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ORIGINAL ARTICLE

# Necessity and sufficiency for hypergeometric functions to be in a subclass of analytic functions

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**Abstract** The purpose of this paper is to introduce necessary and sufficient condition of (Gaussian) hypergeometric functions to be in a subclass of uniformly starlike and uniformly convex functions. Operators related to hypergeometric functions are also considered. Some of our results correct previously known results.

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## 1. Introduction

Let $\mathcal{A}$ denote the class of functions $f(z)$ of the form:

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n, \tag{1.1}$$

which are analytic in the open unit disc $\mathbb{U} = \{z : z \in \mathbb{C} \text{ and } |z| < 1\}$, and let $\mathcal{S}$ be the subclass of all functions in $\mathcal{A}$, which are univalent. Let $g(z) \in \mathcal{A}$, be given by

$$g(z) = z + \sum_{n=2}^{\infty} g_n z^n, \tag{1.2}$$

then, the integral convolution of two power series $f(z)$ and $g(z)$ is given by (see [1]):

$$(f \circledast g)(z) = z + \sum_{n=2}^{\infty} \frac{a_n g_n}{n} z^n = (g \circledast f)(z). \tag{1.3}$$

Let $\mathcal{S}^*(\alpha)$ and $\mathcal{K}(\alpha)$ denote the subclasses of starlike and convex functions of order $\alpha$, respectively. We note that $\mathcal{S}^*(0) = \mathcal{S}^*$ and $\mathcal{K}(0) = \mathcal{K}$, the subclasses of starlike and convex functions (see, for example, Srivastava and Owa [2]).

Goodman [3,4] introduced the classes $\mathcal{UCV}$ and $\mathcal{UST}$ of uniformly convex and uniformly starlike functions. Following Goodman, Rønning [5] (see also [6]) gave one variable analytic characterization for $\mathcal{UCV}$, that is, a function $f(z)$ of the form (1.1) is in the class $\mathcal{UCV}$ if and only if

$$\Re\left\{1 + \frac{zf''(z)}{f'(z)}\right\} > \left|\frac{zf''(z)}{f'(z)}\right| (z \in \mathbb{U}). \tag{1.4}$$

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Goodman proved the classical Alexander's result $f(z) \in \mathcal{UCV} \iff zf'(z) \in \mathcal{UST}$, does not hold. On later, Rønning [7] introduced the class $\mathcal{S}_p$ which consists of functions such that $f(z) \in \mathcal{UCV} \iff zf'(z) \in \mathcal{S}_p$. Also in [5], Rønning generalized the classes $\mathcal{UCV}$ and $\mathcal{S}_p$ by introducing a parameter $\alpha$ in the following.

**Definition 1** [5]. A function $f(z)$ of the form (1.1) is in the class $\mathcal{S}_p(\alpha)$, if it satisfies the following condition:

$$\Re\left\{\frac{zf'(z)}{f(z)}-\alpha\right\} > \left|\frac{zf'(z)}{f(z)}-1\right|(-1 \leqslant \alpha < 1;\ z \in \mathbb{U}), \tag{1.5}$$

and $f(z) \in \mathcal{UCV}(\alpha)$, the class of uniformly convex functions of order $\alpha$ if and only if $zf'(z) \in \mathcal{S}_p(\alpha)$.

Also in [8], Bharati et al. introduced the classes $\mathcal{UCV}(\alpha,\beta)$ and $\mathcal{S}_p(\alpha,\beta)$ as follows:

**Definition 2** [8]. A function $f(z)$ of the form (1.1) is said to be in the class $\mathcal{S}_p(\alpha,\beta)$, if it satisfies the following condition:

$$\Re\left\{\frac{zf'(z)}{f(z)}-\alpha\right\} > \beta\left|\frac{zf'(z)}{f(z)}-1\right| \ (-1 \leqslant \alpha < 1; \beta \geqslant 0; z \in \mathbb{U}), \tag{1.6}$$

and $f(z) \in \mathcal{UCV}(\alpha,\beta)$ if and only if $zf'(z) \in \mathcal{S}_p(\alpha,\beta)$.

