Assiut University Journal of Multidisciplinary Scientific Research (AUNJMSR)
Faculty of Science, Assiut University, Assiut, Egypt.
Printed ISSN 2812-5029
Online ISSN 2812-5037
Vol. 53(3): 413- 429 (2024)
https://aunj.journals.ekb.eg

Statistical Inference for the Modified Burr XII Distribution under Progressive Type-II Censoring

Sara M.A.M. Ali [1], Abd EL-Baset A. Ahmad [1] and Safaa Z. Soliman [1,*]
Assiut University, Faculty of Science, Mathematics Department

[1,*]safaaabdelaziz@science.aun.edu.eg

ARTICLE INFO
Article History:
Received: 2024-07-04
Accepted: 2024-08-05
Online: 2024-08-29

Keywords:
Progressive Type II, Maximum likelihood estimator, Bayesian estimation, Modified Burr XII, Monte Carlo simulation.

ABSTRACT
This paper investigates the estimation of parameters for the modified Burr XII distribution under a progressive Type-II censoring scheme. We employ both Bayesian and maximum likelihood approaches to estimate the model's parameters. Approximate confidence intervals (ACIs) are constructed to quantify the uncertainty associated with the unknown parameters. Subsequently, the Markov chain Monte Carlo (MCMC) method is used to obtain Bayesian estimates. The credible intervals are then computed in turn. Finally, we illustrate the methodology's practical application by analyzing a real data set.

1. INTRODUCTION

Burr [1] introduced twelve cumulative distribution functions with the primary purpose of fitting distributions to real data. One of the most important of them is the Burr XII (BXII) distribution with parameters c, k. BXII is an important distribution in statistics and operations research. It has a wide range of applications in several areas such as chemical engineering, quality control, business, meteorology, hydrology, medical, and reliability studies for more details see Al-Hussaini and Jaheen [2], Wu et al. [3] and Soliman et al. [4]. To significantly expand the flexibility of the BXII distribution, Jamal et al. [5] suggested a new three-parameter modified Burr XII distribution (MBXII). The distribution's adaptability and usefulness are demonstrated by fitting it to two separate data sets. The findings showed that the distribution closely matches the data sets. Burr XII, Logistic, Log-logistic, Modified Log-logistic, Lomax, Modified Lomax and

Modified Logistic are special cases from MBXII distribution, for more details see Jamal et al. [5].

The MBXII distribution has a probability density function (PDF), cumulative distribution function (CDF), and hazard rate function as follows:

$$f_{MBXII}(x) = k\, e^{\lambda x} x^{c} \left[\frac{c}{x} + \lambda\right] (1 + e^{\lambda x} x^{c})^{-k-1}, \qquad x > 0, \qquad (1)$$
$$k > 0, \quad c, \lambda \geq 0, \quad \max(c, \lambda) > 0,$$

$$F_{MBXII}(x) = 1 - \left(1 + e^{\lambda x} x^{c}\right)^{-k}, \qquad (2)$$

$$h_{MBXII}(x) = \left[\frac{c}{x} + \lambda\right] \frac{k\, e^{\lambda x} x^{c}}{1 + e^{\lambda x} x^{c}}.$$

Reliability studies and survival analysis frequently use censored data. Single-censored observations occur when a sample is taken across a complete time period but either the last or the first observations are unknown add to that the censoring may be justified in order to reduce the time and expense of testing. Various techniques can result in censored data. Type-I and Type-II censoring are the most two popular censoring techniques. So-called Type-II censoring that describes the experiment would be continued until a predetermined number of items are failed. As opposed to, when using Type-I censoring, the experiment is over at a pre-specified time. Type-I and Type-II censoring are not flexible enough to allow units to be removed from the experiment at any point other than the termination point. To solve this problem, we discuss a progressive Type-II censoring scheme, which is a generalization of Type-II censoring and a more flexible censoring method. In the following, a description of the specific scenarios in this scheme.

Assume that n identical and independent units are put on a life test, and that at the start of the experiment, the researcher determines that pre-fixed integer m $(m \leq n)$ failures are to be recorded. At first failure (say $x_{1:m:n}$), r_1 of the alive units $n-1$ are randomly taken away from the test. At the second failure (say $x_{2:m:n}$) r_2 of the alive units $n - r_1 - 2$ are randomly taken away from the test. And so it continues until the m^{th}failure then $r_m = n - m - \sum_{i=1}^{m-1} r_i$ are taken away from the test and the experiment is terminated. More information about progressive Type-II censorship and its various advantages can be found in monographs by Aggarwala and Balakrishnan [6], Balakrishnan [7] and Balakrishnan and Cramer [8]. Several authors, including Wu [9], Wu and Chang [10], Asgharzadeh [11], Wu et al. [12], Kang et al. [13], Wu and Gui [14], Abu-Moussa et al. [15] have investigated inference in progressively Type-II censored samples with various lifetime distributions, including Weibull, Pareto, generalized logistic, Burr XII, Half Logistic, Nadarajah-Haghighi, Rayleigh distributions, respectively. However, depending on the observed sample $(x_{1:m:n}, x_{2:m:n}, x_{3:m:n}, \dots, x_{m:m:n})$ from a progressive Type-II censoring scheme the likelihood function of the progressive Type-II censored data is

$$L = A \prod_{i=1}^{m} f(x_{i:m:n})\ (1 - F(x_{i:m:n}))^{r_i}, \tag{3}$$

where $A = n\,(n - r_1 - 1)(n - r_1 - r_2 - 2) \dots (n - m - r_1 - r_2 - \dots - r_{m-1} + 1)$.

For more details see Balakrishnan and Cramer [8].

According to this study, the estimation of the parameters of the MBXII distribution is considered depending on progressive Type-II using maximum likelihood and Bayesian techniques. The maximum likelihood estimation and the Fisher information matrix is discussed in Sections 2 and 3, respectively. Section 4 presents the Bayesian inference method. Section 5 presents a simulation study that compares the performance of the model as well as a real-life example for demonstration purposes. Finally, in Section 6, we give a final conclusion for the paper.

