x_i^2}\right)^{\theta+1}}e^{-\left(e^{\frac{\beta}{2}x_i{}^2}-1\right)^{\theta}}\alpha^{1-e^{-\left(e^{\frac{\beta}{2}x_i^2}-1\right)^{\theta}}}$$

$$\times\left[1-\frac{\alpha^{1-e^{-\left(e^{\frac{\beta}{2}x_i^2}-1\right)^{\theta}}}-1}{\alpha-1}\right]^{R_i}$$

$$L\left(\underline{\varphi}\,;\underline{x}\right)=C(n,m-1)\ \times\prod_{i=1}^{m}\log\,(\alpha)\theta\beta x_i e^{-\frac{\beta}{2}x_i^2}$$

$$\times\frac{\left(1-e^{-\frac{\beta}{2}x_i^2}\right)^{\theta-1}}{(\alpha-1)\left(e^{-\frac{\beta}{2}x_i^2}\right)^{\theta+1}}e^{-\left(e^{\frac{\beta}{2}x_i^2}-1\right)^{\theta}}\alpha^{1-e^{-\left(e^{\frac{\beta}{2}x_i^2}-1\right)^{\theta}}}$$

$$\times\left[\frac{\alpha-\alpha^{1-e^{-\left(e^{\frac{\beta}{2}x_i^2}-1\right)^{\theta}}}}{\alpha-1}\right]^{R_i}. \tag{7}$$

Therefore, the log-likelihood function may be written as

$$\ell\propto m\,log\,[\log(\alpha)]+m\,log(\theta)+m\,log(\beta)$$
$$+\sum_{i=1}^{m}\log(x_i)-\frac{\beta}{2}\sum_{i=1}^{m}x_i^2+(\theta-1)$$
$$\times\sum_{i=1}^{m}log\left(1-e^{-\frac{\beta}{2}x_i^2}\right)-\sum_{i=1}^{m}(1+R_i)\ log\,(\alpha-1)$$
$$+(\theta+1)\sum_{i=1}^{m}\left(\frac{\beta}{2}x_i^2\right)-\sum_{i=1}^{m}\left(e^{\frac{\beta}{2}x_i^2}-1\right)^{\theta}+\sum_{i=1}^{m}\left(1-e^{-\left(e^{\frac{\beta}{2}x_i^2}-1\right)^{\theta}}\right)$$
$$\times\log(\alpha)+\sum_{i=1}^{m}(R_i)log\left[\alpha-\alpha^{1-e^{-\left(e^{\frac{\beta}{2}x_i^2}-1\right)^{\theta}}}\right]. \tag{8}$$

ML estimators can be obtained by differentiating (8) with respect toα, β and θ and then setting to zeros. Hence

$$\frac{\partial \ell}{\partial \alpha}=\frac{m}{\alpha \log(\alpha)}-\frac{\sum_{i=1}^{m}(1+R_i)}{\alpha-1}+\frac{\sum_{i=1}^{m}\left(1-e^{-\left(e^{\frac{\beta}{2}x_i^2}-1\right)^{\theta}}\right)}{\alpha}$$
$$+\frac{\sum_{i=1}^{m}(R_i)\left(1-\left(1-e^{-\left(e^{\frac{\beta}{2}x_i^2}-1\right)^{\theta}}\right)\alpha^{-e^{-\left(e^{\frac{\beta}{2}x_i^2}-1\right)^{\theta}}}\right)}{\alpha-\alpha^{1-e^{-\left(e^{\frac{\beta}{2}x_i^2}-1\right)^{\theta}}}},$$

(9)

$$\frac{\partial \ell}{\partial \beta}=\frac{m}{\beta}-\frac{1}{2}\sum_{i=1}^{m}x_i^2+(\theta-1)\sum_{i=1}^{m}\frac{-e^{-\frac{\beta}{2}{x_i}^2}\left(-\frac{x_i^2}{2}\right)}{\left(1-e^{-\frac{\beta}{2}x_i^2}\right)}$$
$$+(\theta+1)\sum_{i=1}^{m}\left(\frac{x_i^2}{2}\right)-\theta\sum_{i=1}^{m}\left(e^{\frac{\beta}{2}x_i^2}-1\right)^{\theta-1}\left(\frac{{x_i}^2}{2}e^{\frac{\beta}{2}x_i^2}\right)$$
$$+\sum_{i=1}^{m}\left(-e^{-\left(e^{\frac{\beta}{2}x_i^2}-1\right)^{\theta}}\right)\left(-\theta\left(e^{\frac{\beta}{2}x_i^2}-1\right)^{\theta-1}\left(\frac{{x_i}^2}{2}e^{\frac{\beta}{2}x_i^2}\right)\right)\log(\alpha)$$
$$+\sum_{i=1}^{m}(R_i)\left[\frac{\alpha^{1-e^{-\left(e^{\frac{\beta}{2}x_i^2}-1\right)^{\theta}}}(\log(\alpha))\left(-e^{-\left(e^{\frac{\beta}{2}x_i^2}-1\right)^{\theta}}\right)\left(-\theta\left(e^{\frac{\beta}{2}x_i^2}-1\right)^{\theta-1}\left(\frac{{x_i}^2}{2}e^{\frac{\beta}{2}x_i^2}\right)\right)}{\alpha-\alpha^{1-e^{-\left(e^{\frac{\beta}{2}x_i^2}-1\right)^{\theta}}}}\right]$$

(10)

and

$$\frac{\partial \ell}{\partial \theta}=\frac{m}{\theta}+\sum_{i=1}^{m}log\left(1-e^{-\frac{\beta}{2}x_i^2}\right)+\sum_{i=1}^{m}\left(\frac{\beta}{2}x_i^2\right)$$
$$-\sum_{i=1}^{m}\left(e^{\frac{\beta}{2}x_i^2}-1\right)^{\theta}\log\left(e^{\frac{\beta}{2}x_i^2}-1\right)$$
$$+\sum_{i=1}^{m}\left(-e^{-\left(e^{\frac{\beta}{2}x_i^2}-1\right)^{\theta}}\right)\log(\alpha)\left(-\left(e^{\frac{\beta}{2}x_i^2}-1\right)^{\theta}\right).$$

(11)

The ML estimators can be derived by setting the partial first derivatives of (9) – (11) to zeros after replacing α, β and θ by $\hat{\alpha}$, $\hat{\beta}$ and $\hat{\theta}$. Since the system of the non-linear equations cannot be solved analytically. Therefore, numerical methods such as Newton-Raphson method should be employed to solve them and get the ML estimators.

By using the invariance property, the ML estimators of rf and hrf are given by replacing α, β and θ with their corresponding ML estimators; $\hat{\alpha}, \hat{\beta}$ and $\hat{\theta}$ respectively, in (3) and (4) s follows:

$$\hat{R}_{APTW-R}(x_0) = \begin{cases} \dfrac{\hat{\alpha} - \hat{\alpha}^{1-e^{-\left(e^{\frac{\hat{\beta}}{2}x_0^2}-1\right)^{\hat{\theta}}}}}{\hat{\alpha}-1}, & \alpha > 0,\ \alpha \neq 1, \\ e^{-\left(e^{\frac{\hat{\beta}}{2}x_0^2}-1\right)^{\hat{\theta}}}, & \alpha = 1 \end{cases} \tag{12}$$

and

$$\hat{h}_{APTW-R}(x_0) = \begin{cases} \log(\hat{\alpha})\hat{\theta}\hat{\beta}x_0 e^{-\frac{\hat{\beta}}{2}x^2} \dfrac{\hat{\alpha}^{1-e^{-\left(e^{\frac{\hat{\beta}}{2}x_0^2}-1\right)^{\hat{\theta}}}}\left(1-e^{-\frac{\hat{\beta}}{2}x_0^2}\right)^{\hat{\theta}-1}}{\left(\hat{\alpha}-\hat{\alpha}^{1-e^{-\left(e^{\frac{\hat{\beta}}{2}x_0^2}-1\right)^{\hat{\theta}}}}\right)\left(e^{-\frac{\hat{\beta}}{2}x_0^2}\right)^{\hat{\theta}+1}} e^{-\left(e^{\frac{\hat{\beta}}{2}x_0^2}-1\right)^{\hat{\theta}}}, & \alpha > 0,\ \alpha \neq 1, \\ \hat{\theta}\hat{\beta}x_0 e^{-\frac{\hat{\beta}}{2}x_0^2} \dfrac{\left(1-e^{-\frac{\hat{\beta}}{2}x_0^2}\right)^{\hat{\theta}-1}}{\left(e^{-\frac{\hat{\beta}}{2}x_0^2}\right)^{\hat{\theta}+1}}, & \alpha = 1. \end{cases} \tag{13}$$

The Asymptotic confidence intervals

The asymptotic variance-covariance matrix for the estimators $\hat{\alpha}, \hat{\beta}$ and $\hat{\theta}$ can be obtained by inverting the Fisher information matrix with elements, which is the negative expected values of second derivatives of the log-likelihood function as follows

$$I = \left[E - \frac{\partial^2 \ell}{\partial \varphi_k \varphi_j}\right] \qquad k, j = 1,2,3,$$

where $\varphi_1 = \alpha, \varphi_2 = \beta\ and\ \varphi_3 = \theta$.