Denote by $\mathcal{T}$, the subclass of $\mathcal{S}$ consisting of functions of the form:

$$f(z) = z - \sum_{n=2}^{\infty} a_n z^n \ (a_n \geqslant 0). \tag{1.7}$$

Denote also by $\mathcal{T}^*(\alpha) = \mathcal{S}^*(\alpha) \cap \mathcal{T}$, $\mathcal{C}(\alpha) = \mathcal{K}(\alpha) \cap \mathcal{T}$, the subclasses of starlike and convex functions of order $\alpha$ with negative coefficients, which were introduced and studied by Silverman (see [9]). Also let $\mathcal{UCT}(\alpha) = \mathcal{UCV}(\alpha) \cap T$, $\mathcal{S}_p\mathcal{T}(\alpha) = \mathcal{S}_p(\alpha) \cap \mathcal{T}$, $\mathcal{UCT}(\alpha,\beta) = \mathcal{UCV}(\alpha,\beta) \cap T$ and $\mathcal{S}_p\mathcal{T}(\alpha,\beta) = \mathcal{S}_p(\alpha,\beta) \cap \mathcal{T}$.

Let $\mathcal{S}_\gamma(f;\alpha,\beta)$ $(-1 \leqslant \alpha < 1,\ \beta \geqslant 0$ and $0 \leqslant \gamma \leqslant 1)$ be the subclass of $\mathcal{S}$ consisting of functions of the form (1.1) and satisfying the analytic criterion:

$$\Re\left\{\frac{zf'(z)+\gamma z^2 f''(z)}{(1-\gamma)f(z)+\gamma zf'(z)}-\alpha\right\} > \beta\left|\frac{zf'(z)+\gamma z^2 f''(z)}{(1-\gamma)f(z)+\gamma zf'(z)}-1\right| \ (z \in \mathbb{U}). \tag{1.8}$$

The class $\mathcal{S}_\gamma(f;\alpha,\beta)$ was introduced and studied by Aouf et al. [10, with $g(z) = \frac{z}{1-z}$]. Further, we define the class $\mathcal{TS}_\gamma(f;\alpha,\beta)$ by

$\mathcal{TS}_\gamma(f;\alpha,\beta) = \mathcal{S}_\gamma(f;\alpha,\beta) \cap \mathcal{T}$.

Let $F(a,b;c;z)$ be the (Gaussian) hypergeometric function defined by

$$F(a,b;c;z) = \sum_{n=0}^{\infty} \frac{(a)_n(b)_n}{(c)_n(1)_n} z^n,$$

where $c \neq 0, -1, -2, \ldots$ and

$$(\lambda)_n = \begin{cases} 1 & \text{if } n = 0, \\ \lambda(\lambda+1)(\lambda+2)\cdots(\lambda+n-1) & \text{if } n \in \mathbb{N} = \{1,2,\ldots\}. \end{cases}$$

We note that $F(a,b;\ c;\ 1)$ converges for $\Re(c-a-b) > 0$ and is related to Gamma functions by

$$F(a,b;\ c;\ 1) = \frac{\Gamma(c)\Gamma(c-a-b)}{\Gamma(c-a)\Gamma(c-b)}. \tag{1.9}$$

Also, we define the functions

$$g(a,b;\ c;\ z) = zF(a,b;\ c;\ z), \tag{1.10}$$

and

$$h_\mu(a,b;\ c;\ z) = (1-\mu)(g(a,b;c;z)) + \mu z(g(a,b;\ c;\ z))' (\mu \geqslant 0). \tag{1.11}$$

The mapping properties of a function $h_\mu(a,b;\ c;\ z)$ was studied by Shukla and Shukla [11].