2. MAXIMUM LIKELIHOOD ESTIMATION

The likelihood function of the parameters c, k and λ of the MBXII distribution are obtained from Eqs. (1), (2) and using (3) as follows:

$$L(c,k,\lambda) = A \prod_{i=1}^{m} k\, e^{\lambda x_i} {x_i}^{c} \left[\frac{c}{x_i} + \lambda\right] (1 + e^{\lambda x_i} {x_i}^{c})^{-k(r_i+1)-1}. \tag{4}$$

The natural logarithm say, ℓ, of the likelihood function is

$$\ell(c,k,\lambda) = \ln A + m \ln k + \lambda \sum_{i=1}^{m} x_i + \sum_{i=1}^{m} c \ln x_i + \sum_{i=1}^{m} \ln\left[\frac{c}{x_i} + \lambda\right] + \sum_{i=1}^{m} \left[(-k(r_i + 1) - 1) \ln\left[1 + e^{\lambda x_i} {x_i}^{c}\right]\right]. \tag{5}$$

By taking the first partial derivatives of ℓ, with respect to c, k and λ we have

$$\frac{\partial \ell}{\partial c} = \sum_{i=1}^{m} \ln x_i + \sum_{i=1}^{m} \frac{1}{c + \lambda\, x_i} - \sum_{i=1}^{m} \left[(k(r_i + 1) + 1)\ \frac{e^{\lambda x_i} {x_i}^{c} \ln x_i}{1 + e^{\lambda x_i} {x_i}^{c}}\right], \tag{6}$$

$$\frac{\partial \ell}{\partial k} = \frac{m}{k} - \sum_{i=1}^{m} \left[(r_i + 1) \ln\left[1 + e^{\lambda x_i} {x_i}^{c}\right]\right], \tag{7}$$

$$\frac{\partial \ell}{\partial \lambda} = \sum_{i=1}^{m} x_i + \sum_{i=1}^{m} \frac{x_i}{c + \lambda\, x_i} - \sum_{i=1}^{m} \left[(k(r_i + 1) + 1) \frac{e^{\lambda x_i} {x_i}^{c+1}}{1 + e^{\lambda x_i} {x_i}^{c}} \right]. \tag{8}$$

By putting Eqs.(6−8)equal to zero, we get

$$\sum_{i=1}^{m} ln x_i + \sum_{i=1}^{m} \frac{1}{c + \lambda x_i} - \sum_{i=1}^{m} \left[(k(r_i + 1) + 1) \frac{e^{\lambda x_i} x_i^{c} \ln x_i}{1 + e^{\lambda x_i} x_i^{c}} \right] = 0,$$

$$\frac{m}{k} - \sum_{i=1}^{m} \left[(r_i + 1) \ln\left[1 + e^{\lambda x_i} {x_i}^{c}\right]\right] = 0,$$

$$\sum_{i=1}^{m} x_i + \sum_{i=1}^{m} \frac{x_i}{c + \lambda x_i} - \sum_{i=1}^{m} \left[(k(r_i + 1) + 1) \frac{e^{\lambda x_i} x_i^{c+1}}{1 + e^{\lambda x_i} x_i^{c}} \right] = 0.$$

The solution of these equations gives the maximum likelihood estimates (MLEs) for the unknown population parameters c, k, and λ. These solutions cannot be obtained in a simple closed form. However, some numerical techniques can be used in this proposal, e.g., Newton-Raphson method.

3. APPROXIMATE CONFIDENCE INTERVALS

The asymptotic variances-covariances of the MLEs for the parameters c, k, and λ are supplied by elements of the inverse of the Fisher information matrix, which is defined by

$$I_{ij} = E\left[\frac{-\partial^2 \ell}{\partial \Theta_i \partial \Theta_j}\right]; \quad \text{i, j=1,2,3 and} \quad (\Theta_1, \Theta_2, \Theta_3) = (c, k, \lambda).$$

Unfortunately, the exact mathematical expressions for the above expectations are quite difficult to be obtained. Consequently, we provide the MLEs' approximate (observed) asymptotic variance-covariance matrix, produced by removing the expectation operator E. The observed Fisher information matrix $I^{-1}(c, k, \lambda)$ can be given by

$$I^{-1}(c,k,\lambda)=\begin{pmatrix}\frac{-\partial^2 l}{\partial c^2} & \frac{-\partial^2 l}{\partial c\partial k} & \frac{-\partial^2 l}{\partial c\partial \lambda}\\ \frac{-\partial^2 l}{\partial k\partial c} & \frac{-\partial^2 l}{\partial k^2} & \frac{-\partial^2 l}{\partial k\partial \lambda}\\ \frac{-\partial^2 l}{\partial \lambda\partial c} & \frac{-\partial^2 l}{\partial \lambda\partial k} & \frac{-\partial^2 l}{\partial \lambda^2}\end{pmatrix}^{-1}$$

And then $I^{-1}(\hat{c},\hat{k},\hat{\lambda})$ is given by

$$I^{-1}(\hat{c},\hat{k},\hat{\lambda})=\begin{pmatrix} var(\hat{c}) & cov(\hat{c},\hat{k}) & cov(\hat{c},\hat{\lambda})\\ cov(\hat{k},\hat{c}) & var(\hat{k}) & cov(\hat{k},\hat{\lambda})\\ cov(\hat{\lambda},\hat{c}) & cov(\hat{\lambda},\hat{k}) & var(\hat{\lambda})\end{pmatrix}$$