Practically the exact expressions for these expectations are difficult, so the above matrix will be estimated by the observed Fisher information matrix which can be obtained by dropping the expectation operator E [See Cohen (1965)]

$$\hat{I}(\hat{\varphi}) = \left[\frac{\partial^2 \ell}{\partial \varphi_k \partial \varphi_j}\right]_{(\varphi_k=\hat{\varphi}_k,\varphi_j=\hat{\varphi}_j\ )}, \qquad k,j = 1,2,3.$$

The asymptotic normality distribution of the ML estimators can be used to compute the asymptotic confidence intervals for the parameters α, β andθ. Under regularity conditions, the ML estimators $(\hat{\alpha}, \hat{\beta}, \hat{\theta})$ have approximately multivariate normal with mean $(\hat{\alpha}, \hat{\beta}, \hat{\theta})$ and variance–covariance matrix$\hat{I}^{-1}(\hat{\alpha}, \hat{\beta}, \hat{\theta})$, where $\hat{I}(\hat{\alpha}, \hat{\beta}, \hat{\theta})$ is the observed Fishers information matrix and is defined as

$$\hat{I}(\hat{\varphi}) \approx \begin{bmatrix} -\left(\frac{\partial^2 \ell}{\partial \alpha^2}\right) & -\left(\frac{\partial^2 \ell}{\partial \alpha \partial \beta}\right) & -\left(\frac{\partial^2 \ell}{\partial \alpha \partial \theta}\right) \\ -\left(\frac{\partial^2 \ell}{\partial \alpha \partial \beta}\right) & -\left(\frac{\partial^2 \ell}{\partial \beta^2}\right) & -\left(\frac{\partial^2 \ell}{\partial \beta \partial \theta}\right) \\ -\left(\frac{\partial^2 \ell}{\partial \alpha \partial \theta}\right) & -\left(\frac{\partial^2 \ell}{\partial \beta \partial \theta}\right) & -\left(\frac{\partial^2 \ell}{\partial \theta^2}\right) \end{bmatrix}_{(\alpha=\hat{\alpha},\beta=\hat{\beta},\theta=\hat{\theta})}. \tag{14}$$

The diagonal elements of $\hat{I}^{-1}(\hat{\varphi})$ are the asymptotic variances for $\hat{\alpha}, \hat{\beta}$ and $\hat{\theta}$ respectively.

Therefore, the two sided approximate $100(1-\tau)\%$ confidence intervals for$\underline{\varphi}$, are:

$$L_{\varphi_k} = \hat{\varphi}_k - z_{\frac{\tau}{2}} \sigma_{\hat{\varphi}_k} \quad \text{and } U_{\varphi_k} = \underline{\hat{\varphi}} - z_{\frac{\tau}{2}} \sigma_{\hat{\varphi}_k}, \quad k = 1,2,3, \tag{15}$$

where L_{φ_k} and $U_{\varphi_k}$are the lower and upper bounds respectively, $\sigma_{\hat{\varphi}_k}$ is the standard deviation, $z_{\frac{\tau}{2}}$ is the percentile of the standard normal distribution with right-tail probability $\frac{\tau}{2}$ and $\hat{\varphi}_k$ is, respectively, $\hat{\alpha}, \hat{\beta}$ or $\hat{\theta}$.

3. Bayesian Estimation under Progressive Type-II Censoring Scheme

In this section, Based on progressive Type-II censored sample Bayesian estimation under SE and LINEX loss functions is considered to obtain the Bayes estimators of the parameters(α, β, θ), rf and hrf and the corresponding credible intervals. Assuming independent gamma prior distribution for the parameters (α, β, θ) with pdf as

$$\pi(\varphi_k; a_k, b_k) = \frac{{b_k}^{a_k}}{\Gamma a_k} {\varphi_k}^{a_k-1} exp(-b_k\varphi_k),$$

$$\varphi_k > 0,\ \varphi_1 \neq 1, (a_k, b_k) > 0,\ k = 1,2,3, \quad (16)$$

where $\varphi_1 = \alpha, \varphi_2 = \beta, \varphi_3 = \theta,\ a_k$ and $b_k$are the hyper parameters of the prior distribution assumed to be known and assumed to be chosen in which have minimal or no effect on posterior distribution.

Therefore, the joint prior distribution of $\underline{\varphi} = (\alpha, \beta, \theta)$ can be expressed as follows

$$\pi\left(\underline{\varphi}; \underline{a}, \underline{b}\right) = \alpha^{a_1-1}\, \beta^{a_2-1}\, \theta^{a_3-1} exp[-(b_1\alpha + b_2\beta + \quad b_3\theta)],$$

$$\underline{\varphi} > 0,\ \varphi_1 \neq 1, (\underline{a}, \underline{b}) > \underline{0}. \quad (17)$$

Combining the likelihood function in (7) and the joint prior distribution given by (17), then the joint posterior distribution of the parameters, $\underline{\varphi} = (\alpha, \beta, \theta)$ can be written as,

$$\pi\left(\underline{\varphi}|\underline{x}\right) = A\, L\left(\underline{\varphi}|\underline{x}\right) \pi\left(\underline{\varphi}; \underline{a}, \underline{b}\right)$$

$$= A\, \alpha^{a_1-1}\, \beta^{m+a_2-1}\, \theta^{m+a_3-1} exp[-(b_1\alpha + b_2\beta + b_3\theta)]$$

$$\times \prod_{i=1}^{m} \log(\alpha) x_i e^{-\frac{\beta}{2}x_i^2} \frac{\left(1-e^{-\frac{\beta}{2}x_i^2}\right)^{\theta-1}}{(\alpha-1)\left(e^{-\frac{\beta}{2}x_i^2}\right)^{\theta+1}} e^{-\left(e^{\frac{\beta}{2}x_i^2}-1\right)^{\theta}} \alpha^{1-e^{-\left(e^{\frac{\beta}{2}x_i^2}-1\right)^{\theta}}}$$

$$\times \left[\frac{\alpha-\alpha^{1-e^{-\left(e^{\frac{\beta}{2}x_i^2}-1\right)^{\theta}}}}{\alpha-1}\right]^{R_i}, \quad (18)$$

where A is a normalizing constant,

$$A^{-1} = \int_{\underline{\varphi}} L\left(\underline{\varphi}|\underline{x}\right) \pi\left(\underline{\varphi}; \underline{a}, \underline{b}\right)\, d\underline{\varphi}$$

$$= \int_{\underline{\varphi}} \alpha^{a_1-1}\ \beta^{m+a_2-1}\, \theta^{m+a_3-1} exp[-(b_1\alpha + b_2\beta + b_3\theta)]$$

$$\times \prod_{i=1}^{m} \log(\alpha) x_i e^{-\frac{\beta}{2}{x_i}^2} \frac{\left(1-e^{-\frac{\beta}{2}x_i^2}\right)^{\theta-1}}{(\alpha-1)\left(e^{-\frac{\beta}{2}x_i^2}\right)^{\theta+1}} e^{-\left(e^{\frac{\beta}{2}x_i^2}-1\right)^{\theta}} \alpha^{1-e^{-\left(e^{\frac{\beta}{2}x_i^2}-1\right)^{\theta}}}$$

$$\times\left[\frac{\alpha-\alpha^{1-e^{-\left(e^{\frac{\beta}{2}x_i^2}-1\right)^{\theta}}}}{\alpha-1}\right]^{R_i} d\underline{\varphi}, \tag{19}$$

where $\int_{\underline{\varphi}}=\int_{\alpha}\int_{\beta}\int_{\theta}$ and $d\underline{\varphi}=d\alpha\, d\beta\, d\theta.$ (20)

The marginal posterior distributions of the parameters $\underline{\varphi}$ can be expressed as follows:

$$\pi(\varphi_k|\underline{x})=\int_{\underline{\varphi}_j}\pi\left(\underline{\varphi}|\underline{x}\right)d\underline{\varphi}_j, \quad k\neq j,\ \ k,j=1,2,3. \tag{21}$$

3.1 Bayes estimators of the parameters, reliability and hazard rate functions under squared error loss function

In this subsection the Bayes estimators of the parameters $\underline{\varphi}=(\alpha,\beta,\theta)$, rf and hrf are obtained under SE loss function. The Bayes estimators of each α,β and θ under the SE loss function are the means of their marginal posteriors and are defined by

$$\tilde{\varphi}_{k(SE)}=E\left(\varphi_k|\underline{x}\right)=\int_{\underline{\varphi}}\varphi_k\,\pi\left(\underline{\varphi}|\underline{x}\right)d\underline{\varphi}$$

$$=\int_{\underline{\varphi}}\varphi_k\,A\,\alpha^{a_1-1}\ \beta^{m+a_2-1}\ \theta^{m+a_3-1}exp[-(b_1\alpha+b_2\beta+b_3\theta)]$$

$$\times\prod_{i=1}^{m}\log(\alpha)x_i e^{-\frac{\beta}{2}x_i^2}\frac{\left(1-e^{-\frac{\beta}{2}x_i^2}\right)^{\theta-1}}{(\alpha-1)\left(e^{-\frac{\beta}{2}x_i^2}\right)^{\theta+1}}\,e^{-\left(e^{\frac{\beta}{2}x_i^2}-1\right)^{\theta}}\alpha^{1-e^{-\left(e^{\frac{\beta}{2}x_i^2}-1\right)^{\theta}}}$$

$$\times\left[\frac{\alpha-\alpha^{1-e^{-\left(e^{\frac{\beta}{2}x_i^2}-1\right)^{\theta}}}}{\alpha-1}\right]^{R_i} d\underline{\varphi}, \quad (\varphi_1,\varphi_2,\varphi_3)>0, \varphi_1\neq 1, k=1,2,3. \tag{22}$$