Corresponding to the Gaussian hypergeometric function ${}_2F_1(a,b;\ c;\ z)$, we define the linear operator $\mathcal{M}_{a,b,c} : \mathcal{A} \to \mathcal{A}$ by the integral convolution

$$[\mathcal{M}_{a,b,c}(f)](z) = g(a,b;\ c;\ z) \circledast f(z) = z + \sum_{n=2}^{\infty} \frac{(a)_{n-1}(b)_{n-1}}{(c)_{n-1}(1)_{n-1}} \frac{a_n}{n} z^n \ (c \neq 0,-1,-2,\ldots), \tag{1.12}$$

and the linear operator $\mathcal{N}_\mu : \mathcal{A} \to \mathcal{A}$ by the integral convolution

$$[\mathcal{N}_\mu(f)](z) = h_\mu(a,b;\ c;\ z) \circledast f(z) = z + \sum_{n=2}^{\infty}[1+\mu(n-1)] \frac{(a)_{n-1}(b)_{n-1}}{(c)_{n-1}(1)_{n-1}} \frac{a_n}{n} z^n \times (c \neq 0,-1,-2,\ldots). \tag{1.13}$$

Merkes and Scott [12] and Ruscheweyh and Singh [13] used continued fractions to find sufficient conditions for $zF(a,b;\ c;\ z)$ to be in the class $\mathcal{S}^*(\alpha)$ $(0 \leqslant \alpha < 1)$ for various choices of the parameters $a$, $b$ and $c$. Carlson and Shaffer [14] showed how some convolution results about the class $\mathcal{S}^*(\alpha)$ may be expressed in terms of a linear operator acting on hypergeometric functions. Recently, Silverman [15] gave a necessary and sufficient conditions for $zF(a,b;\ c;\ z)$ to be in the classes $\mathcal{S}^*(\alpha)$ and $\mathcal{K}(\alpha)$.

## 2. Main results

Unless otherwise mentioned, we assume throughout this paper that $-1 \leqslant \alpha < 1$, $\beta \geqslant 0$ and $0 \leqslant \gamma \leqslant 1$.
To establish our results, we need the following lemmas due to Aouf et al. [10].

**Lemma 2.1** [10, Theorem 1, with $g(z) = \frac{z}{1-z}$]. *A sufficient condition for $f(z)$ defined by (1.1) to be in the class $\mathcal{S}_\gamma(f;\ \alpha,\beta)$ is*

$$\sum_{n=2}^{\infty}[n(1+\beta)-(\alpha+\beta)][1+\gamma(n-1)]|a_n| \leqslant 1-\alpha. \tag{2.1}$$

**Lemma 2.2** [10, Theorem 2, with $g(z) = \frac{z}{1-z}$]. *A necessary and sufficient condition for $f(z)$ defined by (1.7) to be in the class $\mathcal{TS}_\gamma(f;\ \alpha,\beta)$ is*

$$\sum_{n=2}^{\infty}[n(1+\beta)-(\alpha+\beta)][1+\gamma(n-1)]a_n \leqslant 1-\alpha. \tag{2.2}$$

*By using Lemmas 2.1 and 2.2, we get the following results.*

**Theorem 2.1.** *Let $a,\ b>0$ and $c>a+b+2$, then the sufficient condition for $g(a,b;\ c;\ z)$ to be in the class $\mathcal{S}_\gamma(g;\ \alpha,\beta)$ is that*

$$\frac{\Gamma(c)\Gamma(c-a-b)}{\Gamma(c-a)\Gamma(c-b)}\left[1+\frac{ab[(1+\beta)-\gamma(\alpha-\beta-2)]}{(1-\alpha)(c-a-b-1)}+\frac{\gamma(1+\beta)(a)_2(b)_2}{(1-\alpha)(c-a-b-2)_2}\right]\leqslant 2. \tag{2.3}$$

*Also, condition (2.3) is necessary and sufficient for $F_1(a,b;\ c;\ z)=z(2-F(a,b;\ c;\ z))$ to be in the class $\mathcal{TS}_\gamma(F_1;\alpha,\beta)$.*