With

$$\frac{-\partial^2\ell}{\partial c^2}=\sum_{i=1}^{m}\frac{1}{(c+\lambda x_i)^2}+\sum_{i=1}^{m}\left[(k(r_i+1)+1)\frac{e^{\lambda x_i}x_i^c(\ln x_i)^2}{(1+e^{\lambda x_i}x_i^c)^2}\right],$$

$$\frac{-\partial^2\ell}{\partial c^2}=\frac{m}{k^2},$$

$$\frac{-\partial^2\ell}{\partial \lambda^2}=\sum_{i=1}^{m}\frac{x_i^2}{(c+\lambda x_i)^2}+\sum_{i=1}^{m}\left[(k(r_i+1)+1)\frac{e^{\lambda x_i}x_i^{c+2}}{(1+e^{\lambda x_i}x_i^c)^2}\right],$$

$$\frac{-\partial^2\ell}{\partial c\partial k}=\frac{-\partial^2\ell}{\partial k\partial c}=\sum_{i=1}^{m}\left[(r_i+1)\frac{e^{\lambda x_i}x_i^c\ln x_i}{1+e^{\lambda x_i}x_i^c}\right],$$

$$\frac{-\partial^2\ell}{\partial c\partial \lambda}=\frac{-\partial^2\ell}{\partial \lambda\partial c}=\sum_{i=1}^{m}\frac{x_i}{(c+\lambda x_i)^2}+\sum_{i=1}^{m}\left[(k(r_i+1)+1)\frac{e^{\lambda x_i}x_i^{c+1}\ln x_i}{(1+e^{\lambda x_i}x_i^c)^2}\right],$$

$$\frac{-\partial^2\ell}{\partial k\partial \lambda}=\frac{-\partial^2\ell}{\partial \lambda\partial k}=\sum_{i=1}^{m}\left[(r_i+1)\frac{e^{\lambda x_i}x_i^{c+1}}{1+e^{\lambda x_i}x_i^c}\right].$$

The asymptotic normality of the MLEs can be employed to compute the approximate confidence intervals for parameters c, k, and λ. As a result, $(1-\xi)100\%$ confidence intervals (CIs) for parameters c, k, and λ become $\left[\hat{c}\pm z_{\frac{\xi}{2}}\sqrt{var(\hat{c})}\right]$, $\left[\hat{k}\pm z_{\frac{\xi}{2}}\sqrt{var(\hat{k})}\right]$, and $\left[\hat{\lambda}\pm z_{\frac{\xi}{2}}\sqrt{var(\hat{\lambda})}\right]$,where $z_{\frac{\xi}{2}}$ is a normal standard value.

4. BAYES ESTIMATION

Modern products often have highly reliable and protracted lifespans, which can occasionally result in a lack of data availability in lifetime studies, where a small sample size may significantly impact the correctness of the inferential results. Therefore, Bayesian estimation is more practical than classical estimation methods due to its capacity to include more information in the inferential approach. The Bayesian estimation has drawn the attention of many researchers in recent years.

In this section, we obtain the Bayesian inference of the unknown parameters of the MBXII distribution based on progressive Type-II censoring under squared error (SE) and LINEX loss functions. It is assumed here that the parameters c, k, and λ are independent and follow the gamma prior distributions with hyperparameters $(a_1, b_1 > 0)$ for c, $(a_2, b_2 > 0)$ for k, and $(a_3, b_3 > 0)$ for λ .i.e.

$$\pi_1(c) \propto c^{a_1-1} e^{-b_1 c}, \quad c > 0,$$

$$\pi_2(k) \propto k^{a_2-1} e^{-b_2 k}, \quad k > 0,$$

$$\pi_3(\lambda) \propto \lambda^{a_3-1} e^{-b_3 \lambda}, \quad \lambda > 0.$$

The joint prior density function for c, k, and λ is

$$\pi(c, k, \lambda) \propto c^{a_1-1} k^{a_2-1} \lambda^{a_3-1} e^{-(b_1 c + b_2 k + b_3 \lambda)}, \qquad c, k, \lambda > 0. \tag{9}$$

Combining Eq.(4) with Eq.(9), the joint posterior density function of c, k, and λ can be written as

$$\pi^*(c, k, \lambda \mid x) \propto c^{a_1-1} k^{m+a_2-1} \lambda^{a_3-1} e^{-\left(b_1 c + b_2 k + \lambda\left(b_3 - \sum_i^m x_i\right)\right)} \prod_{i=1}^{m} \left[x_i^c \left[\frac{c}{x_i} + \lambda \right] \left(1 + e^{\lambda x_i} x_i^c\right)^{-k(r_i+1)-1} \right]. \tag{10}$$

Therefore, the Bayes estimate of any function of c, k, and λ say $\Psi = \Psi(c, k, \lambda)$ under SE and LINEX loss functions are given respectively by

$$\hat{\Psi}_{BS}(c,k,\lambda) = E[\Psi(c,k,\lambda) \mid x] = \int_\lambda \int_k \int_c \Psi(c,k,\lambda) \times \pi^*(c,k,\lambda \mid x)\, dc\, dk\, d\lambda, \tag{11}$$

$$\hat{\Psi}_{BL}(c,k,\lambda) = -\frac{1}{a}\ln\left[E\left(e^{-a\Psi(c,k,\lambda)} \mid x\right)\right] = -\frac{1}{a}\ln\left[\int_\lambda \int_k \int_c e^{-a\Psi(c,k,\lambda)} \times \pi^*(c,k,\lambda \mid x)\, dc\, dk\, d\lambda\right], \quad a \neq 0. \tag{12}$$

It is evident that, it is not possible to compute 11 and 12 analytically because it cannot be obtained in a simple closed form. Then, we propose using the MCMC method to compute Bayes estimates for $c, k,$ and λ by generating samples from the posterior distribution using the Metropolis-Hastings technique. A lot of papers dealt with MCMC techniques such as Gupta [16] and Wu and Gui [14], among others.