Also, the Bayes estimator of $\lambda(\alpha,\beta,\theta)$ (any function of parameters α,β,θ) under the SE loss function, can be derived as

$$\tilde{\lambda}_{(SE)}(\alpha,\beta,\theta)=\int_{\underline{\varphi}}g(\alpha,\beta,\theta)\,\pi\left(\alpha,\beta,\theta|\underline{x}\right)d\underline{\varphi}.$$

Therefore, the Bayes estimators of the rf and hrf under SE loss function can be obtained using (3), (4) and (18) as follows:

$$\tilde{R}_{(SE)} = E\left(R(x_0)\middle|\underline{x}\right)$$

$$= \int_{\underline{\varphi}} \frac{\alpha-\alpha^{1-e^{-\left(e^{\frac{\beta}{2}x_0^2}-1\right)^{\theta}}}}{\alpha-1} A\,\alpha^{a_1-1}\ \beta^{m+a_2-1}\ \theta^{m+a_3-1}$$

$$\times exp[-(b_1\alpha+b_2\beta+b_3\theta)]$$

$$\times \prod_{i=1}^{m} \log(\alpha) x_i e^{-\frac{\beta}{2}x_i^2} \frac{\left(1-e^{-\frac{\beta}{2}x_i^2}\right)^{\theta-1}}{(\alpha-1)\left(e^{-\frac{\beta}{2}x_i^2}\right)^{\theta+1}}\ e^{-\left(e^{\frac{\beta}{2}x_i^2}-1\right)^{\theta}}\ \alpha^{1-e^{-\left(e^{\frac{\beta}{2}x_i^2}-1\right)^{\theta}}}$$

$$\times \left[\frac{\alpha-\alpha^{1-e^{-\left(e^{\frac{\beta}{2}x_i^2}-1\right)^{\theta}}}}{\alpha-1}\right]^{R_i} d\underline{\varphi} \tag{23}$$

and

$$\tilde{h}_{(SE)} = E\left(h(x_0)\middle|\underline{x}\right)$$

$$= \int_{\underline{\varphi}} \log(\alpha)\,\theta\beta x_0 e^{-\frac{\beta}{2}x_0^2} \frac{\left(1-e^{-\frac{\beta}{2}x_0^2}\right)^{\theta-1} e^{-\left(e^{\frac{\beta}{2}x_0^2}-1\right)^{\theta}}}{\left(\alpha-\alpha^{1-e^{-\left(e^{\frac{\beta}{2}x_0^2}-1\right)^{\theta}}}\right)\left(e^{-\frac{\beta}{2}x_0^2}\right)^{\theta+1}}$$

$$\times \alpha^{1-e^{-\left(e^{\frac{\beta}{2}x_0^2}-1\right)^{\theta}}}\ A\,\alpha^{a_1-1}\ \beta^{m+a_2-1}\ \theta^{m+a_3-1}$$

$$\times exp[-(b_1\alpha+b_2\beta+b_3\theta)]$$

$$\times$$

$$\prod_{i=1}^{m} \log(\alpha) x_i e^{-\frac{\beta}{2}x_i^2} \frac{\left(1-e^{-\frac{\beta}{2}x_i^2}\right)^{\theta-1}}{(\alpha-1)\left(e^{-\frac{\beta}{2}x_i^2}\right)^{\theta+1}}\ e^{-\left(e^{\frac{\beta}{2}x_i^2}-1\right)^{\theta}}\ \alpha^{1-e^{-\left(e^{\frac{\beta}{2}x_i^2}-1\right)^{\theta}}}$$

$$\times \left[\frac{\alpha-\alpha^{1-e^{-\left(e^{\frac{\beta}{2}x_i^2}-1\right)^{\theta}}}}{\alpha-1}\right]^{R_i} d\underline{\varphi}. \qquad (24)$$

3.2 Bayes estimators of the parameters, reliability and hazard rate functions under linear exponential loss function

The Bayes estimators for the parametersα, β and θ under the LINEX loss function are given, respectively, by

$$\tilde{\varphi}_{k(LINEX)} = \left(\frac{-1}{v}\right) ln\{E(e^{-v\varphi_k}|\underline{x})\}$$

$$= \left(\frac{-1}{v}\right) ln\left\{\int_{\underline{\varphi}} e^{-v\varphi_k}\,\pi\left(\underline{\varphi}|\underline{x}\right) d\underline{\varphi}\right\}$$

$$= \left(\frac{-1}{v}\right) ln\left\{\int_{\underline{\varphi}} e^{-v\varphi_k} A\,\alpha^{a_1-1}\ \beta^{m+a_2-1}\ \theta^{m+a_3-1}\right.$$

$$\times exp[-(b_1\alpha + b_2\beta + b_3\theta)] \times$$

$$\prod_{i=1}^{m} \log(\alpha) x_i e^{-\frac{\beta}{2}x_i^2} \frac{\left(1-e^{-\frac{\beta}{2}x_i^2}\right)^{\theta-1}}{(\alpha-1)\left(e^{-\frac{\beta}{2}x_i^2}\right)^{\theta+1}}$$

$$\left.\times\ e^{-\left(e^{\frac{\beta}{2}x_i^2}-1\right)^{\theta}} \alpha^{1-e^{-\left(e^{\frac{\beta}{2}x_i^2}-1\right)^{\theta}}} \times \left[\frac{\alpha-\alpha^{1-e^{-\left(e^{\frac{\beta}{2}x_i^2}-1\right)^{\theta}}}}{\alpha-1}\right]^{R_i} d\underline{\varphi}\right\},$$

$$\varphi_1, \varphi_2, \varphi_3 > 0, \varphi_1 \neq 1\,, k = 1,2,3, \qquad (25)$$

where v is the loss parameter and $v \neq 0.$ If $v > 0$, then overestimation is more serious than under estimation and vice versa.

Similarly, the Bayes estimators of the rf and hrf under LINEX loss function can be obtained using (3), (4) and (18) as follows:

$$\tilde{R}_{(LINEX)}(x_0) = \left(\frac{-1}{v}\right) ln\{E(e^{-vr(x_0)}|\underline{x})\}$$

$$= \left(\frac{-1}{v}\right) ln\left[\int_{\underline{\varphi}} e^{-v\frac{\alpha-\alpha^{1-e^{-\left(e^{\frac{\beta}{2}x_0^2}-1\right)^{\theta}}}}{\alpha-1}} A\,\alpha^{a_1-1}\right.$$

$$\times exp[-(b_1\alpha + b_2\beta + b_3\theta)]$$

$$\times \prod_{i=1}^{m} \log(\alpha) x_i e^{-\frac{\beta}{2}x_i^2} \frac{\left(1-e^{-\frac{\beta}{2}x_i^2}\right)^{\theta-1}}{(\alpha-1)\left(e^{-\frac{\beta}{2}x_i^2}\right)^{\theta+1}} e^{-\left(e^{\frac{\beta}{2}x_i^2}-1\right)^{\theta}}$$

$$\times \alpha^{1-e^{-\left(e^{\frac{\beta}{2}x_i^2}-1\right)^{\theta}}} \left[\left[\frac{\alpha-\alpha^{1-e^{-\left(e^{\frac{\beta}{2}x_i^2}-1\right)^{\theta}}}}{\alpha-1}\right]^{R_i}\right] d\underline{\varphi}$$

(26)

and

$$\tilde{h}_{(LINEX)}(x_0) = \left(\frac{-1}{v}\right) ln\{E(e^{-vh(x_0)}|\underline{x})\}$$

$$= \left(\frac{-1}{v}\right) ln\left[\int_{\underline{\varphi}} exp\left[-v * \log(\alpha)\, \theta\beta x_0 e^{-\frac{\beta}{2}x_0^2} \frac{\left(1-e^{-\frac{\beta}{2}x_0^2}\right)^{\theta-1} e^{-\left(e^{\frac{\beta}{2}x_0^2}-1\right)^{\theta}}}{\left(\alpha-\alpha^{1-e^{-\left(e^{\frac{\beta}{2}x_0^2}-1\right)^{\theta}}}\right)\left(e^{-\frac{\beta}{2}x_0^2}\right)^{\theta+1}} \alpha^{1-e^{-\left(e^{\frac{\beta}{2}x_0^2}-1\right)^{\theta}}}\right]\right.$$

$$\times A\, \alpha^{a_1-1}\ \beta^{m+a_2-1}\ \theta^{m+a_3-1} exp[-(b_1\alpha + b_2\beta + b_3\theta)]$$

$$\times \prod_{i=1}^{m} \log(\alpha) x_i e^{-\frac{\beta}{2}x_i^2} \frac{\left(1-e^{-\frac{\beta}{2}x_i^2}\right)^{\theta-1}}{(\alpha-1)\left(e^{-\frac{\beta}{2}x_i^2}\right)^{\theta+1}} e^{-\left(e^{\frac{\beta}{2}x_i^2}-1\right)^{\theta}}$$

$$\times \alpha^{1-e^{-\left(e^{\frac{\beta}{2}x_i^2}-1\right)^{\theta}}} \left[\left[\frac{\alpha-\alpha^{1-e^{-\left(e^{\frac{\beta}{2}x_i^2}-1\right)^{\theta}}}}{\alpha-1}\right]^{R_i}\right] d\underline{\varphi}. \quad (27)$$

Such expectations cannot be obtained in explicit form because of the intractable integrals; therefore, numerical methods such as MCMC method can be used. MCMC technique is a Monte Carlo integration method which draws samples from the posterior distribution. Metropolis-Hastings and Gibbs sampling techniques are the most widely used MCMC samplers. For more Details See Hastings (1970), Smith and Roberts (1993) and Green *et al* (1994).