**Proof.** Since

$$g(a,b;\ c;\ z)=z+\sum_{n=2}^{\infty}\frac{(a)_{n-1}(b)_{n-1}}{(c)_{n-1}(1)_{n-1}}z^n,$$

then, according to Lemma 2.1, we need only to show that

$$\sum_{n=2}^{\infty}[n(1+\beta)-(\alpha+\beta)][1+\gamma(n-1)]\frac{(a)_{n-1}(b)_{n-1}}{(c)_{n-1}(1)_{n-1}}\leqslant 1-\alpha.$$

Thus

$$\begin{aligned}&\sum_{n=2}^{\infty}[n(1+\beta)-(\alpha+\beta)][1+\gamma(n-1)]\frac{(a)_{n-1}(b)_{n-1}}{(c)_{n-1}(1)_{n-1}}\\&\quad=(1-\alpha)\sum_{n=2}^{\infty}\frac{(a)_{n-1}(b)_{n-1}}{(c)_{n-1}(1)_{n-1}}+[(1+\beta)-\gamma(\alpha-\beta-2)]\\&\qquad\times\sum_{n=2}^{\infty}\frac{(a)_{n-1}(b)_{n-1}}{(c)_{n-1}(1)_{n-2}}+\gamma(1+\beta)\sum_{n=3}^{\infty}\frac{(a)_{n-1}(b)_{n-1}}{(c)_{n-1}(1)_{n-3}}.\end{aligned} \tag{2.4}$$

Since $(\lambda)_n=\lambda(\lambda+1)_{n-1}$, then from (1.9), we may express (2.4) as

$$\begin{aligned}&(1-\alpha)\left[\sum_{n=0}^{\infty}\frac{(a)_n(b)_n}{(c)_n(1)_n}-1\right]+[(1+\beta)-\gamma(\alpha-\beta-2)]\frac{ab}{c}\sum_{n=0}^{\infty}\frac{(a+1)_n(b+1)_n}{(c+1)_n(1)_n}\\&\quad+\gamma(1+\beta)\frac{(a)_2(b)_2}{(c)_2}\sum_{n=0}^{\infty}\frac{(a+2)_n(b+2)_n}{(c+2)_n(1)_n}=(1-\alpha)\left[\frac{\Gamma(c)\Gamma(c-a-b)}{\Gamma(c-a)\Gamma(c-b)}-1\right]\\&\quad+[(1+\beta)-\gamma(\alpha-\beta-2)]\frac{ab}{c}\frac{\Gamma(c+1)\Gamma(c-a-b-1)}{\Gamma(c-a)\Gamma(c-b)}\\&\quad+\gamma(1+\beta)\frac{(a)_2(b)_2}{(c)_2}\frac{\Gamma(c+2)\Gamma(c-a-b-2)}{\Gamma(c-a)\Gamma(c-b)}=\frac{\Gamma(c)\Gamma(c-a-b)}{\Gamma(c-a)\Gamma(c-b)}\\&\quad\times\left[(1-\alpha)+\frac{ab[(1+\beta)-\gamma(\alpha-\beta-2)]}{(c-a-b-1)}+\frac{\gamma(1+\beta)(a)_2(b)_2}{(c-a-b-2)_2}\right]-(1-\alpha).\end{aligned}$$

But this last expression is bounded above by $(1-\alpha)$ if (2.3) holds. Since

$$F_1(a,b;\ c;\ z)=z-\sum_{n=2}^{\infty}\frac{(a)_{n-1}(b)_{n-1}}{(c)_{n-1}(1)_{n-1}}z^n,$$

the necessity of (2.3) for $F_1(a,b;\ c;\ z)$ to be in the class $\mathcal{TS}_\gamma(F_1;\ \alpha,\beta)$ follows from Lemma 2.2. This completes the proof of Theorem 2.1. □