4.1 MCMC method

From Eq.(10), the conditional posterior density distributions of $c, k,$ and λ are given, respectively, by

$$\pi_1^*(c \mid k,\lambda,x) \propto c^{a_1-1} e^{-b_1 c} \prod_{i=1}^m x_i^c \left[\frac{c}{x_i} + \lambda\right] \left(1 + e^{\lambda x_i} x_i^c\right)^{-k(r_i+1)-1},$$

$$\pi_2^*(k \mid c,\lambda,x) \propto k^{m+a_2-1} e^{-b_2 k} \prod_{i=1}^m \left(1 + e^{\lambda x_i} x_i^c\right)^{-k(r_i+1)},$$

$$\pi_3^*(\lambda \mid c,k,x) \propto \lambda^{a_3-1} e^{-\lambda\left(b_3 - \sum_i^m x_i\right)} \prod_{i=1}^m \left[\frac{c}{x_i} + \lambda\right] \left(1 + e^{\lambda x_i} x_i^c\right)^{-k(r_i+1)-1}.$$

• The Metropolis-Hasting algorithm proceeds as follows:

1. Take the initial guess of $c, k,$ and λ, say $c^{(0)}, k^{(0)}$, and $\lambda^{(0)}$ respectively.
2. Set j=1.
3. Generate $c^{(j)}, k^{(j)}$, and $\lambda^{(j)}$ from $\pi_1^*(c \mid k,\lambda,x)$, $\pi_2^*(k \mid c,\lambda,x)$, and $\pi_3^*(\lambda \mid c,k,x)$ with $N\left(c^{(j-1)}, Var(c)\right)$, $N\left(k^{(j-1)}, Var(k)\right)$, and $N\left(c^{(j-1)}, Var(\lambda)\right)$ as normal proposal distribution, where Var(c), Var(k), and Var(λ) can be obtained from the main diagonal of the inverse Fisher information matrix.
4. Set j=j+1.
5. Repeat steps 3-4 N times.
6. Obtain the Bayesian estimates of $\Psi = \Psi(c,k,\lambda)$ under squared error loss function of

$$\widehat{\Psi}_{BS} = E[\Psi \mid x] = \frac{1}{N-M}\sum_{j=M+1}^{N} \Psi^{(j)}, \qquad M = burn-in.$$

The Bayes estimates of $\Psi = \Psi(c, k, \lambda)$, under LINEX loss function are given by

$$\widehat{\Psi}_{BL} = \frac{-1}{a}\ln\left[E\left(\sum_{j=M+1}^{N} exp(-a\Psi) \mid x\right)\right]$$
$$= \frac{-1}{a}\ln\left[\frac{1}{N-M}\sum_{j=M+1}^{N} exp\left(-a\Psi^{(j)}\right)\right].$$

5. SIMULATION STUDY AND DATA ANALYSIS

To evaluate the practical utility of the proposed estimation methodologies, the following two subsections will present their performance assessment. This will be achieved through a simulation study and subsequent analysis of a real data.

5.1 Simulation study

In this section, Monte Carlo simulations are conducted to compare between MLEs and Bayesian estimates of the MBXII parameters. Samples are generated under progressive Type-II censoring with two sample sizes $n = 25$ and 50, different numbers of failures m and different schemes. At first, we generate c, k and λ from the Gamma(a_1, b_1), Gamma(a_2, b_2) and Gamma (a_3, b_3) prior densities, respectively, with hyperparameters $a_1 = b_1 = 2$, $a_2 = b_2 = a_3 = b_3 = 3$. These generated values are:

Case 1: $c = 0.27358$, $k = 0.5$ and $\lambda = 0.4789$,

Case 2: $c = 2.2134$, $k = 0.61$, and $\lambda = 1{,}23$.

We generate 1000 random samples from the MBXII distribution for the previous parameter combinations. The average estimate (AE) and mean squared error (MSE) are the tools used to test the point estimate, but the lower interval (L) and upper interval (U) and then the average length estimate are those that are used to test the interval estimate. Also in the tables, BSEL denotes the Bayes estimates under the squared error loss function and BLINEX denotes the Bayes estimates under the LINEX loss function.

Tables 1-4 contain list of all the outcomes of the criteria quantities, where $\xi = 0.05$ is the significance level for all interval estimates. Moreover, for Bayesian estimation, the Metropolis-Hasting sampling size N is taken to be 11,000 abandoning the first 1000 iterations and we take two values for a $(a = 3, -3)$for the LINEX loss function. Where the scheme $(0^{*6}, 5^{*3}, 0^{*6}) = (0,0,0,0,0,0,5,5,5,0,0,0,0,0,0)$.

Table 1: AEs and MSEs of MLE and the Bayes estimates of $\boldsymbol{c} = \mathbf{0.27358}, \boldsymbol{k} = \mathbf{0.5}$ and $\boldsymbol{\lambda} = \mathbf{0.4789}$