The Bayes estimates under SE and LINEX loss functions of the parameters, rf and hrf can be obtained by the generated samples from the posterior densities by using Gibbs sampling and then used Metropolis Algorithms in following steps

Start with J=1 and initial values$(\alpha^0, \beta^0, \theta^0)$.
Generate posterior sample for α, β and θ from (18).

Repeat step 2, for all $J = 1,2,3, \dots N$ and obtained$(\alpha_1, \beta_1, \theta_1), (\alpha_2, \beta_2, \theta_2), \dots, (\alpha_N, \beta_N, \theta_N)$, where N is the *number of replications* (NRs).

After obtaining the posterior samples, the Bayes estimates under SE loss function are given by

$$\tilde{\varphi}_{k(SE)} \approx \frac{1}{N}\sum_{J=1}^{N}(\varphi_k)_J, \qquad \tilde{R}_{(SE)} \approx \frac{1}{N}\sum_{J=1}^{N}\frac{\alpha_J - \alpha_J^{1-e^{-\left(e^{\frac{\beta_J}{2}x_0^2}-1\right)^{\theta_J}}}}{\alpha_J - 1}$$

and

$$\tilde{h}_{(SE)} \approx \frac{1}{N}\sum_{J=1}^{N}\log(\alpha_J)\,\theta_J\beta_J x_0 e^{-\frac{\beta_J}{2}x_0^2} \times e^{-\left(e^{\frac{\beta_J}{2}x_0^2}-1\right)^{\theta_J}}$$

$$\times \frac{\alpha_J^{1-e^{-\left(e^{\frac{\beta_J}{2}x_0^2}-1\right)^{\theta_J}}}\left(1-e^{-\frac{\beta_J}{2}x_0^2}\right)^{\theta_J-1}}{\left(\alpha_J - \alpha_J^{1-e^{-\left(e^{\frac{\beta_J}{2}x_0^2}-1\right)^{\theta_J}}}\right)\left(e^{-\frac{\beta_J}{2}x_0^2}\right)^{\theta_J+1}}, \qquad (28)$$

where $k = 1,2,3.$

The Bayes estimates under LINEX loss function are given by

$$\tilde{\varphi}_{k(LINEX)} \approx \left(\frac{-1}{v}\right) ln\left[\frac{1}{N}\sum_{J=1}^{N} e^{-v(\varphi_k)_J}\right],$$

$$\tilde{R}_{(LINEX)} \approx \left(\frac{-1}{v}\right) ln\left[\frac{1}{N}\sum_{J=1}^{N} e^{-v\frac{\alpha_J - \alpha_J^{1-e^{-\left(e^{\frac{\beta_J}{2}x_0^2}-1\right)^{\theta_J}}}}{\alpha_J - 1}}\right]$$

and

$$\tilde{h}_{(SE)} \approx \left(\frac{-1}{v}\right) ln\left[\frac{1}{N}\sum_{J=1}^{N} exp\left[-v\log(\alpha_J)\,\theta_J\beta_J x_0 e^{-\frac{\beta_J}{2}x_0^2}\right.\right.$$

$$\times \frac{e^{-\left(e^{\frac{\beta_J}{2}x_0^2}-1\right)^{\theta_J}}\left(1-e^{-\frac{\beta_J}{2}x_0^2}\right)^{\theta_J-1}}{\left(\alpha_J-\alpha_J^{1-e^{-\left(e^{\frac{\beta_J}{2}x_0^2}-1\right)^{\theta_J}}}\right)\left(e^{-\frac{\beta_J}{2}x_0^2}\right)^{\theta_J+1}}\,\alpha_J^{1-e^{-\left(e^{\frac{\beta_J}{2}x_0^2}-1\right)^{\theta_J}}}\Bigg]\Bigg]. \tag{29}$$

Credible interval

Chen and Shao (1999) developed an algorism to calculate Bayesian credible interval when closed form of marginal posterior are known or MCMC samples can be generated from the posterior densities.

In general, a two-sided $100(1-\tau)$ % credible interval of the parameters $\underline{\varphi}$ is given by

$$P\left[L(\underline{x}) < \varphi_k < U(\underline{x})\right] = \int_{L(\underline{x})}^{U(\underline{x})} \pi(\varphi_k|\underline{x})\, d\underline{\varphi} = 1-\tau, \tag{30}$$

where $L(\underline{x})$ and $U(\underline{x})$, are the *lower limit* (LL) and *upper limit* (UL).

From the marginal posterior distributions of the parameters $\underline{\varphi}$ in (20), the $100(1-\tau)$ % credible interval for φ_k are given by

$$P\left[\varphi_k > L(\underline{x})|\underline{x}\right] = 1-\frac{\tau}{2} \text{ and } P\left[\varphi_k > U(\underline{x})|\underline{x}\right] = \frac{\tau}{2}. \tag{31}$$

on the other hands, when the posterior distribution cannot be obtained in explicit form the credible intervals of the parameters can be constructed by using their corresponding ordered MCMC samples according to the algorism developed by Chen and Shao (1999). Then the best credible interval is that interval which has the shortest length.

4. Numerical Results

In this section, a simulation study and a real data set are conducted under Progressive Type-II censoring scheme for estimating the parameters of APTW-R distribution in life time by mathematica 9 and R programming language. The simulation results are investigated to explore and assess performance of the ML and Bayesian estimation by evaluating relative absolute biases (RABs), relative errors (REs), *estimated risks* (ERs), variances and length of confidence and credible intervals (L.CIs).

4.1 Simulation study

This subsection focused on demonstrating the theoretical estimation results using simulated data to evaluate the performance of the ML and Bayesian estimation procedures under progressive Type-II censoring scheme on the basis of generated data from the APTW-R distribution. ML and Bayes averages of the estimates of the parameters based on progressive Type-II censoring are computed. Moreover, RABs, Res, ERs, variances and L.CIs for the parameters are calculated. All simulation studies are performed using Mathematica 9 and R programming language. For this purpose, applying the algorithm given by Balakrishnan and Sandhu (1995), different sample sizes n = 30, 50 and 100, different effective sample sizes m and set of different samples schemes are generated from APTW-R distribution for fixed values of $\alpha = 0.5$, $\beta = 0.5$ and $\theta = 0.5$. The NRs is restricted to 1000 In ML estimation and 10000 in Bayesian estimation. The scheme which minimizes the measurements of accuracy (RABs, REs, ERs, variances and L.CIs) of the estimates can be considered the best scheme. The RABs, ERs and REs of ML and Bayes estimates of the parameters are computed as follows:

$$\text{RAB (estimate)} = \frac{|\text{bias (estimate)}|}{\text{true value}}, \qquad \text{ER} = \frac{\sum_{i=1}^{NR}(estimated\ value - true\ value)^2}{NR}$$

and

$$\text{RE} = \frac{\sqrt{\text{ER}\ (estimated\ value)}}{true\ value}.$$

Table 1 summarizes the simulation results including the REs, variance for the ML estimates and L.CIs for the parameters based on progressive Type-II censoring under different samples schemes.

Table 2 shows the Bayes averages, ERs and L.CIs of the unknown parameters under SE and LINEX loss functions based on progressive Type-II censoring under different samples schemes.

4.2 Application to real data

This subsection aims to demonstrate how the proposed APTW-R distribution can be used in practice through analyzing a real lifetime data set. The real data set is obtained from Bekker *et al* (2000) it represents the survival times in years of a group of patients given chemotherapy treatment. The data consisting of 46 survival times (in years) for 46 patients are: 0.047, 0.115, 0.121, 0.132, 0.164, 0.197, 0.203, 0.260, 0.282, 0.296, 0.334, 0.395, 0.458, 0.466, 0.501,0.507, 0.529, 0.534, 0.540, 0.570, 0.641, 0.644, 0.696, 0.841, 0.863, 1.099, 1.219, 1.271, 1.326, 1.447, 1.485, 1.553, 1.581, 1.589, 2.178, 2.343, 2.416, 2.444, 2.825, 2.830, 3.578, 3.658, 3.743, 3.978, 4.003 and 4.033.

To check the validity of proposed model, Kolmogorov-Smirnov (K-S) goodness of fit test is performed for the data set. The p value is 0.8344. The p value shows that APTW-R fits the data very well.

Different progressive censoring schemes were considered for the real data set. The results are displayed in Table 3 and 4. Table 3 presents the ML estimates and corresponding *standard errors* (Ses) of the unknown parameters for the real data set based on progressive Type-II censoring under different samples schemes.