**Remark 2.1.**

(i) Putting $\beta=\gamma=0$ in Theorem 2.1, we obtain the result obtained by Silverman [15, Theorem 1].
(ii) Putting $\beta=0$ and $\gamma=1$ in Theorem 2.1, we obtain the result obtained by Silverman [15, Theorem 3].
(iii) Putting $\beta=1$ and $\gamma=0$ in Theorem 2.1, we obtain the result obtained by Cho et al. [16, Theorem 2.1].
(iv) Putting $\beta=\gamma=1$ in Theorem 2.1, we obtain the result obtained by Cho et al. [16, Theorem 2.3].
(v) Putting $\gamma=0$ in Theorem 2.1, we obtain the result obtained by Swaminathan [17, Theorem 2.1] (see also Kwon and Cho [18, (ii) of Theorem 2.3]).
(vi) Putting $\gamma=1$ in Theorem 2.1, we obtain the result obtained by Swaminathan [17, Theorem 2.3] (see also Kwon and Cho [18, (ii) of Theorem 2.4]).

**Theorem 2.2.** *Let $a,\ b>0$ and $c>a+b+3$, then the sufficient condition for $h_\mu(a,b;\ c;\ z)$ to be in the class $\mathcal{S}_\gamma(h_\mu;\ \alpha,\beta)$ is that*

$$\begin{aligned}&\frac{\Gamma(c)\Gamma(c-a-b)}{\Gamma(c-a)\Gamma(c-b)}\left[1+\frac{ab[(1+\beta)(1+2\gamma+2\mu+2\gamma\mu)-(\alpha+\beta)(\gamma+\mu+\gamma\mu)]}{(1-\alpha)(c-a-b-1)}\right.\\&\quad\left.+\frac{(a)_2(b)_2[(1+\beta)(\gamma+\mu+4\gamma\mu)-\gamma\mu(\alpha+\beta)]}{(1-\alpha)(c-a-b-2)_2}+\frac{\gamma\mu(1+\beta)(a)_3(b)_3}{(1-\alpha)(c-a-b-3)_3}\right]\leqslant 2.\end{aligned} \tag{2.5}$$

*Also, condition (2.5) is necessary and sufficient for $h^*(a,b;\ c;\ z)=z\left(2-\frac{h_\mu(a,b;\ c;\ z)}{z}\right)$ to be in the class $\mathcal{TS}_\gamma(h^*;\alpha,\beta)$.*

**Proof.** Since

$$h_\mu(a,b;\ c;\ z)=z+\sum_{n=2}^{\infty}[1+\mu(n-1)]\frac{(a)_{n-1}(b)_{n-1}}{(c)_{n-1}(1)_{n-1}}z^n,$$

then, according to Lemma 2.1, we need only to show that

$$\sum_{n=2}^{\infty}[n(1+\beta)-(\alpha+\beta)][1+\gamma(n-1)][1+\mu(n-1)]\frac{(a)_{n-1}(b)_{n-1}}{(c)_{n-1}(1)_{n-1}}\leqslant 1-\alpha.$$

