Electronic Journal of Mathematical Analysis and Applications LOGO.png
Vol. 12(2) July 2024, No.10.
ISSN: 2090-729X (online)
ISSN: 3009-6731(print)
http://ejmaa.journals.ekb.eg/

THE HYERS-ULAM STABILITY OF AN ADDITIVE AND QUADRATIC FUNCTIONAL EQUATION IN 2-BANACH SPACE

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Abstract. In 1940, the stability problem of functional equations was arose due to a question of Stanisaw Ulam concerning the stability of group homomorphisms. Significant work was done by Donald H. Hyers about HYERS-ULAM STABILITY and obtained a partial affirmative answer to the question of Ulam in the context of banach spaces in the case of additive mappings. In 1978, T. M. Rassias expanded Hyers's theorem for mappings between Banach spaces by considering an unbounded Cauchy difference subject to a continuity condition upon the mapping. After that, Many Researchers had studied about Hyers-Ulam stability of an additive quadratic type functional equations. In this research article , the Hyers-Ulam stability of an additive quadratic type functional equation was discussed and obtained the generalization of Hyers-Ulam stability of an additive quadratic type functional equation

$$f(x+ay)+af(x-y)=f(x-ay)+af(x+y)$$

for any integer a with $a \neq -1, 0, 1$ in 2-Banach space.

1. Introduction

Stability of a function for a function from normed space to banach space has been studied by Hyers [4]. He has proved that for a function $f : X \longrightarrow Y$, a function between normed space X and Banach space Y satisfying

$$\|f(x+y)-f(x)-f(y)\| \leq \delta$$

for each $x, y \in X$ and $\delta > 0$. Then there exists a unique additive function $T : X \longrightarrow Y$ such that

$$\|f(x)-T(x)\| \leq \delta$$

1991 *Mathematics Subject Classification.* 46K05, 46H05.
Key words and phrases. Quadratic function, Additive function, Hyers-Ulam stability, 2-Banach space.
Submitted March 24, 2024. Revised June 27, 2024.

for each $x \in X$. It is a positive answer to a problem raised by Ulam [13] for a functional equation on a metric group. The stability of functional equations for functions from normed space to Banach space have been extensively studied by many authors. The terminology generalized Hyers-Ulam stability originates from these historical backgrounds.
The functional equation

$$f(x+y)+f(x-y)=2f(x)+2f(y) \tag{1.1}$$

is called a quadratic functional equation and every solution of the quadratic functional equation (1.1) is said to be a quadratic function. A stability problem for the quadratic functional equation (1.1) was solved by many authors [2]. In this paper, we investigate the Hyers-Ulam Stability of the following functional equations,

$$f(x+2y)+2f(x-y)=f(x-2y)+2f(x+y), \tag{1.2}$$

$$f(x+ay)+af(x-y)=f(x-ay)+af(x+y), \tag{1.3}$$

for any fixed integer a with $a \neq -1, 0, 1$, introduced by Jun and Kim [6], for a function from 2-normed space (normed space) to 2-Banach space. In fact, It is shown that in [6] that these two functional equations (1.2) and (1.3) are equivalent and such f satisfies $f(x) = B(x,x) + A(x) + f(0)$, for each $x \in E_1$, where $B : E_1 \times E_2 \longrightarrow E_2$ is symmetric biadditive, and $A : E_1 \longrightarrow E_2$ is additive. In the 1960s, S. Gähler [3] introduced the concept of 2-normed spaces.

Definition 1.1. *Let X be a linear space over $\mathbb{R}$ with $\dim X > 1$ and let $\|\cdot,\cdot\| : X \times X \longrightarrow \mathbb{R}$ be a function satisfying the following properties:*

(1) $\|x,y\| = 0$ *if and only if x and y are linearly dependent,*
(2) $\|x,y\| = \|y,x\|$,
(3) $\|ax,y\| = |a|\|x,y\|$,
(4) $\|x,y+z\| \leq \|x,y\| + \|x,z\|$

for each $x,y,z \in X$ and $a \in \mathbb{R}$. Then the function $\|\cdot,\cdot\|$ is called a 2-norm on X and $(X,\|\cdot,\cdot\|)$ is called a 2-normed space.