Table 4 presents the Bayes estimates and Ses of the unknown parameters for the real data set under SE and LINEX loss functions based on progressive Type-II censoring under different samples schemes.

Figure 2 presents the PP and QQ plots and fitted pdf for the real data set, which indicates that the APW-R distribution provides better fit to the data set.

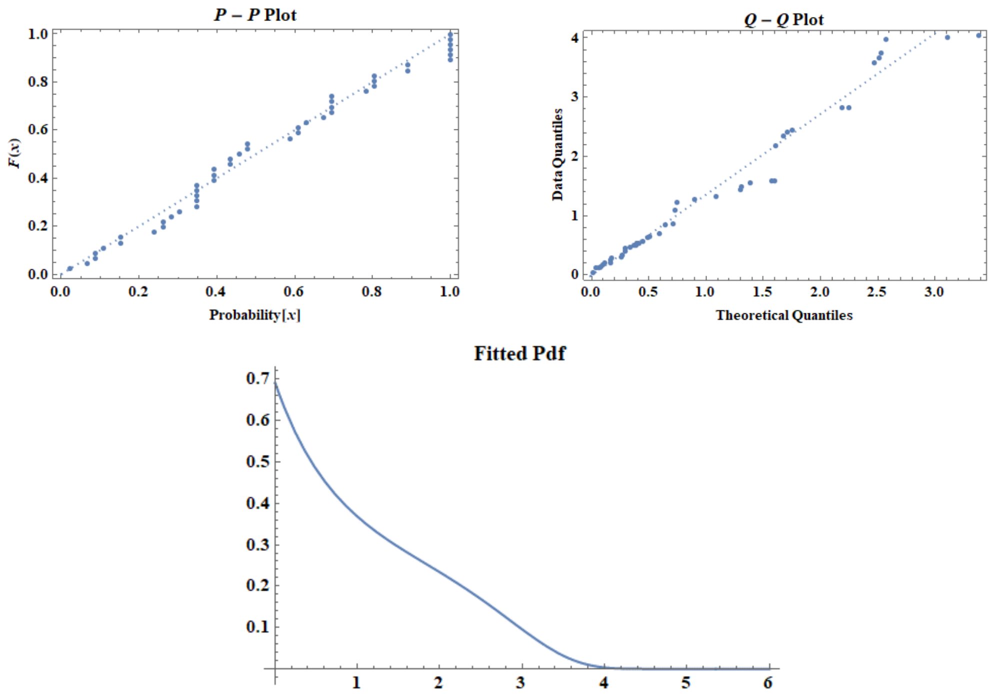

Figure 2 PP-plot, QQ-plot and fitted pdf of real data

4.3 Concluding remarks

From Tables 1 and 2, the following observations can be summarized as follows

The ML and Bayes averages of the estimates perform better when the sample size n increases.
The REs, and variances of the ML estimates decrease in most cases when the sample size n or effective sample size m increases. Also, the L.CIs gets shorter when the sample size n or effective sample size m increases.
The ERs and RABs of Bayes estimates decrease in most cases when the sample size n or effective sample size m increases. Also, the L.CIs gets shorter when the sample size n or effective sample size m increases.
The ERs, RABs and L.CIs of the Bayes estimates under SE loss function are in most cases less than the ERs, RABs and L.CIs of the Bayes estimates under LINEX loss functions, so the Bayes estimators under SE loss function are efficient than the Bayes estimators under LINEX loss functions.

It is note that the usual Type- II censoring scheme is a special case of the progressive type-II censoring scheme that can be obtained simply by taking $R_1 = R_2 = \cdots, R_{m-1}$= 0 and $R_m = n - m$.
In general the progressive type-II censoring scheme is more efficient than type-II censoring.

5. Conclusions

In this paper, the ML and Bayesian estimation methods are adopted for estimating the unknown parameters of the APTW-R distribution under progressive Type-II censoring scheme. The performance of the two proposed methods of estimation is investigated numerically for different sample sizes and different schemes using simulation study. The performances of the ML and Bayes estimators are quite satisfactory. The Bayes estimators are considered under two different loss functions, the SE loss function; as a symmetric loss function and LINEX loss function; as an asymmetric loss function under the assumption of independent gamma priors. A real data set from medicine field is explored to illustrate how the scheme can be used in practice. It is noted that the usual Type-II censoring scheme is a special case of the progressive Type-II censoring scheme which can be obtained simply by taking $R_1 = R_2 = \cdots, R_{m-1}$= 0 and $R_m = n - m$. The numerical study showed that when the sample size n and the effective sample size m increases, the accuracy metrics and the L.CIs decreases. Based on our study, the ERs, RABs and the L.CIs of the Bayes estimates under LINEX loss function are in most cases less than the ERs, RABs and the L.CIs of the Bayes estimates under SE loss functions, so the Bayes estimators under LINEX loss function are efficient than the Bayes estimators under SE loss functions.

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Table 1
ML averages, relative errors, variances of ML estimates and 95%confidence intervals of the parameters from APTW-R distribution based on Type-II progressive censoring scheme
($\alpha = 0.5$, $\beta = 0.5$, $\theta = 0.5$, $x_0 = 0.3$ and $NR = 1000$)

n	m	Scheme	Parameter	Average	RE	Variance	UL	LL	Length
30	10	$(0^{*9}, 20)$	α	0.6591	1.0717	0.9437	2.5405	−1.222	2.5404
			β	0.8277	1.1238	0.9326	2.7589	−0.1338	2.7589
			θ	0.4534	0.2954	0.0172	0.6499	0.2619	0.3830
			$R(x)$	0.7786	0.0881	0.0060	0.9769	0.6804	0.2965
			$h(x)$	0.5969	0.1900	0.0196	0.9113	0.3824	0.5289
		$(20,0^{*9})$		0.6641	0.4745	0.0468	1.0350	0.2931	0.7419
				0.6769	0.6519	0.0505	1.0040	0.3498	0.6542
				0.6258	0.2318	0.0076	0.7979	0.4836	0.3143
				0.8665	0.0626	0.0015	0.9392	0.6937	0.2455
				0.5794	0.1893	0.0137	0.7727	0.3859	0.3868
		(2^{*10})		0.5706	0.4101	0.0429	0.9767	0.1647	0.8120
				0.5334	0.3151	0.0266	0.8535	0.2135	0.6400
				0.4743	0.1578	0.0039	0.5965	0.3520	0.2445
				0.7992	0.0704	0.0024	0.8943	0.7042	0.1901
				0.6836	0.1378	0.0064	0.8405	0.5266	0.3139
		$(2^{*6}, 4^{*2}, 0^{*2})$		0.6911	0.3966	0.0195	1.0628	0.3194	0.7434
				0.6434	0.4612	0.0498	1.1548	0.1320	1.0228
				0.5427	0.2145	0.0071	0.6942	0.3911	0.3031
				0.8392	0.0777	0.0011	0.9050	0.7734	0.1316
				0.6219	0.1172	0.0063	0.7904	0.4534	0.3370
	20	$(0^{*19}, 10)$	α	0.6452	0.5890	0.0637	1.1038	0.1867	0.9172
			β	0.6587	0.4867	0.0398	1.0495	0.2679	0.7816
			θ	0.6165	0.2098	0.0052	0.7580	0.4751	0.2829
			$R(x)$	0.7809	0.0807	0.0024	0.8765	0.6853	0.1912
			$h(x)$	0.66614	0.1759	0.0119	0.8672	0.4555	0.4117
		$(10, 0^{*19})$	α	0.6360	0.4591	0.0363	1.073	0.3990	0.6740
			β	0.5141	0.2734	0.0195	0.7926	0.2357	0.5569
			θ	0.4280	0.1933	0.0050	0.5728	0.3131	0.2597
			$R(x)$	0.8617	0.0616	0.0015	0.9459	0.7876	0.1583
			$h(x)$	0.7615	0.1498	0.0105	0.8356	0.5847	0.2509
		$(0^{*5}, 1^{*10}, 0^{*5})$	α	0.5916	0.3678	0.0285	0.9226	0.2605	0.6621
			β	0.6554	0.4578	0.0380	1.0377	0.2731	0.7646
			θ	0.5636	0.1522	0.0032	0.6737	0.4535	0.2202
			$R(x)$	0.8364	0.0450	0.0011	0.9019	0.7708	0.1311
			$h(x)$	0.6845	0.1282	0.0110	0.8852	0.4837	0.4015
		$(2^{*3}, 0^{*13}, 1^{*4})$		0.7190	0.3207	0.0400	1.1116	0.3268	0.7843
				0.6207	0.2517	0.0225	0.9150	0.3263	0.5887
				0.4987	0.2077	0.0018	0.5823	0.4152	0.1671
				0.8195	0.0370	0.0009	0.8780	0.7609	0.1171
				0.6561	0.1579	0.0066	0.8159	0.4963	0.3196