n	m	scheme		MLE AE	MLE MSE	BSEL AE	BSEL MSE	BLINEX $a=3$ AE	BLINEX $a=3$ MSE	BLINEX $a=-3$ AE	BLINEX $a=-3$ MSE
30	**15**	(1^{*15})	$\hat{c}$	0.30562	0.01766	0.33971	0.01849	0.35824	0.02631	0.32342	0.01345
			$\hat{k}$	0.51911	0.04876	0.54002	0.02297	0.59314	0.03805	0.49867	0.01657
			$\hat{\lambda}$	0.65676	0.25859	0.64536	0.06111	0.85395	0.23969	0.54464	0.02436
		$(0^{*6}, 5^{*3}, 0^{*6})$	$\hat{c}$	0.31445	0.02096	0.34808	0.02126	0.36801	0.03246	0.33096	0.01499
			$\hat{k}$	0.54646	0.06243	0.55655	0.02938	0.61464	0.05145	0.51253	0.01991
			$\hat{\lambda}$	0.6442	0.2953	0.63965	0.06261	0.85173	0.24894	0.54005	0.02565
	25	$(0^{*10}, 1^{*5}, 0^{*10})$	$\hat{c}$	0.30188	0.01428	0.33559	0.01679	0.3535	0.02408	0.31989	0.01212
			$\hat{k}$	0.50184	0.03751	0.51666	0.01737	0.55907	0.02496	0.48203	0.01444
			$\hat{\lambda}$	0.6227	0.20298	0.60097	0.0418	0.75365	0.15216	0.525	0.01927
		$(0^{*12}, 5, 0^{*12})$	$\hat{c}$	0.30361	0.01399	0.33508	0.01557	0.35222	0.0217	0.31982	0.01147
			$\hat{k}$	0.50591	0.0394	0.51796	0.01868	0.56137	0.02702	0.4827	0.01539
			$\hat{\lambda}$	0.60809	0.2005	0.59857	0.04243	0.74758	0.14847	0.52226	0.01942
50	**25**	(1^{*25})	$\hat{c}$	0.29266	0.00754	0.31224	0.00807	0.32134	0.00978	0.30365	0.00673
			$\hat{k}$	0.51294	0.02922	0.52142	0.01609	0.55582	0.02225	0.49229	0.01323
			$\hat{\lambda}$	0.56542	0.09844	0.59395	0.04151	0.72324	0.11939	0.52094	0.02073
		$(0^{*10}, 5^{*5}, 0^{*10})$	$\hat{c}$	0.29459	0.00748	0.31382	0.00832	0.32308	0.0102	0.3051	0.00685
			$\hat{k}$	0.52483	0.0333	0.52983	0.01813	0.56463	0.02567	0.50045	0.01433
			$\hat{\lambda}$	0.58137	0.24365	0.59642	0.04634	0.72822	0.13985	0.52433	0.02308
	40	$(0^{*15}, 1^{*10}, 0^{*15})$	$\hat{c}$	0.28269	0.00686	0.23052	0.00764	0.31356	0.00916	0.29727	0.00644
			$\hat{k}$	0.50945	0.00686	0.51514	0.01436	0.54386	0.01824	0.49015	0.01257
			$\hat{\lambda}$	0.55691	0.11166	0.5683	0.0344	0.65738	0.08494	0.51424	0.0192
		$(0^{*19}, 5^{*2}, 0^{*19})$	$\hat{c}$	0.28917	0.00664	0.30757	0.00747	0.31591	0.00901	0.29969	0.00627
			$\hat{k}$	0.51414	0.02625	0.51722	0.01499	0.54567	0.01911	0.49232	0.01299
			$\hat{\lambda}$	0.5473	0.09207	0.56239	0.03325	0.64983	0.08142	0.50954	0.01924

Table 2: AEs and MSEs of MLE and the Bayes estimates of $c = 2.2134, k = 0.61$ and $\lambda = 1.23$

n	m	scheme		MLE AE	MLE MSE	BSEL AE	BSEL MSE	BLINEX $a=3$ AE	BLINEX $a=3$ MSE	BLINEX $a=-3$ AE	BLINEX $a=-3$ MSE
30	15	(1^{*15})	$\hat{c}$	2.0886	0.27271	1.95354	0.10107	2.22443	0.12102	1.82246	0.1647
			$\hat{k}$	0.76526	0.05681	0.53024	0.01392	0.56164	0.01229	0.50373	0.01729
			$\hat{\lambda}$	0.52561	0.56486	1.3078	0.00634	1.4101	0.03283	1.23724	0.00023
		$(0^{*6}, 5^{*3}, 0^{*6})$	$\hat{c}$	2.05759	0.19881	1.93559	0.09867	2.17038	0.08035	1.81561	0.16593
			$\hat{k}$	0.84988	0.11122	0.57779	0.01373	0.61595	0.01686	0.54608	0.01406
			$\hat{\lambda}$	0.50168	0.59204	1.3161	0.00763	1.42011	0.03647	1.24345	0.0003
	25	$(0^{*10}, 1^{*5}, 0^{*10})$	$\hat{c}$	2.19512	0.1716	1.97771	0.08773	2.19751	0.08212	1.85229	0.14429
			$\hat{k}$	0.91162	0.14406	0.61729	0.01317	0.64657	0.01726	0.59142	0.01135
			$\hat{\lambda}$	0.55691	0.53958	1.32432	0.00921	1.42956	0.04028	1.24966	0.00056
		$(0^{*12}, 5, 0^{*12})$	$\hat{c}$	2.28409	0.24516	2.01573	0.10551	2.26731	0.16891	1.87533	0.14245
			$\hat{k}$	0.88673	0.12573	0.60272	0.01133	0.63047	0.01416	0.57824	0.01049
			$\hat{\lambda}$	0.54451	0.53434	1.32146	0.00863	1.42603	0.0388	1.24765	0.00047
50	25	(1^{*25})	$\hat{c}$	2.16377	0.11221	1.95427	0.08698	2.12594	0.05148	1.84533	0.14489
			$\hat{k}$	0.86832	0.09937	0.57686	0.00798	0.60273	0.00849	0.55394	0.00889
			$\hat{\lambda}$	0.52355	0.55139	1.31484	0.00748	1.41779	0.03564	1.24303	0.00035
		$(0^{*10}, 5^{*5}, 0^{*10})$	$\hat{c}$	2.17358	0.1035	1.96836	0.08425	2.1381	0.05501	1.85848	0.13791
			$\hat{k}$	0.89898	0.14417	0.60339	0.01066	0.63263	0.01382	0.5779	0.00976
			$\hat{\lambda}$	0.56298	0.51939	1.32317	0.00892	1.42825	0.03964	1.2486	0.00049
	40	$(0^{*15}, 1^{*10}, 0^{*15})$	$\hat{c}$	2.33853	0.13951	2.06338	0.06515	2.24092	0.07748	1.93697	0.09983
			$\hat{k}$	0.90208	0.1383	0.60256	0.00677	0.62249	0.00783	0.5841	0.00658
			$\hat{\lambda}$	0.58483	0.56571	1.32599	0.00955	1.43083	0.04081	1.2513	0.00065
		$(0^{*19}, 5^{*2}, 0^{*19})$	$\hat{c}$	2.35334	0.1392	2.06957	0.07345	2.25042	0.09047	1.9422	0.10439
			$\hat{k}$	0.90015	0.12748	0.60994	0.00718	0.63022	0.00864	0.59115	0.00667
			$\hat{\lambda}$	0.60115	0.53282	1.3277	0.00993	1.43224	0.04144	1.25288	0.00074

