

EXISTENCE OF A UNIQUE POSITIVE CONTINUOUS SOLUTION OF AN URYSOHN QUADRATIC INTEGRAL EQUATION

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ABSTRACT. In this work, we concerned with the nonlinear Urysohn quadratic integral equation

$$x(t) = a(t) + \int_0^t f_1(t, s, x(s)) ds \int_0^t f_2(t, s, x(s)) ds, \quad t \in [0, T].$$

The existence of a unique positive continuous solution will be proved. The fractional orders quadratic integral equation will be considered as an application.

1. INTRODUCTION

Quadratic integral equations(QIES) are often applicable in the theory of radiative transfer, kinetic theory of gases, in the theory of neutron transport and in the traffic theory. The quadratic integral equations can be very often encountered in many applications(see[1]-[15]). Here we are concerned with the nonlinear Urysohn quadratic integral equation

$$x(t) = a(t) + \int_0^t f_1(t, s, x(s)) ds \int_0^t f_2(t, s, x(s)) ds, \quad t \in [0, T]. \quad (1)$$

The existence of a unique positive continuous solution of will be proved.

2. MAIN RESULTS

Consider the quadratic integral equation (1) under the following assumptions

- (i) $a : I = [0, T] \rightarrow R_+$ is continuous, $a = \sup_{t \in [0, T]} |a(t)|$.
- (ii) $f_i : [0, T] \times [0, T] \times R \rightarrow R_+$ are measurable in $s \in [0, T]$ for all $(t, x) \in [0, T] \times R$, nonincreasing in $t \in [0, T]$ for all $(s, x) \in [0, T] \times R$ and satisfy the Lipschitz condition

$$|f_i(t, s, x(s)) - f_i(t, s, y(s))| \leq |k_i(t, s)| |x(s) - y(s)|, \quad i = 1, 2 \quad (2)$$

for every $(t, s, x(s)) \in [0, T] \times [0, T] \times R_+$ and

$$\int_0^t |k_i(t, s)| ds \leq K_i, \quad i = 1, 2, \quad \forall t \in [0, T].$$

(iii)

$$\int_0^t f_i(t, s, 0) ds \leq m_i, i = 1, 2.$$

Remark From assumptions (ii) and (iii) we can get

$$f_i(t, s, x(s)) \leq k_i(t, s)|x(s)| + f_i(t, s, 0).$$

Define the set

$$Q_r = \{x \in R : |x(t)| \leq r\} \subset C[0, T].$$

The following lemma can be easily proved.

Lemma 1. Let $x \in Q_r$. Then we have

$$\int_0^t |f_i(t, s, x(s))| \leq K_i r + m_i = M_i$$

Definition By a solution of the quadratic integral equation (1) we mean a function $x \in Q_r$. This function satisfies equation (1).

Now for the existence of a unique positive continuous solution of the quadratic integral equation (1) we have the following theorem.

Theorem 1. Let the assumptions (i)-(iii) be satisfied. If $(M_1K_2 + M_2K_1) = K < 1$, then the quadratic integral equation (1) has a unique positive solution $x \in C[0, T]$.

Proof. Define the operator F associated with the quadratic integral equation (1), by

$$Fx(t) = a(t) + \int_0^t f_1(t, s, x(s))ds \cdot \int_0^t f_2(t, s, x(s))ds. \quad (3)$$

The operator F_2 maps $C[0, T]$ into itself
for this, if we let $x \in C[0, T]$, $t_1, t_2 \in [0, T]$, $t_1 < t_2$, and $t_2 - t_1 \leq \delta$, then