Thus

$$\begin{aligned}&\sum_{n=2}^{\infty}[n(1+\beta)-(\alpha+\beta)][1+\gamma(n-1)][1+\mu(n-1)]\frac{(a)_{n-1}(b)_{n-1}}{(c)_{n-1}(1)_{n-1}}\\&=[(1+\beta)(1+2\gamma+2\mu+2\gamma\mu)-(\alpha+\beta)(\gamma+\mu+\gamma\mu)]\sum_{n=2}^{\infty}\frac{(a)_{n-1}(b)_{n-1}}{(c)_{n-1}(1)_{n-2}}\\&\quad+[(1+\beta)(\gamma+\mu+4\gamma\mu)-\gamma\mu(\alpha+\beta)]\sum_{n=3}^{\infty}\frac{(a)_{n-1}(b)_{n-1}}{(c)_{n-1}(1)_{n-3}}\\&\quad+\gamma\mu(1+\beta)\sum_{n=4}^{\infty}\frac{(a)_{n-1}(b)_{n-1}}{(c)_{n-1}(1)_{n-4}}+(1-\alpha)\sum_{n=2}^{\infty}\frac{(a)_{n-1}(b)_{n-1}}{(c)_{n-1}(1)_{n-1}}\\&=[(1+\beta)(1+2\gamma+2\mu+2\gamma\mu)\\&\quad-(\alpha+\beta)(\gamma+\mu+\gamma\mu)]\frac{ab}{c}\sum_{n=0}^{\infty}\frac{(a+1)_n(b+1)_n}{(c+1)_n(1)_n}\\&\quad+[(1+\beta)(\gamma+\mu+4\gamma\mu)-\gamma\mu(\alpha+\beta)]\frac{(a)_2(b)_2}{(c)_2}\sum_{n=0}^{\infty}\frac{(a+2)_n(b+2)_n}{(c+2)_n(1)_n}\\&\quad+\gamma\mu(1+\beta)\frac{(a)_3(b)_3}{(c)_3}\sum_{n=0}^{\infty}\frac{(a+3)_n(b+3)_n}{(c+3)_n(1)_n}+(1-\alpha)\left[\sum_{n=0}^{\infty}\frac{(a)_n(b)_n}{(c)_n(1)_n}-1\right]\\&=[(1+\beta)(1+2\gamma+2\mu+2\gamma\mu)\\&\quad-(\alpha+\beta)(\gamma+\mu+\gamma\mu)]\frac{ab}{c}\frac{\Gamma(c+1)\Gamma(c-a-b-1)}{\Gamma(c-a)\Gamma(c-b)}\end{aligned}$$

$$
\begin{aligned}
&+[(1+\beta)(\gamma+\mu+4\gamma\mu)\\
&-\gamma\mu(\alpha+\beta)]\frac{(a)_2(b)_2}{(c)_2}\frac{\Gamma(c+2)\Gamma(c-a-b-2)}{\Gamma(c-a)\Gamma(c-b)}\\
&+\gamma\mu(1+\beta)\frac{(a)_3(b)_3}{(c)_3}\frac{\Gamma(c+3)\Gamma(c-a-b-3)}{\Gamma(c-a)\Gamma(c-b)}\\
&+(1-\alpha)\left[\frac{\Gamma(c)\Gamma(c-a-b)}{\Gamma(c-a)\Gamma(c-b)}-1\right]\\
=&\frac{\Gamma(c)\Gamma(c-a-b)}{\Gamma(c-a)\Gamma(c-b)}\Big[(1-\alpha)\\
&+\frac{ab[(1+\beta)(1+2\gamma+2\mu+2\gamma\mu)-(\alpha+\beta)(\gamma+\mu+\gamma\mu)]}{(c-a-b-1)}\\
&+\frac{(a)_2(b)_2[(1+\beta)(\gamma+\mu+4\gamma\mu)-\gamma\mu(\alpha+\beta)]}{(c-a-b-2)_2}+\frac{\gamma\mu(1+\beta)(a)_3(b)_3}{(c-a-b-3)_3}\Big]-(1-\alpha).
\end{aligned}
$$

But this last expression is bounded above by $(1-\alpha)$ if (2.5) holds. Since

$$h^*(a,b;\ c;\ z)=z-\sum_{n=2}^{\infty}[1+\mu(n-1)]\frac{(a)_{n-1}(b)_{n-1}}{(c)_{n-1}(1)_{n-1}}z^n,$$

the necessity of (2.5) for $h^*(a,b;c;z)$ to be in the class $\mathcal{TS}_\gamma(h^*;\ \alpha,\beta)$ follows from Lemma 2.2. This completes the proof of Theorem 2.2. □

**Remark 2.2.** Putting $\alpha=\gamma=0$ in Theorem 2.2, we obtain the result obtained by Ramachandran et al. [19, Theorem 2.1, with $p=2$ and $q=1$].
Putting $\alpha=0$ and $\gamma=1$ in Theorem 2.2, we get the correct form of the result obtained by Ramachandran et al. [19, Theorem 2.3, with $p=2$ and $q=1$].