Table 3: CIs and Credible intervals of $\boldsymbol{c = 0.27358}, \boldsymbol{k = 0.5}$ and $\boldsymbol{\lambda = 0.4789}$

$\boldsymbol{n}$	$\boldsymbol{m}$	scheme		Approximate Intervals L	Approximate Intervals U	Approximate Intervals length	Credible Intervals L	Credible Intervals U	Credible Intervals length
30	**15**	(1^{*15})	$\hat{c}$	0.08824	0.523	0.43476	0.16746	0.55205	0.38459
			$\hat{k}$	0.07704	0.96117	0.88412	0.26234	0.93035	0.66801
			$\hat{\lambda}$	0.27504	1.58857	1.86361	0.22288	1.34048	1.1176
		$(0^{*6}, 5^{*3}, 0^{*6})$	$\hat{c}$	0.09281	0.53609	0.44327	0.17295	0.56432	0.39136
			$\hat{k}$	0.09177	1.00116	0.90939	0.27163	0.95734	0.68571
			$\hat{\lambda}$	-0.29777	1.58617	1.88394	0.22514	1.33898	1.11384
	25	$(0^{*10}, 1^{*5}, 0^{*10})$	$\hat{c}$	-0.18386	0.78762	0.97148	0.1676	0.54269	0.37509
			$\hat{k}$	-0.406	1.40967	1.81567	0.26083	0.86926	0.60842
			$\hat{\lambda}$	-1.53982	2.78521	4.32503	0.24448	1.20212	0.95765
		$(0^{*12}, 5, 0^{*12})$	$\hat{c}$	0.09417	0.51306	0.41889	0.16735	0.54257	0.37522
			$\hat{k}$	0.10915	0.90267	0.79353	0.25995	0.87355	0.6136
			$\hat{\lambda}$	-0.15705	1.37322	1.53027	0.24102	1.20252	0.9615
50	**25**	(1^{*25})	$\hat{c}$	0.13461	0.4507	0.31609	0.18136	0.46349	0.28213
			$\hat{k}$	0.17404	0.85185	0.67781	0.28287	0.83695	0.55409
			$\hat{\lambda}$	0.03167	1.16251	1.19418	0.23573	1.16758	0.93185
		$(0^{*10}, 5^{*5}, 0^{*10})$	$\hat{c}$	0.13702	0.45217	0.31515	0.18224	0.46726	0.28502
			$\hat{k}$	0.18686	0.86281	0.67595	0.29108	0.84676	0.55567
			$\hat{\lambda}$	-0.06212	1.22485	1.28697	0.24453	1.16305	0.91852
	40	$(0^{*15}, 1^{*10}, 0^{*15})$	$\hat{c}$	0.13848	0.43411	0.29563	0.18035	0.44955	0.2692
			$\hat{k}$	0.20361	0.81529	0.61168	0.29259	0.80441	0.51182
			$\hat{\lambda}$	0.06252	1.05131	0.98879	0.26641	1.04683	0.78041
		$(0^{*19}, 5^{*2}, 0^{*19})$	$\hat{c}$	0.14067	0.43768	0.29701	0.1827	0.45097	0.26827
			$\hat{k}$	0.20725	0.82103	0.61378	0.29447	0.8041	0.50963
			$\hat{\lambda}$	0.05771	1.0369	0.97919	0.26468	1.03566	0.77098

Table 4: CIs and Credible intervals of $c = 2.2134, k = 0.61$ and $\lambda = 1.23$

n	m	scheme		Approximate Intervals L	Approximate Intervals U	Approximate Intervals length	Credible Intervals L	Credible Intervals U	Credible Intervals length
30	15	(1^{*15})	$\hat{c}$	0.60177	3.57544	2.97368	1.52166	2.75239	1.23073
			$\hat{k}$	-0.60757	2.13809	2.74565	0.30236	0.83145	0.52908
			$\hat{\lambda}$	-2.08503	3.13625	5.22128	1.0104	1.88036	0.86996
		$(0^{*6}, 5^{*3}, 0^{*6})$	$\hat{c}$	0.65741	3.45776	2.80035	1.5203	2.6952	1.1749
			$\hat{k}$	-0.64237	2.34213	2.9845	0.32887	0.90694	0.57807
			$\hat{\lambda}$	-2.00976	3.01311	5.02287	1.01066	1.88844	0.87778
	25	$(0^{*10}, 1^{*5}, 0^{*10})$	$\hat{c}$	0.91855	3.47169	2.55314	1.53102	2.72125	1.19022
			$\hat{k}$	-0.55104	2.37428	2.92533	0.38626	0.90351	0.51725
			$\hat{\lambda}$	-1.89237	3.00618	4.89856	1.0112	1.8949	0.8837
		$(0^{*12}, 5, 0^{*12})$	$\hat{c}$	0.96608	3.60209	2.63601	1.53994	2.80659	1.26665
			$\hat{k}$	-0.51079	2.28426	2.79505	0.37765	0.88302	0.50537
			$\hat{\lambda}$	-1.87992	2.96894	4.84887	1.01134	1.89317	0.88183
50	25	(1^{*25})	$\hat{c}$	0.95303	3.3745	2.42146	1.52973	2.6306	1.10087
			$\hat{k}$	0.61184	2.34848	2.96033	0.35909	0.84907	0.48998
			$\hat{\lambda}$	-2.02312	3.07022	5.09333	1.01124	1.88688	0.87564
		$(0^{*10}, 5^{*5}, 0^{*10})$	$\hat{c}$	1.00846	3.33869	2.33023	1.53581	2.64175	1.10595
			$\hat{k}$	-0.58087	2.37884	2.95971	0.37497	0.89025	0.51528
			$\hat{\lambda}$	-1.90569	3.03164	4.93733	1.01134	1.89333	0.88199
	40	$(0^{*15}, 1^{*10}, 0^{*15})$	$\hat{c}$	1.27725	3.39982	2.12257	1.56843	2.74723	1.1788
			$\hat{k}$	-0.43933	2.24349	2.68282	0.40287	0.83836	0.43549
			$\hat{\lambda}$	-1.73161	2.90126	4.63287	1.011173	1.89343	0.8817
		$(0^{*19}, 5^{*2}, 0^{*19})$	$\hat{c}$	1.29601	3.41066	2.11465	1.573	2.75958	2.75958
			$\hat{k}$	-0.42326	2.22355	2.64681	0.40842	0.8468	0.43838
			$\hat{\lambda}$	-1.71049	2.91278	4.62327	1.01214	1.89565	0.88352