We introduce a basic property of 2-normed spaces as follows. Let $(X,\|\cdot,\cdot\|)$ be a linear 2-normed space, $x \in X$ and $\|x,y\| = 0$ for each $y \in X$. Suppose $x \neq 0$, since $\dim X > 1$, choose $y \in X$ such that $\{x,y\}$ is linearly independent so we have $\|x,y\| \neq 0$, which is a contradiction. Therefore, we have the following lemma.

Lemma 1.2. *Let $(X,\|\cdot,\cdot\|)$ be a 2-normed space. If $x \in X$ and $\|x,y\| = 0$, for each $y \in X$, then $x = 0$.*

Let $(X,\|\cdot,\cdot\|)$ be a 2-normed space. For $x,z \in X$, let $p_z(x) = \|x,z\|$, $x \in X$. Then for each $z \in X$, p_z is a real-valued function on X such that $p_z(x) = \|x,z\| \geq 0$, $p_z(\alpha x) = |\alpha|\|x,z\| = |\alpha|p_z(x)$ and
$p_z(x+y) = \|x+y,z\| = \|z,x+y\| \leq \|z,x\|+\|z,y\| = \|x,z\|+\|y,z\| = p_z(x)+p_z(y)$, for each $\alpha \in \mathbb{R}$ and all $x,y \in X$. Thus p_z is a a semi-norm for each $z \in X$. For $x \in X$, let $\|x,z\| = 0$, for each $z \in X$. By Lemma 1.2, $x = 0$. Thus for $0 \neq x \in X$, there is $z \in X$ such that $p_z(x) = \|x,z\| \neq 0$. Hence the family $\{p_z(x) : z \in X\}$ is a separating family of semi-norms. It will give us locally convex topology on X which makes X to be a locally convex topological vector space.

Definition 1.3. *A sequence* $\{x_n\}$ *in a* 2*-normed space* X *is called a* 2*-Cauchy sequence if*

$$\lim_{m,n\to\infty} \|x_n - x_m, x\| = 0$$

for each $x \in X$.

Definition 1.4. *A sequence* $\{x_n\}$ *in a* 2*-normed space* X *is called a* 2*-convergent sequence if there is an* $x \in X$ *such that*

$$\lim_{n\to\infty} \|x_n - x, y\| = 0$$

for each $y \in X$. *If* $\{x_n\}$ *converges to* x, *we write* $\lim_{n\to\infty} x_n = x$.

Definition 1.5. *We say that a* 2*-normed space* $(X, \|\cdot,\cdot\|)$ *is a* 2*-Banach space if every* 2*-Cauchy sequence in* X *is* 2*-convergent in* X.

By using (2) and (4) of Definition 1.1 one can see that $\|\cdot,\cdot\|$ is continuous in each component. More precisely for a convergent sequence $\{x_n\}$ in a 2-normed space X,

$$\lim_{n\to\infty} \|x_n, y\| = \Big\| \lim_{n\to\infty} x_n, y \Big\|$$

for each $y \in X$.

2. Stability of a functional equation for functions $f : (X, \|\cdot\|) \longrightarrow (X, \|\cdot,\cdot\|)$

Throughout this section, consider X a real normed linear space. We also consider that there is a 2-norm on X which makes $(X, \|\cdot,\cdot\|)$ a 2-Banach space. For a function $f : (X, \|\cdot\|) \longrightarrow (X, \|\cdot,\cdot\|)$, define $D_f : X \times X \longrightarrow X$ by

$$D_f(x, y) = f(x + ay) + af(x - y) - f(x - ay) + af(x + y) \tag{2.1}$$

for each $x, y \in X,\ a \neq -1, 0, 1$.