Table 1 continued

n	m	Scheme	Parameter	Average	RE	Variance	UL	LL	Length
50	20	$(0^{*19}, 30)$	α	0.7532	0.9942	0.2129	1.6575	−0.1513	1.6575
			β	0.7103	0.8868	0.2468	1.8014	−0.1459	1.8014
			θ	0.4721	0.2719	0.0140	0.6378	0.3064	0.3314
			$R(x)$	0.7852	0.0818	0.0043	0.9136	0.6567	0.2569
			$h(x)$	0.6906	0.1896	0.0143	0.9075	0.4738	0.4337
		$(30, 0^{*19})$	α	0.6947	0.4994	0.0499	0.8876	0.2019	0.6857
			β	0.6826	0.3401	0.0320	0.9899	0.3753	0.6146
			θ	0.6794	0.2113	0.0094	0.8480	0.5348	0.3132
			$R(x)$	0.8858	0.0652	0.0025	0.9554	0.8011	0.1543
			$h(x)$	0.5314	0.1620	0.0135	0.6545	0.3083	0.3462
		$(0^{*5}, 10^{*1}, 0^{*5}, 20^{*1}, 0^{*8})$	α	0.6225	0.6645	0.0153	0.8650	0.3800	0.4850
			β	0.6763	0.4657	0.0443	1.0888	0.2638	0.8250
			θ	0.5484	0.1490	0.0093	0.7375	0.3593	0.3782
			$R(x)$	0.8273	0.0632	0.0026	0.9282	0.7264	0.2018
			$h(x)$	0.6615	0.0925	0.0059	0.8117	0.5114	0.30043
		$(2^{*9}, 0^{*4}, 4^{*3}, 0^{4})$		0.6757	0.8343	0.1535	1.4438	−0.0925	1.4438
				0.6830	0.7561	0.1746	1.5020	−0.1361	1.5020
				0.5689	0.2376	0.0112	0.7764	0.3612	0.4151
				0.8410	0.0486	0.0012	0.9080	0.7740	0.1340
				0.6409	0.1661	0.0116	0.8522	0.4297	0.4226
	40	$(0^{*39}, 10)$	α	0.7015	0.5103	0.0547	0.9964	0.3066	0.6898
			β	0.5395	0.2948	0.0212	0.8249	0.2540	0.5709
			θ	0.4547	0.2058	0.0052	0.5958	0.3137	0.2821
			$R(x)$	0.7993	0.0764	0.0017	0.8797	0.7189	0.1608
			$h(x)$	0.6470	0.1630	0.0092	0.8360	0.4581	0.3779
		$(10,0^{*39})$		0.5589	0.3222	0.0146	0.8766	0.2212	0.6554
				0.5720	0.2558	0.0136	0.8002	0.3438	0.4564
				0.5568	0.1779	0.0034	0.6750	0.4185	0.2565
				0.8354	0.0545	0.0014	0.9122	0.7587	0.1535
				0.6589	0.1234	0.0042	0.7757	0.5421	0.2336
		$(0^{*10}, 1^{*6}, 0^{*5}, 2^{*2}, 0^{*17})$		0.5414	0.3850	0.0377	0.9224	0.2604	0.6620
				0.5754	0.2859	0.0190	0.8454	0.3054	0.5400
				0.5573	0.1549	0.0044	0.6866	0.4280	0.2586
				0.8359	0.0444	0.0011	0.9023	0.7695	0.1328
				0.6498	0.1318	0.0075	0.8191	0.4805	0.3386
		$(1^{*10}0^{*30})$		0.5978	0.2310	0.0068	0.7597	0.4360	0.3237
				0.6057	0.2658	0.0065	0.7637	0.4478	0.3159
				0.5913	0.1923	0.0046	0.7243	0.4584	0.2659
				0.8568	0.0518	0.0007	0.9088	0.8049	0.1039
				0.5970	0.1196	0.0027	0.6980	0.4961	0.2019

Table 1 continued

n	m	Scheme	Parameter	Average	RE	Variance	UL	LL	Length
100	40	$(0^{*39}, 60)$	α	0.5422	0.3401	0.0291	0.8763	0.2081	0.6682
			β	0.5215	0.3067	0.0243	0.8273	0.2158	0.6115
			θ	0.4534	0.1569	0.0034	0.5586	0.3482	0.2104
			$R(x)$	0.7827	0.0662	0.0019	0.8677	0.6977	0.1700
			$h(x)$	0.6678	0.1433	0.0092	0.7811	0.5544	0.2267
		$(60, 0^{*39})$	α	0.4812	0.2983	0.0220	0.7722	0.1902	0.5820
			β	0.5445	0.1687	0.0051	0.6850	0.4039	0.2811
			θ	0.5499	0.1422	0.0019	0.66526	0.4346	0.2306
			$R(x)$	0.8304	0.0329	0.0006	0.8764	0.7844	0.09204
			$h(x)$	0.7160	0.0864	0.0031	0.8260	0.6060	0.2200
		$(0^{*5}, 3^{*5}, 4^{*5}, 5^{*5}, 0^{*20})$		0.4830	0.3038	0.0228	0.7788	0.1872	0.5917
				0.5260	0.1139	0.0026	0.6254	0.4267	0.1987
				0.5529	0.1496	0.0033	0.6665	0.4392	0.2273
				0.8356	0.0375	0.0005	0.8798	0.7915	0.0883
				0.6491	0.0823	0.0024	0.7453	0.5528	0.1925
		$(0^{*5}, 2^{*15}, 0^{*5}, 6^{*5}, 0^{*10}$		0.6138	0.3199	0.0100	0.8341	0.3935	0.4406
				0.5296	0.2289	0.0163	0.7463	0.3129	0.4334
				0.4705	0.1548	0.0023	0.6108	0.3303	0.2805
				0.8020	0.0540	0.0010	0.8844	0.7195	0.1649
				0.6686	0.0979	0.0034	0.7975	0.5397	0.2578
	80	$(0^{*79}, 20)$	α	0.7495	0.3177	0.0202	1.0279	0.4711	0.5568
			β	0.5733	0.2411	0.0091	0.7608	0.3858	0.3750
			θ	0.4737	0.1243	0.0025	0.5722	0.3752	0.1970
			$R(x)$	0.8144	0.0347	0.0009	0.8670	0.7618	0.1052
			$h(x)$	0.6406	0.0675	0.0018	0.7227	0.5585	0.1642
		$(20,0^{*79})$		0.5691	0.1542	0.0135	0.7964	0.3418	0.4546
				0.5621	0.1563	0.0051	0.7032	0.4209	0.2822
				0.5806	0.0164	0.0006	0.6276	0.5337	0.0939
				0.8573	0.0032	0.0007	0.8761	0.8385	0.0376
				0.5937	0.0138	0.0004	0.6352	0.5522	0.0830
		$(0^{*10}, 2^{*10}, 0^{*60})$		0.4523	0.0507	0.0174	0.6989	0.2057	0.4932
				0.5652	0.0284	0.0060	0.6744	0.4560	0.2184
				0.5624	0.0223	0.0101	0.6757	0.4491	0.2266
				0.8312	0.0292	0.0005	0.8655	0.7970	0.0685
				0.6816	0.0550	0.0004	0.7532	0.6100	0.1432
		$(0^{*20}5^{*4}, 0^{*56})$		0.6848	0.2655	0.0112	0.8037	0.5659	0.2378
				0.6980	0.0549	0.0005	0.8507	0.5454	0.3053
				0.6654	0.1530	0.0015	0.7106	0.6202	0.0904
				0.8865	0.0206	0.0001	0.9029	0.8702	0.0327
				0.5370	0.0886	0.0029	0.5900	0.4841	0.1059

Table 2: Bayes averages, relative absolute biases, estimated risks of the Bayes estimates and 95% credible intervals of the parameters based on progressive Type- II censoring scheme ($\boldsymbol{\alpha}$ = 0.5, $\boldsymbol{\beta}$ = 0.5, $\boldsymbol{\theta}$ = 0.5 **and** NR=10000)

n	m	Scheme	Par	SE						LINEX(ν= 0.5)					
				Average	RAB	ER	UL	LL	Length	Average	RAB	ER	UL	LL	Length
30	10	$(0^{*9}, 20)$	α	0.5635	0.1271	1.6176	0.6131	0.4847	0.1284	0.4874	0.0250	0.0626	0.5135	0.4476	0.0659
			β	0.5472	0.0944	0.8912	0.5929	0.5014	0.0915	0.4790	0.0418	0.1751	0.5032	0.4327	0.0705
			θ	0.4658	0.0682	0.4664	0.5268	0.3922	0.1345	0.5337	0.0675	0.4562	0.5727	0.5015	0.0712
		$(20, 0^{*9})$	α	0.5146	0.0293	0.0861	0.5408	0.4816	0.0592	0.4923	0.0153	0.0235	0.5041	0.4804	0.0236
			β	0.4788	0.0422	0.1783	0.5149	0.4311	0.0838	0.4883	0.0232	0.0542	0.4977	0.4797	0.0180
			θ	0.4730	0.0539	0.2915	0.5154	0.4392	0.0762	0.4886	0.0227	0.0517	0.5001	0.4789	0.0212
		(2^{*10})	α	0.5083	0.0167	0.0279	0.5259	0.4880	0.0379	0.4985	0.0029	0.0008	0.5125	0.4807	0.0318
			β	0.4668	0.0662	0.4383	0.5031	0.4397	0.0634	0.5166	0.0332	0.1106	0.5249	0.5063	0.0186
			θ	0.4879	0.0240	0.0578	0.5077	0.4667	0.0410	0.5095	0.0191	0.0367	0.5188	0.4995	0.0193
		$(2^{*6}, 4^{*2}, 0^{*2})$	α	0.5331	0.0662	0.4383	0.5681	0.5012	0.0669	0.5080	0.0161	0.0260	0.5135	0.5043	0.0091
			β	0.4631	0.0736	0.5418	0.4874	0.4207	0.0667	0.4952	0.0095	0.0090	0.5044	0.4891	0.0152
			θ	0.5061	0.0122	0.0149	0.5353	0.4828	0.0525	0.5021	0.0042	0.0017	0.5112	0.4965	0.0146