5.2 Real data example

To demonstrate how the proposed method works in practice, we examined one data set as an example. These data represent the total annual rainfall (in inches) during January from 1880 to 1916 recorded at Los Angeles Civic Center (see the website of Los Angeles Almanac: www.laalmanac.com/weather/we08aa.htm), and further analyzed by Selim [17] and Wu and Gui [14]. The data is shown in Table 5 below. The Kolmogorov-Smirnov (K-S) goodness of fit test is used to determine the validity of the MBXII model in fitting this data. Kolmogorov-Smirnov test results show that K-S distance = 0.1072 and p-value = 0.8742. As a result, the MBXII model fits the data well. Furthermore, supporting this conclusion are the empirical CDF plot in Figure 1 and the Quantile-Quantile (Q-Q) plot. There are 37 values in the original data set, $n = 37$, we take $m = 30$ and $a = 1$ for the LINEX loss function. The censoring schemes that have been used are:

Scheme I: $r_1 = 4,\ r_j = 0,\ \text{if } j = 2,3,\ldots,m-1, \text{and } r_m = 3.$

Scheme II: $r_j = 1,\ \text{if } j = 1,2,3,\ldots,n-m,\ r_j = 0 \text{ if } j > n-m.$

We propose the progressive censored sample, as illustrated in Table 6. We use the same hyper-parameters for Bayesian estimation as those in Subsection 5.1. The estimates depend on progressive Type-II censoring can be found in Table 7. In Table 7, the results obtained with scheme I are represented by the values in the top cell and the values in the lower cell represent the outcomes that were reached using Scheme II. The proposed estimations for a given parameter have pretty similar values.

Table 5: A real data set recorded by Selim [17].

1.33	1.43	1.01	1.62	3.15	1.05
7.72	0.20	6.03	0.25	7.83	0.25
0.88	6.29	0.94	5.84	3.23	3.70
1.26	2.64	1.17	2.49	1.62	2.10
0.14	2.57	3.85	7.02	5.04	7.27
1.53	6.70	0.07	2.01	10.35	5.42
13.3					

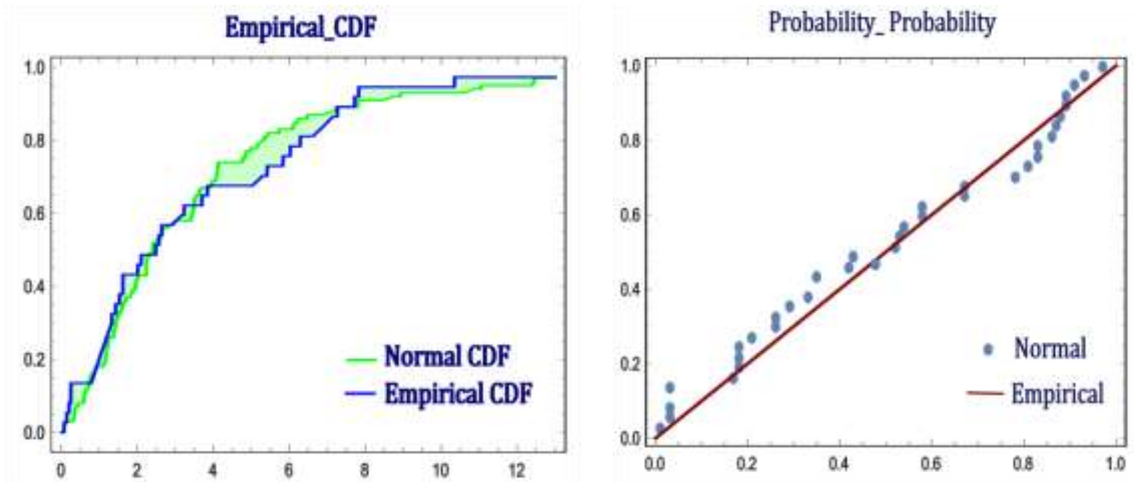

Figure 1: The K-S test for generated and empirical data for MBXII $(\boldsymbol{c}, \boldsymbol{k}, \boldsymbol{\lambda})$.

Table 6: The censored data set.