Theorem 2.1. *Let* $\varepsilon \geq 0,\ 0 < p < 2,\ r > 0$ *and* $f : X \longrightarrow X$ *be a mapping satisfying*

$$\|D_f(x, y), z\| \leq \varepsilon(\|x\|^p + \|y\|^p)\|z\|^r \tag{2.2}$$

for each $x, y, z \in X$. *Then there exists a unique quadratic mapping* $Q : X \longrightarrow X$ *which satisfies* (1.3) *and*

$$\Big\| \frac{f(x) + f(-x)}{2} - Q(x) - f(0), z \Big\| \leq \varepsilon \|x\|^p\ \|z\|^r \frac{|a|^p + 2|a| + 1}{2^p(|a|^2 - |a|^p)} \tag{2.3}$$

for each $x, z \in X$.

Proof. Let $f_1 : X \longrightarrow X$ be a function defined by

$$f_1(x) = \frac{1}{2}[f(x) + f(-x)] - f(0) \tag{2.4}$$

$for\ each\ x \in X.$ Then $f_1(0) = 0,\ f_1(x) = f_1(-x),\ for\ each\ x \in X.$ Also for $x, y \in X$,

$$\begin{aligned} \|D_{f_1}(x, y), z\| &= \|f_1(x + ay) + af_1(x - y) - f_1(x - ay) - af_1(x + y), z\| \\ &\leq \varepsilon(||x||^p + ||y||^p)||z||^r \end{aligned} \tag{2.5}$$

for each $z \in X$. Letting $y = x$ in (2.5), we get

$$\|f_1((a + 1)x) - f_1((a - 1)x) - af_1(2x), z\| \leq 2\varepsilon||x||^p||z||^r \tag{2.6}$$

for each $z \in X$. Replacing x by ay in (2.5), we get

$$\|f_1(2ay) + af_1((a-1)y) - af_1((a+1)y), z\| \leq \varepsilon[|a|^p + 1]||y||^p||z||^r \tag{2.7}$$

for each $z \in X$. Replacing y by x in (2.7), we get

$$\|f_1(2ax) + af_1((a-1)x) - af_1((a+1)x), z\| \leq \varepsilon[|a|^p + 1]||x||^p||z||^r \tag{2.8}$$

for each $z \in X$. Multiplying $|a|$ on both sides of (2.6) and adding to (2.8), for $x \in X$, we get

$$\|f_1(2ax) - a^2 f_1(2x), z\| \leq 2|a|\varepsilon||x||^p||z||^r + \varepsilon[|a|^p + 1]||x||^p||z||^r \tag{2.9}$$

for each $z \in X$. Replacing x by $\frac{x}{2}$ in (2.9), we get

$$\left\|\frac{f_1(ax)}{a^2} - f_1(x), z\right\| \leq \frac{2 \cdot 2^{-p}\varepsilon}{|a|}||x||^p||z||^r + \varepsilon\frac{2^{-p}}{|a|^2}[|a|^p + 1]||x||^p||z||^r \tag{2.10}$$

for each $z \in X$. Replacing x by ax in (2.10), we get

$$\left\|\frac{f_1(a^2x)}{a^2} - f_1(ax), z\right\| \leq \frac{2 \cdot 2^{-p}\varepsilon|a|^p}{|a|}||x||^p||z||^r + \varepsilon\frac{2^{-p}}{|a|^2}[|a|^{2p} + |a|^p]||x||^p||z||^r \tag{2.11}$$

for each $x \in X$. By (2.10) and (2.11), for $x \in X$, we get

$$\begin{aligned}
&\left\|\frac{f_1(a^2x)}{a^4} - f_1(x), z\right\| \\
&\leq \left\|\frac{f_1(a^2x)}{a^4} - \frac{f_1(ax)}{a^2}, z\right\| + \left\|\frac{f_1(ax)}{a^2} - f_1(x), z\right\| \\
&\leq \frac{2 \cdot 2^{-p}\varepsilon}{|a|}\Big[1 + \frac{|a|^p}{|a|^2}\Big]||x||^p||z||^r + \frac{\varepsilon 2^{-p}}{|a|^2}\Big[(1 + |a|^p) + (|a|^p + |a|^{2p})\frac{1}{|a|^2}\Big]||x||^p||z||^r
\end{aligned}$$