Table 2 continued

n	m	Scheme	Par	SE						LINEX($\nu = 0.5$)					
				Average	RAB	ER	UL	LL	Length	Average	RAB	ER	UL	LL	Length
30	20	$(0^{*19}, 10)$	α	0.4575	0.0848	0.0847	0.5067	0.4072	0.0995	0.5097	0.0195	0.0381	0.5256	0.4912	0.0344
			β	0.4635	0.0729	0.5327	0.5100	0.4233	0.0867	0.5194	0.0389	0.1518	0.5406	0.4935	0.0472
			θ	0.4762	0.0475	0.2263	0.5010	0.4515	0.0495	0.4782	0.0435	0.1897	0.4963	0.4667	0.0296
		$(10, 0^{*19})$	α	0.5082	0.0164	0.0270	0.5419	0.4772	0.0647	0.5049	0.0099	0.0098	0.5173	0.4942	0.0231
			β	0.5009	0.0019	0.0003	0.5164	0.4849	0.0314	0.4986	0.0027	0.0008	0.5061	0.4910	0.0150
			θ	0.4779	0.0440	0.1938	0.4966	0.4618	0.0348	0.5053	0.0106	0.0112	0.5114	0.4996	0.0117
		$(0^{*5}, 1^{*10}, 0^{*5})$	α	0.5145	0.0291	0.7207	0.5291	0.4983	0.0308	0.5075	0.0150	0.0226	0.5114	0.5034	0.0080
			β	0.5162	0.0324	0.1055	0.5239	0.5010	0.0229	0.4970	0.0058	0.0034	0.5015	0.4917	0.0098
			θ	0.5086	0.0173	0.0299	0.5172	0.4975	0.0196	0.4973	0.0052	0.0027	0.5054	0.4888	0.0166
		$(2^{*3}, 0^{*13}, 1^{*4})$	α	0.5157	0.0315	0.0995	0.5300	0.4974	0.0325	0.5015	0.0031	0.0010	0.5048	0.4970	0.0078
			β	0.4961	0.0076	0.0059	0.5038	0.4869	0.0169	0.4929	0.0140	0.0197	0.5000	0.4869	0.0131
			θ	0.4824	0.0350	0.1228	0.4986	0.4662	0.0324	0.4979	0.0040	0.0016	0.5024	0.4931	0.0092

Table 2 continued

n	m	Scheme	Par	SE						LINEX(ν= 0.5)					
				Average	RAB	ER	UL	LL	Length	Average	RAB	ER	UL	LL	Length
50	20	$(0^{*29}, 30)$	α	0.4904	0.0191	0.0368	0.5219	0.4651	0.0568	0.5039	0.0079	0.0062	0.5070	0.5008	0.0062
			β	0.4866	0.0266	0.0710	0.5226	0.4439	0.0786	0.4987	0.0025	0.0007	0.5031	0.4953	0.0077
			θ	0.4938	0.0124	0.0155	0.5153	0.4708	0.0444	0.5021	0.0042	0.0018	0.5048	0.4990	0.0058
		$(30, 0^{*29})$	α	0.4937	0.0125	0.0157	0.5002	0.4806	0.0196	0.5026	0.0054	0.0029	0.5064	0.4993	0.0071
			β	0.5052	0.0104	0.0109	0.5123	0.4993	0.0130	0.4988	0.0022	0.0004	0.5008	0.4966	0.0042
			θ	0.4937	0.0123	0.0152	0.5020	0.4845	0.0174	0.4988	0.0022	0.0006	0.5011	0.4957	0.0054
		$(0^{*5}, 10^{*1}, 0^{*5}, 20^{*1}, 0^{*8})$	α	0.5017	0.0035	0.0011	0.5060	0.4988	0.0072	0.5016	0.0033	0.0011	0.5045	0.4995	0.0049
			β	0.5002	0.0005	2.5051×10^{-5}	0.5065	0.4925	0.0140	0.5002	0.0004	1.9042×10^{-5}	0.5040	0.4978	0.0062
			θ	0.4970	0.0059	0.0034	0.5008	0.4930	0.0078	0.4977	0.0043	0.0019	0.5002	0.4955	0.0047
		$(2^{*9}, 0^{*4}, 4^{*3}, 0^{4})$	α	0.5078	0.0156	0.0246	0.5137	0.5014	0.0123	0.5035	0.0071	0.0050	0.5062	0.5006	0.0056
			β	0.4925	0.0149	0.0224	0.5027	0.4795	0.0231	0.5011	0.0022	0.0004	0.5046	0.4989	0.0057
			θ	0.4967	0.0065	0.0043	0.5022	0.4895	0.0126	0.4986	0.0026	0.0006	0.5005	0.4963	0.0042

Table 2continued

n	m	Scheme	Par	SE						LINEX(ν = 0.5)					
				Average	RAB	ER	UL	LL	Length	Average	RAB	ER	UL	LL	Length
50	40	$(0^{*39}, 10)$	α	0.4906	0.0186	0.0348	0.5009	0.4802	0.0207	0.5021	0.0043	0.0018	0.5045	0.5000	0.0045
			β	0.5115	0.0231	0.0534	0.5204	0.5004	0.0200	0.4989	0.0021	0.0005	0.5008	0.4965	0.0043
			θ	0.4945	0.0109	0.0119	0.5036	0.4859	0.0177	0.4987	0.0024	0.0006	0.5012	0.4960	0.0052
		$(10, 0^{*39})$	α	0.4939	0.0121	0.0148	0.4973	0.4906	0.0067	0.5020	0.0041	0.0016	0.5044	0.4994	0.0050
			β	0.4954	0.0090	0.0081	0.5028	0.4877	0.0151	0.5009	0.0019	0.0003	0.5024	0.4994	0.0030
			θ	0.4963	0.0073	0.0054	0.5032	0.4912	0.0119	0.4993	0.0012	0.0003	0.5007	0.4979	0.0028
		$(0^{*10}, 1^{*6}, 0^{*5}, 2^{*2}, 0^{*17})$	α	0.4963	0.0072	0.0052	0.4998	0.4922	0.0075	0.5011	0.0023	0.0005	0.5022	0.4992	0.0030
			β	0.5024	0.0050	0.0024	0.5070	0.4983	0.0087	0.4992	0.0015	0.0002	0.5004	0.4964	0.0040
			θ	0.5031	0.0064	0.0041	0.5092	0.4985	0.0107	0.5010	0.0019	0.0004	0.5033	0.4980	0.0053
		$(1^{*10}0^{*30})$	α	0.4956	0.0087	0.0075	0.5010	0.4891	0.0119	0.4987	0.0024	0.0006	0.5010	0.4970	0.0040
			β	0.4961	0.0077	0.0059	0.5002	0.4919	0.0083	0.5007	0.0015	0.0002	0.5024	0.4983	0.0041
			θ	0.4984	0.0031	0.0009	0.5024	0.4945	0.0079	0.4991	0.0016	0.0003	0.5019	0.4968	0.0051

Table 2 continued

n	m	Scheme	Par	SE						LINEX(ν = 0.5)					
				Average	RAB	ER	UL	LL	Length	Average	RAB	ER	UL	LL	Length
100	40	$(0^{*39}, 60)$	α	0.4929	0.0141	0.0201	0.5037	0.4833	0.0204	0.5028	0.0058	0.0033	0.5061	0.5001	0.0060
			β	0.4901	0.0196	0.0386	0.5022	0.4792	0.0229	0.4989	0.0021	0.0004	0.5016	0.4955	0.0061
			θ	0.4945	0.0109	0.0119	0.5037	0.4859	0.0178	0.5035	0.0038	0.0015	0.5065	0.5008	0.0057
		$(60, 0^{*39})$	α	0.4922	0.0116	0.0133	0.5019	0.4865	0.0154	0.5024	0.0049	0.0024	0.5053	0.4992	0.0061
			β	0.4949	0.0101	0.0103	0.5001	0.4899	0.0102	0.5008	0.0017	0.0003	0.5027	0.4992	0.0035
			θ	0.5052	0.0104	0.0108	0.5087	0.4990	0.0097	0.5006	0.0014	0.0002	0.5028	0.4993	0.0035
		$(0^{*5}, 3^{*5}, 4^{*5}, 5^{*5}, 0^{*20})$	α	0.4970	0.0060	0.0035	0.5012	0.4931	0.0080	0.5019	0.0038	0.0014	0.5034	0.5004	0.0030
			β	0.5073	0.0147	0.0216	0.5122	0.4992	0.0130	0.4985	0.0029	0.0008	0.4995	0.4973	0.0022
			θ	0.5048	0.0097	0.0093	0.5134	0.4964	0.0170	0.5015	0.0031	0.0009	0.5034	0.4995	0.0039
		$(0^{*5}, 2^{*15}, 0^{*5}, 6^{*5}, 0^{*10})$	α	0.4974	0.0051	0.0026	0.5020	0.4915	0.0105	0.5008	0.0017	0.0003	0.5020	0.4998	0.0022
			β	0.5071	0.0142	0.0202	0.5120	0.4990	0.0130	0.5015	0.0031	0.0010	0.5030	0.5006	0.0024
			θ	0.5018	0.0036	0.0013	0.5057	0.4981	0.0076	0.4991	0.0017	0.0002	0.5001	0.4979	0.0022