Scheme I												
i	1	2	3	4	5	6	7	8	9	10	11	12
$x_{i:30:37}$	0.07	0.14	0.2	0.25	0.88	0.94	1.05	1.17	1.26	1.33	1.43	1.53
r_i	4	0	0	0	0	0	0	0	0	0	0	0
i	13	14	15	16	17	18	19	20	21	22	23	24
$x_{i:30:37}$	1.62	2.01	2.49	2.57	2.64	3.15	3.23	3.7	3.85	5.04	5.42	5.84
r_i	0	0	0	0	0	0	0	0	0	0	0	0
i	25	26	27	28	29	30						
$x_{i:30:37}$	6.03	6.29	6.70	7.02	7.27	7.72						
r_i	0	0	0	0	0	3						
Scheme II												
i	1	2	3	4	5	6	7	8	9	10	11	12
$x_{i:30:37}$	0.07	0.14	0.2	0.25	0.88	0.94	1.01	1.17	1.26	1.33	1.43	1.53
r_i	1	1	1	1	1	1	1	0	0	0	0	0
i	13	14	15	16	17	18	19	20	21	22	23	24
$x_{i:30:37}$	1.62	2.01	2.49	2.57	2.64	3.15	3.23	3.7	3.85	5.42	5.84	6.03
r_i	0	0	0	0	0	0	0	0	0	0	0	0
i	25	26	27	28	29	30						
$x_{i:30:37}$	6.29	6.70	7.02	7.83	10.35	13.3						
r_i	0	0	0	0	0	0						

Table 7: The MLE and Bayes estimates for real data example

Scheme I	MLE	BSEL	BLINEX
$\hat{c}$	0.76763	0.83385	0.87449
$\hat{k}$	0.16698	0.2627	0.26648
$\hat{\lambda}$	1.30983	0.8835	0.98298
Scheme II	MLE	BSEL	BLINEX
$\hat{c}$	0.79474	0.90292	0.95465
$\hat{k}$	0.16954	0.26756	0.27094
$\hat{\lambda}$	1.41168	0.91187	1.0057

CONCLUSION

The purpose of this research is to look into parameter point estimation and interval estimation. Maximum likelihood and Bayesian estimation procedures are used to estimate the unknown parameters of the MBXII distribution based on progressive Type-II censored sample. We also use the observed Fisher information matrix to calculate the approximate confidence intervals. We apply the MCMC method to get the processes for Bayesian estimation under different loss functions. After that, the pertinent credible intervals are computed in turn. Based on the results listed in Tables 1 and 2, one can observe the following:

1. The MSE associated with Bayes estimates are less than the MLE_s.
2. The MSE of Bayes estimates and MLEs of unknown parameters c, k, and λ decreases as the values of n and m increase.

From the results in Tables 3 and 4,

1. When n or m is increased, the average length of credible intervals and approximate confidence intervals decrease.
2. The average length of credible intervals is relatively smaller than the average length of approximate confidence intervals when n and m are fixed.

According to simulation results, it is discovered that Bayesian estimation performs better than MLEs in the vast majority of circumstances. Finally, a real data set is provided to confirm the proposed estimates.

REFERENCES

[1] I. W. Burr, Cumulative frequency functions, The Annals of mathematical statistics, 13(2) (1942), 215–232.

[2] E. K. Al-Hussaini, Z. F. Jaheen, Bayesian estimation of the parameters, reliability and failure rate functions of the Burr Type XII failure model, Journal of Statistical Computation and Simulation, 41(1-2) (1992), 31–40.

[3] S. F. Wu, C. C. Wu, Y. L. Chen, Y. R. Yu, and Y. P. Lin, Interval estimation of a two-parameter Burr-XII distribution under progressive censoring, Statistics, 44(1) (2010), 77–88.

[4] A. A. Soliman, N. A. Abou-Elheggag, A. H. A. Ellah, A. A. Modhesh, Bayesian and non-Bayesian inferences of the Burr-XII distribution for progressive first-failure censored data, Metron, 70(2012), 1–25.

[5] F. Jamal, C. Chesneau, M. A. Nasir, A. Saboor, E. Altun, M. A. Khan, On a modified Burr XII distribution having flexible hazard rate shapes, Mathematica Slovaca, 70(1) (2020), 193–212.

[6] R. Aggarwala, N. Balakrishnan, Some properties of progressive censored order statistics from arbitrary and uniform distributions with applications to inference and simulation, Journal of Statistical Planning and Inference, 70(1) (1998), 35–49.

[7] N. Balakrishnan, Progressive censoring methodology: an appraisal, Test, 16(2007), 211–259.

[8] N. Balakrishnan, E. Cramer, The Art of Progressive Censoring, Statistics for Industry and Technology, (2014).

[9] S. J. Wu, Estimations of the parameters of the Weibull distribution with progressively censored data, Journal of the Japan Statistical Society, 32(2) (2002), 155–163.

[10] S. J. Wu, C. T. Chang, Inference in the Pareto distribution based on progressive Type-II censoring with random removals, Journal of Applied Statistics, 30(2) (2003), 163–172.

[11] A. Asgharzadeh, Point and interval estimation for a generalized Logistic distribution under progressive Type-II censoring, Communications in Statistics - Theory and Methods, 35(9) (2006), 1685–1702.

[12] S. J. Wu, Y. J. Chen, C. T. Chang, Statistical inference based on progressively censored samples with random removals from the Burr Type XII distribution, Journal of Statistical Computation and Simulation, 77(1) (2007), 19–27.

[13] S. B. Kang, Y. S. Cho, J. T. Han, Estimation for the half logistic distribution under progressive Type-II censoring, Communications of the Korean Statistical Society, 15(2008), 815–823.

[14] M. Wu, W. Gui, Estimation and prediction for Nadarajah-Haghighi distribution under progressive Type-II censoring, Symmetry, 13(6) (2021), 999.

[15] M. Abu-Moussa, A. Abd-Elfattah, E. Hafez, Estimation of stress-strength parameter for Rayleigh distribution based on progressive Type-II censoring, Information Sciences Letters, 10(1) (2021), 101–110.

[16] S. Upadhyay, A. Gupta, A Bayes analysis of modified Weibull distribution via Markov chain Monte Carlo simulation, Journal of Statistical Computation and Simulation, 80(3) (2010), 241–254.

[17] M. A. Selim, Estimation and prediction for Nadarajah-Haghighi distribution based on record values, Pakistan Journal of Statistics, 34(1) (2018), 77–90.