for each $x, z \in X$. For $x \in X$, by using mathematical induction on n in (2.12), we get

$$\begin{aligned}
\left\|\frac{f_1(a^nx)}{a^{2n}} - f_1(x), z\right\| &\leq \frac{2 \cdot 2^{-p}\varepsilon}{|a|}||x||^p||z||^r\frac{1 - |a|^{(p-2)n}}{1 - |a|^{p-2}} \\
&\quad + \varepsilon\frac{2^{-p}}{|a|^2}||x||^p||z||^r\Big[\frac{1 - |a|^{(p-2)n}}{1 - |a|^{p-2}} + \frac{|a|^p(1 - |a|^{(p-2)n})}{1 - |a|^{p-2}}\Big]
\end{aligned} \tag{2.12}$$

for each $z \in X$. For $x \in X$, $m, n \in \mathbb{N}$, we get

$$\begin{aligned}\Big\|\frac{f_1(a^m x)}{a^{2m}} - \frac{f_1(a^n x)}{a^{2n}}, z\Big\| &= \frac{1}{|a|^{2n}}\Big\|\frac{f_1(a^{m-n}a^n x)}{a^{2(m-n)}} - f_1(a^n x), z\Big\| \\ &\le \frac{1}{|a|^{2n}}\frac{2\cdot 2^{-p}\varepsilon}{|a|}\|a^n x\|^p||z||^r \sum_{j=0}^{m-n-1}|a|^{(p-2)j} \\ &\quad + \frac{1}{|a|^{2n}}\frac{\varepsilon 2^{-p}}{|a|^2}\|a^n x\|^p||z||^r \sum_{j=0}^{m-n-1}[|a|^{(p-2)j} + |a|^{(p-2)j+p}] \\ &= \frac{2\cdot 2^{-p}\varepsilon}{|a|}||x||^p||z||^r \frac{|a|^{(p-2)n}(1-|a|^{(p-2)(m-n)})}{1-|a|^{p-2}} \\ &\quad + \varepsilon\frac{2^{-p}}{|a|^2}||x||^p||z||^r\Big[\Big(\frac{|a|^{(p-2)n}(1-|a|^{(p-2)(m-n)})}{1-|a|^{p-2}}\Big) \\ &\quad + \Big(\frac{|a|^{(p-2)n+p}(1-|a|^{(p-2)(m-n)})}{1-|a|^{p-2}}\Big)\Big]\end{aligned}$$

for each $z \in X$.Thus for $x \in X$ and since $p-2<0$, $\Big\|\frac{f_1(a^m x)}{a^{2m}} - \frac{f_1(a^n x)}{a^{2n}}, z\Big\| \longrightarrow 0$ as $m, n \longrightarrow \infty$, each $z \in X$. Therefore $\{\frac{f_1(a^n x)}{a^{2n}}\}$ is a 2-Cauchy sequence in X, for each $x \in X$, for each $x \in X$. Since X is a 2-Banach space, $\{\frac{f_1(a^n x)}{a^{2n}}\}$ 2-converges, for each $x \in X$. Define $Q : X \longrightarrow X$ as

$$Q(x) := \lim_{n\to\infty}\frac{f_1(a^n x)}{a^{2n}}$$

for each $x \in X$. Also, by (2.12), we have for $x \in X$,

$$\|Q(x) - f_1(x), z\| \le \varepsilon||x||^p||z||^r\frac{|a|^p + 2|a| + 1}{2^p(|a|^2 - |a|^p)} \tag{2.13}$$