Table 2 continued

n	m	Scheme	Par	SE						LINEX(ν= 0.5)					
				Average	RAB	ER	UL	LL	Length	Average	RAB	ER	UL	LL	Length
100	80	$(0^{*79}, 20)$	α	0.5017	0.0035	0.0012	0.5106	0.4961	0.0145	0.4991	0.0017	0.0003	0.5005	0.4976	0.0029
			β	0.4950	0.0098	0.0097	0.5003	0.4882	0.0120	0.4994	0.0011	0.0001	0.5001	0.4987	0.0014
			θ	0.5011	0.0022	0.0005	0.5080	0.4914	0.0166	0.5005	0.0010	0.0001	0.5023	0.4992	0.0031
		$(20,0^{*79})$	α	0.4983	0.0033	0.0011	0.5000	0.4973	0.0027	0.4992	0.0014	0.0002	0.5005	0.4983	0.0022
			β	0.5011	0.0023	0.0005	0.5027	0.4994	0.0033	0.5005	0.0010	0.0001	0.5011	0.4998	0.0013
			θ	0.4996	0.0007	5.929×10^{-5}	0.5003	0.4983	0.0020	0.5001	0.0003	9.261×10^{-6}	0.5011	0.4991	0.0020
		$(0^{*10}, 2^{*10}, 0^{*60})$	α	0.4986	0.0026	0.0006	0.5005	0.4968	0.0037	0.5002	0.0004	1.6886×10^{-5}	0.5007	0.4994	0.0013
			β	0.5021	0.0043	0.0019	0.5037	0.5001	0.0036	0.5000	9.8017e-05	9.6075×10^{-7}	0.5006	0.4994	0.0012
			θ	0.4994	0.0011	0.0001	0.5007	0.4981	0.0026	0.4997	0.0005	2.7556×10^{-5}	0.5001	0.4994	0.0007
		$(0^{*79}, 20)$	α	0.5017	0.0035	0.0012	0.5106	0.4961	0.0145	0.4991	0.0017	0.0003	0.5005	0.4976	0.0029
			β	0.4950	0.0098	0.0097	0.5003	0.4882	0.0120	0.4994	0.0011	0.0001	0.5001	0.4987	0.0014
			θ	0.5011	0.0022	0.0005	0.5080	0.4914	0.0166	0.5005	0.0010	0.0001	0.5023	0.4992	0.0031

Table 3
ML estimates and standard errors of the parameters for the real data set based on progressive Type-II censoring scheme under different sample schemes

n	m	Scheme	Par	Estimate	Ses
46	23	Type-II $(0^{*22}, 23)$	α	0.4798	0.2611
			β	0.3476	0.2674
			θ	0.6444	0.2540
			$R(x)$	0.9082	0.2453
			$h(x)$	0.4125	0.2642
		$(23, 0^{*22})$	α	0.3574	0.2600
			β	0.4589	0.2551
			θ	0.5331	0.2518
			$R(x)$	0.8202	0.2409
			$h(x)$	0.6939	0.2452
		$(10, 0^{*4}, 5, 0^{*5}, 5, 0^{*4}, 3, 0^{*6})$	α	0.4920	0.2536
			β	0.3523	0.2603
			θ	0.5957	0.2490
			$R(x)$	0.8893	0.2390
			$h(x)$	0.4635	0.2549
		$(0^{*7}, 15, 0^{*6}, 8, 0^{*8})$	α	0.5619	0.2505
			β	0.3438	0.2607
			θ	0.6349	0.2474
			$R(x)$	0.9113	0.2384
			$h(x)$	0.3926	0.2583
	36	Type-II $(0^{*35}, 10)$	α	0.6444	0.2049
			β	0.2937	0.2181
			θ	0.5232	0.2092
			$R(x)$	0.8801	0.1981
			$h(x)$	0.4437	0.2121
		$(10, 0^{*35})$	α	0.4528	0.1935
			β	0.4888	0.1920
			θ	0.5115	0.19105
			$R(x)$	0.8172	0.1806
			$h(x)$	0.6804	0.1848
		$(0^{*10}, 2, 0^{*9}, 3, 0^{*5}, 5, 0^{*9})$	α	0.5013	0.1915
			β	0.3688	0.1971
			θ	0.5842	0.1882
			$R(x)$	0.8823	0.17903
			$h(x)$	0.4853	0.1921
		$(0^{*19}, 2,8, 0^{*15})$	α	0.4906	0.1918
			β	0.3780	0.1967
			θ	0.6004	0.1876
			$R(x)$	0.8868	0.1789
			$h(x)$	0.4786	0.1924

Table 4

Bayes estimates and standard errors of the parameters for the real data set under squared error and linear exponential loss functions based on progressive Type-II censoring under different sample schemes

n	m	Scheme	Par	SE		LINEX(ν = 0.1)	
				Estimate	Ses	Estimate	Ses
46	23	Type- II $(0^{*22}, 23)$	α	0.5191	0.0013	0.5131	0.0011
			β	0.4654	0.0038	0.4625	0.0031
			θ	0.4878	0.0012	0.4844	0.0010
		$(23, 0^{*22})$	α	0.5087	0.0007	0.4983	0.0003
			β	0.4956	0.0009	0.4930	0.0006
			θ	0.4874	0.0019	0.4916	0.0005
		$(10, 0^{*4}, 5, 0^{*5}, 5, 0^{*4}, 3, 0^{*6})$	α	0.4924	0.0009	0.4982	0.0005
			β	0.5026	0.0012	0.4938	0.0006
			θ	0.4744	0.0011	0.49959	0.0003
		$(0^{*7}, 15, 0^{*6}, 8, 0^{*8})$	α	0.4972	0.0009	0.49933	0.0005
			β	0.4978	0.0007	0.5004	0.0004
			θ	0.5001	0.0008	0.4978	0.0005
	36	Type- II $(0^{*35}, 10)$	α	0.4847	0.0012	0.5095	0.0008
			β	0.4886	0.0013	0.5079	0.0008
			θ	0.4941	0.0010	0.4946	0.0007
		$(10, 0^{*35})$	α	0.5029	0.0011	0.5049	0.0007
			β	0.5132	0.0007	0.5089	0.0005
			θ	0.4912	0.0008	0.5019	0.0004
		$(0^{*10}, 2, 0^{*9}, 3, 0^{*5}, 5, 0^{*9})$	α	0.5130	0.0005	0.4941	0.0003
			β	0.4930	0.0007	0.5062	0.0004
			θ	0.4869	0.0007	0.4979	0.0002
		$(0^{*19}, 2,8, 0^{*15})$	α	0.5029	0.0009	0.4903	0.0007
			β	0.4924	0.0012	0.4974	0.0005
			θ	0.5023	0.0008	0.5041	0.0004

الملخص العربي

في هذا البحث تم استخدام طريقة الإمكان الأعظم وطريقة بييز لتقدير المعالم المجهولة للتوزيع المقترح اعتمادا على نظام المراقبة التدريجي من النوع الثاني. وقد تم استخدام نظام المحاكاة مونت كارلو في هذه الدراسة وذلك باستخدام أحجام عينات مختلفة (n)وأيضا طرق مراقبة تدريجية مختلفة .(Schemes) وقد ثبت من خلال الدراسة أن مقدرات الإمكان الأعظم وأيضا مقدرات بييز جيدة وذلك من خلال حساب بعض المقاييس المستخدمة مثل متوسط مربع الخطأ ومقدار التحيز وطول فترة الثقة. كما تم استخدام نوعين من دوال الخسارة في حالة التقدير البيزي وهما دالة الخسارة التربيعية وكذلك دالة الخسارة الأسية الخطية. وقد أوضحت الدراسة أن متوسط مربع الخطأ في حالة التقدير البييزي تحت دالة الخسارة الأسية الخطية أقل منه في حالة التقدير باستخدام دالة الخسارة التربيعية. كذلك تم استخدام مجموعة من البيانات الحقيقية وذلك لتوضيح كيف يمكن استخدام هذه الطرق في الناحية التطبيقية. وتمثل هذه البيانات أوقات البقاء لعدد٤٦ مريض تم اعطائهم نوع معين من العلاج الكيماوي. وقد خضعت هذه البيانات لاختبار جودة التوفيق. وتم حساب تقديرات المعالم وحساب الخطأ المعياري المناظر لكل تقدير وذلك باستخدام طرق التقدير المذكورة تحت نظام المراقبة التدريجي. وأوضحت الدراسة أن التقدير في حالة نظام المراقبة التدريجي أفضل من نظام المراقبة العادي رغم أن الثاني يعتبر حالة خاصة من الأول.