$$\begin{aligned}
|Fx(t_2) - Fx(t_1)| &= |a(t_2) - a(t_1)| \\
&+ \left| \int_0^{t_2} f_1(t_2, s, x(s)) ds \int_0^{t_2} f_2(t_2, s, x(s)) ds \right. \\
&- \left. \int_0^{t_1} f_1(t_1, s, x(s)) ds \int_0^{t_1} f_2(t_1, s, x(s)) ds \right| \\
&= |a(t_2) - a(t_1)| \\
&+ \left| \left[\int_0^{t_1} f_1(t_2, s, x(s)) ds + \int_{t_1}^{t_2} f_1(t_2, s, x(s)) ds \right] \int_0^{t_2} f_2(t_2, s, x(s)) ds \right. \\
&- \left. \int_0^{t_1} f_1(t_1, s, x(s)) ds \int_0^{t_1} f_2(t_1, s, x(s)) ds \right| \\
&= |a(t_2) - a(t_1)| \\
&+ \left| \int_0^{t_1} f_1(t_2, s, x(s)) ds \int_0^{t_2} f_2(t_2, s, x(s)) ds \right. \\
&+ \left| \int_{t_1}^{t_2} f_1(t_2, s, x(s)) ds \int_0^{t_2} f_2(t_2, s, x(s)) ds \right. \\
&- \left. \int_0^{t_1} f_1(t_1, s, x(s)) ds \int_0^{t_1} f_2(t_1, s, x(s)) ds \right| \\
&\leq |a(t_2) - a(t_1)| \\
&+ \left| \int_0^{t_1} f_1(t_1, s, x(s)) ds \int_0^{t_2} f_2(t_1, s, x(s)) ds \right. \\
&+ \left| \int_{t_1}^{t_2} f_1(t_1, s, x(s)) ds \int_0^{t_2} f_2(t_1, s, x(s)) ds \right. \\
&- \left. \int_0^{t_1} f_1(t_1, s, x(s)) ds \int_0^{t_1} f_2(t_1, s, x(s)) ds \right| \\
&\leq |a(t_2) - a(t_1)| \\
&+ \left| \int_0^{t_1} f_1(t_1, s, x(s)) ds \left[\int_0^{t_2} f_2(t_1, s, x(s)) ds \right. \right. \\
&- \left. \left. \int_0^{t_1} f_2(t_1, s, x(s)) ds \right] \right. \\
&+ \left| \int_{t_1}^{t_2} f_1(t_1, s, x(s)) ds \int_0^{t_2} f_2(t_1, s, x(s)) ds \right| \\
&\leq |a(t_2) - a(t_1)| \\
&+ \left| \int_0^{t_1} f_1(t_1, s, x(s)) ds \int_{t_1}^{t_2} f_2(t_1, s, x(s)) ds \right. \\
&+ \left. \int_0^{t_2} f_2(t_1, s, x(s)) ds \int_{t_1}^{t_2} f_1(t_1, s, x(s)) ds \right|,
\end{aligned}$$

then

$$\begin{aligned} |Fx(t_2) - Fx(t_1)| &\leq |a(t_2) - a(t_1)| \\ &+ (K_1 \|x(t)\| + M_1) \int_{t_1}^{t_2} f_2(t_1, s, x(s)) ds \\ &+ (K_2 \|x(t)\| + M_2) \int_{t_1}^{t_2} f_1(t_1, s, x(s)) ds. \end{aligned}$$

This proves that $F_2 : C[0, T] \rightarrow C[0, T]$.

Now to prove that F_2 is contraction, we have the following.

Let $x, y \in C[0, T]$, then

$$\begin{aligned} |Fx(t) - Fy(t)| &= \left| \int_0^t f_1(t, s, x(s)) ds \int_0^t f_2(t, s, x(s)) ds \right. \\ &\quad \left. - \int_0^t f_1(t, s, y(s)) ds \int_0^t f_2(t, s, y(s)) ds \right| \\ &= \left| \int_0^t f_1(t, s, x(s)) ds \int_0^t f_2(t, s, x(s)) ds \right. \\ &\quad \left. - \int_0^t f_1(t, s, y(s)) ds \int_0^t f_2(t, s, y(s)) ds \right. \\ &\quad \left. + \int_0^t f_1(t, s, x(s)) ds \int_0^t f_2(t, s, y(s)) ds \right. \\ &\quad \left. - \int_0^t f_1(t, s, x(s)) ds \int_0^t f_2(t, s, y(s)) ds \right| \\ &= \left| \int_0^t f_1(t, s, x(s)) ds \int_0^t [f_2(t, s, x(s)) - f_2(t, s, y(s))] ds \right. \\ &\quad \left. + \int_0^t f_2(t, s, y(s)) ds \int_0^t [f_1(t, s, x(s)) - f_1(t, s, y(s))] ds \right| \\ &\leq \int_0^t f_1(t, s, x(s)) ds \int_0^t |f_2(t, s, x(s)) - f_2(t, s, y(s))| ds \\ &\quad + \int_0^t f_2(t, s, y(s)) ds \int_0^t |f_1(t, s, x(s)) - f_1(t, s, y(s))| ds \\ &\leq M_1 \int_0^t |k_2(t, s)| |x(s) - y(s)| ds \\ &\quad + M_2 \int_0^t |k_1(t, s)| |x(s) - y(s)| ds. \end{aligned}$$

Then

$$\begin{aligned} \|Fx - Fy\| &\leq M_1 K_2 \|x - y\| + M_2 K_1 \|x - y\| \\ \|Fx - Fy\| &\leq (M_1 K_2 + M_2 K_1) \|x - y\|. \end{aligned}$$

Hence

$$\|Fx - Fy\| \leq K \|x - y\|. \quad (4)$$

Since $K < 1$, then F_2 is contraction. Then by using Banach fixed point theorem, the operator F has a unique fixed point $x \in C[0, T]$ by second group of the assumptions. ■

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