for each $z \in X$. Therefore for $x \in X$,

$$\Big\|\frac{f(x)+f(-x)}{2} - Q(x) - f(0), z\Big\| \le \varepsilon||x||^p||z||^r\frac{|a|^p + 2|a| + 1}{2^p(|a|^2 - |a|^p)}$$

for each $z \in X$. Now for $x, y \in X$,

$$\begin{aligned}\|D_Q(x,y), z\| &= \lim_{n\to\infty}\frac{1}{a^{2n}}\|D_{f_1}(a^n x, a^n y), z\| \\ &\le \lim_{n\to\infty}\frac{1}{a^{2n}}[\|a^n x\|^p + \|a^n y\|^p]||z||^r \\ &= \lim_{n\to\infty}|a|^{(p-2)n}[||x||^p + ||y||^p]||z||^r \\ &= 0\end{aligned}$$

Therefore for $x, y \in X$, $\|D_Q(x,y), z\| = 0$, for each $z \in X$. Therefore $D_Q(x,y) = 0$, for each $x, y \in X$. Therefore Q is a quadratic. Suppose Q' is another quadratic function satisfying (2.3) and (1.3). Since Q and Q' are quadratic, for $x \in X$,

$Q(a^n x) = a^{2n}Q(x)$, $Q'(a^n x) = a^{2n}Q'(x)$. For each n and for $x \in X$,

$$\begin{aligned}
\|Q(x) - Q'(x), z\| &= \frac{1}{a^{2n}}\|Q(a^n x) - Q'(a^n x), z\| \\
&\le \frac{1}{a^{2n}}[\|Q(a^n x) - f_1(a^n x), z\| + \|f_1(a^n x) - Q'(a^n x), z\|] \\
&\le \frac{2}{a^{2n}}\varepsilon\|a^n x\|^p||z||^r \frac{|a|^p + |a| + 1}{2^p(|a|^2 - |a|^p)} \\
&= 2|a|^{(p-2)n}\varepsilon||x||^p||z||^r \frac{|a|^p + |a| + 1}{2^p(|a|^2 - |a|^p)}
\end{aligned}$$

Thus for $x \in X$, $\|Q(x) - Q'(x), z\| \longrightarrow 0$ as $n \to \infty$, for each $z \in X$. Therefore $\|Q(x) - Q'(x), z\| = 0$, for each $z \in X$. Therefore $Q(x) = Q'(x)$, for each $x \in X$. □

Theorem 2.2. *Let $\varepsilon \ge 0, r > 0, p > 2$ and $f : X \longrightarrow X$ be a mapping satisfying*

$$\|D_f(x, y), z\| \le \varepsilon(\|x\|^p + \|y\|^p)\|z\|^r \tag{2.14}$$

for each $x, y, z \in X$. Then there exists a unique quadratic mapping $Q : X \longrightarrow X$ which satisfies (1.3) *and*

$$\left\|\frac{f(x) + f(-x)}{2} - Q(x) - f(0), z\right\| \le \varepsilon\|x\|^p \ \|z\|^r \frac{|a|^p + 2|a| + 1}{2^p(|a|^p - |a|^2)} \tag{2.15}$$

for each $x, z \in X$.

Proof. Proof is similar to that of the proof of the Theorem 2.1. □

Theorem 2.3. *Let $\varepsilon \ge 0, r > 0, 0 < p < 1$ and $f : X \longrightarrow X$ be a mapping satisfying*

$$\|D_f(x, y), z\| \le \varepsilon(\|x\|^p + \|y\|^p)\|z\|^r \tag{2.16}$$

for each $x, y, z \in X$. Then there exists a unique additive mapping $A : X \longrightarrow X$ satisfying (1.3) *and*

$$\left\|\frac{f(x) - f(-x)}{2} - A(x), z\right\| \le \frac{\varepsilon\|x\|^p \ \|z\|^r}{2(|a| - |a|^p)} \tag{2.17}$$

for each $x, z \in X$.

Proof. Let $f_2 : X \longrightarrow X$ be a function defined by

$$f_2(x) = \frac{1}{2}[f(x) - f(-x)]$$

for each $x \in X$. Then $f_2(0) = 0, f_2(-x) = -f_2(x)$ and for $x, y \in X$

$$\begin{aligned}
\|D_{f_2}(x, y), z\| &= \|f_2(x + ay) + af_2(x - y) - f_2(x - ay) - af_2(x + y), z\| \\
&\le \varepsilon(||x||^p + ||y||^p)||z||^r
\end{aligned} \tag{2.18}$$

for each $z \in X$. Putting $x = 0$ in (2.18), for $y \in Y$, we get,

$$\|2f_2(ay) - 2af_2(y), z\| \le \varepsilon||y||^p||z||^r$$

for each $z \in X$. Therefore for $y \in X$,

$$\|f_2(ay) - af_2(y), z\| \le \frac{\varepsilon}{2}||y||^p||z||^r \tag{2.19}$$

for each $z \in X$. Replacing y by x in (2.19), we get

$$\|f_2(ax) - af_2(x), z\| \le \frac{\varepsilon}{2}||x||^p||z||^r \tag{2.20}$$

for each $z \in X$. Therefore for $x \in X$,

$$\left\|\frac{f_2(ax)}{a} - f_2(x), z\right\| \leq \frac{\varepsilon}{2|a|}||x||^p||z||^r \tag{2.21}$$

for each $z \in X$. Replacing x by ax in (2.21), we get

$$\left\|\frac{f_2(a^2x)}{a} - f_2(ax), z\right\| \leq \frac{\varepsilon}{2|a|}|a|^p||x||^p||z||^r \tag{2.22}$$

for each $z \in X$. Now by (2.21) and (2.22), for $x \in X$, we get

$$\|\frac{f_2(a^2x)}{a^2} - f_2(x), z\| \leq \frac{\varepsilon}{2|a|}\Big[1 + \frac{|a|^p}{|a|}\Big]||x||^p||z||^r$$

for each $z \in X$. By using induction on n, for $x \in X$,

$$\left\|\frac{f_2(a^nx)}{a^n} - f_2(x), z\right\| \leq \frac{\varepsilon}{2|a|}||x||^p||z||^r\frac{1 - |a|^{(p-1)n}}{1 - |a|^{(p-1)}} \tag{2.23}$$

for each $z \in X$. For $m, n \in \mathbb{N}$, by (2.23), for $x \in X$,

$$\begin{aligned}\left\|\frac{f_2(a^mx)}{a^m} - \frac{f_2(a^nx)}{a^n}, z\right\| &= \frac{1}{|a|^n}\left\|\frac{f_2(a^{(m-n)}a^nx)}{a^{(m-n)}} - f_2(a^nx), z\right\| \\ &\leq \frac{\varepsilon}{2|a|}||x||^p||z||^r|a|^{(p-1)n}\frac{1 - |a|^{(p-1)(m-n)}}{1 - |a|^{(p-1)n}}\end{aligned}$$

for each $z \in X$. Therefore for $x \in X$, $\left\|\frac{f_2(a^mx)}{a^m} - \frac{f_2(a^nx)}{a^n}, z\right\| \longrightarrow 0$ as $m, n \to \infty$. Therefore $\{\frac{f_2(a^mx)}{a^m}\}$ is a 2-Cauchy sequence in X, for each $x \in X$. Since X is a 2-Banach space, $\{\frac{f_2(a^mx)}{a^m}\}$ 2-converges, for each $x \in X$. Define $A : X \longrightarrow X$ as

$$A(x) := \lim_{n\to\infty}\frac{f_2(a^nx)}{a^n}$$

for each $x \in X$. Then for $x \in X$, $A(0) = 0$, $A(-x) = -A(x)$. Also by (2.23), for $x \in X$, we get

$$\left\|\frac{f(x) - f(-x)}{2} - A(x), z\right\| \leq \frac{\varepsilon||x||^p||z||^r}{2(|a| - |a|^p)}$$

for each $z \in X$. Now for $x, y \in X$.

$$\begin{aligned}\|D_A(x, y), z\| &= \lim_{n\to\infty}\frac{1}{|a|^n}\|D_{f_2}(a^nx, a^ny), z\| \\ &\leq \lim_{n\to\infty}\frac{1}{|a|^n}\varepsilon[||a^nx\|^p + \|a^ny\|^p]||z||^r \\ &= \lim_{n\to\infty}|a|^{(p-1)n}\varepsilon[||x||^p + ||y||^p]||z||^r\end{aligned}$$

for each $z \in X$. Therefore for $x, y \in X$, $\|D_A(x, y), z\| \longrightarrow 0$ as $n \to \infty$, for each $z \in X$. Therefore $D_A(x, y) = 0$, for each $x, y \in X$. Uniqueness of A is as in the proof of the Theorem 2.1. □

Theorem 2.4. *Let* $\varepsilon \geq 0, r > 0, p > 1$ *and* $f : X \longrightarrow X$ *be a mapping satisfying*

$$\|D_f(x, y), z\| \leq \varepsilon(\|x\|^p + \|y\|^p)\|z\|^r \tag{2.24}$$

for each x, y, z *in* X. *Then there exists a unique additive mapping* $A : X \longrightarrow X$ *which satisfies* (1.3) *and*

$$\left\| \frac{f(x) - f(-x)}{2} - A(x), z \right\| \leq \frac{\varepsilon \|x\|^p \ \|z\|^r}{2(|a|^p - |a|)} \tag{2.25}$$

for each $x, z \in X$.

Proof. Proof is similar to that of the proof of the Theorem 2.3. □

Corollary 2.5. *Let* $\varepsilon \geq 0,\ 0 < p < 1,\ r > 0$ *and* $f : X \longrightarrow X$ *be a mapping satisfying*

$$\|D_f(x, y), z\| \leq \varepsilon(\|x\|^p + \|y\|^p)\|z\|^r \tag{2.26}$$

for each $x, y, z \in X$. *Then there exists unique additive mapping* $A : X \longrightarrow X$ *and unique quadratic mapping* $Q : X \longrightarrow X$ *which satisfy* (1.3) *and*

$$\|f(x) - A(x) - Q(x) - f(0), z\| \leq \varepsilon ||x||^p ||z||^r \left[\frac{|a|^p + 2|a| + 1}{2^p(|a|^2 - |a|^p)} + \frac{1}{2(|a| - |a|^p)} \right] \tag{2.27}$$

for each $x, z \in X$.

Corollary 2.6. *Let* $\varepsilon \geq 0,\ r > 0,\ p > 2$ *and* $f : X \longrightarrow X$ *be a mapping which satisfies*

$$\|D_f(x, y), z\| \leq \varepsilon(\|x\|^p + \|y\|^p)\|z\|^r \tag{2.28}$$

for each $x, y, z \in X$. *Then there exist unique additive mapping* $A : X \longrightarrow X$ *and unique quadratic mapping* $Q : X \longrightarrow X$ *which satisfy* (1.3) *and the inequality*

$$\|f(x) - Q(x) - A(x) - f(0), z\| \leq \varepsilon ||x||^p ||z||^r \left[\frac{|a|^p + 2|a| + 1}{2^p(|a|^2 - |a|^p)} + \frac{1}{2(|a| - |a|^p)} \right] \tag{2.29}$$

for each $x, z \in X$.

3. **Conclusion**

We have proved the generalized Hyers-Ulam stability of the additive quadratic type functional equation

$$f(x + ay) + af(x - y) = f(x - ay) + af(x + y)$$

for any integer a with $a \neq -1, 0, 1$ in 2-Banach space. We hope that this research work is a further improvement in the field of functional equations.

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