**Corollary 2.1.** *Let $a,b>0$ and $c>a+b+3$, then the sufficient condition for $h_\mu(a,b;c;z)$ to be in the class $\mathcal{UCV}(\beta)$ is that*

$$
\begin{aligned}
&\mu(1+\beta)\frac{(a)_3(b)_3}{(c)_3}F(a+3,b+3;\ c+3;\ 1)\\
&\quad+(4\mu\beta+5\mu+\beta+1)\frac{(a)_2(b)_2}{(c)_2}F(a+2,b+2;\ c+2;\ 1)\\
&\quad+(2\mu\beta+4\mu+2\beta+3)\frac{ab}{c}F(a+1,b+1;\ c+1;\ 1)\\
&\quad+F(a,b;\ c;\ 1)\leqslant 2. \qquad (2.6)
\end{aligned}
$$

*Also, the condition (2.6) is necessary and sufficient for $h^*(a,b;\ c;\ z)=z\left(2-\frac{h_\mu(a,b;\ c;\ z)}{z}\right)$ to be in the class $\mathcal{UCT}(\beta)$.*

**Theorem 2.3.** *Let $a,\ b>-1,\ ab<0$ and $c>a+b+2$, then the necessary and sufficient condition for $g(a,b;c;z)$ to be in the class $\mathcal{TS}_\gamma(g;\alpha,\beta)$ is that*

$$
\begin{aligned}
&\gamma(1+\beta)(a)_2(b)_2+[(1+\beta)-\gamma(\alpha-\beta-2)]ab(c-a-b-2)\\
&\quad+(1-\alpha)(c-a-b-2)_2\geqslant 0. \qquad (2.7)
\end{aligned}
$$

**Proof.** Since

$$
\begin{aligned}
g(a,b;\ c;\ z)&=z+\frac{ab}{c}\sum_{n=2}^{\infty}\frac{(a+1)_{n-2}(b+1)_{n-2}}{(c+1)_{n-2}(1)_{n-1}}z^n\\
&=z-\left|\frac{ab}{c}\right|\sum_{n=2}^{\infty}\frac{(a+1)_{n-2}(b+1)_{n-2}}{(c+1)_{n-2}(1)_{n-1}}z^n,
\end{aligned}
$$

then, according to Lemma 2.2, we need only to prove that

$$\sum_{n=2}^{\infty}[n(1+\beta)-(\alpha+\beta)][1+\gamma(n-1)]\frac{(a+1)_{n-2}(b+1)_{n-2}}{(c+1)_{n-2}(1)_{n-1}}\leqslant\left|\frac{c}{ab}\right|(1-\alpha). \qquad (2.8)$$

Thus

$$
\begin{aligned}
&\sum_{n=2}^{\infty}[n(1+\beta)-(\alpha+\beta)][1+\gamma(n-1)]\frac{(a+1)_{n-2}(b+1)_{n-2}}{(c+1)_{n-2}(1)_{n-1}}\\
&=[(1+\beta)-\gamma(\alpha-\beta-2)]\sum_{n=2}^{\infty}\frac{(a+1)_{n-2}(b+1)_{n-2}}{(c+1)_{n-2}(1)_{n-2}}\\
&\quad+\gamma(1+\beta)\sum_{n=3}^{\infty}\frac{(a+1)_{n-2}(b+1)_{n-2}}{(c+1)_{n-2}(1)_{n-3}}+(1-\alpha)\sum_{n=2}^{\infty}\frac{(a+1)_{n-2}(b+1)_{n-2}}{(c+1)_{n-2}(1)_{n-1}}\\
&=[(1+\beta)-\gamma(\alpha-\beta-2)]\sum_{n=0}^{\infty}\frac{(a+1)_n(b+1)_n}{(c+1)_n(1)_n}\\
&\quad+\gamma(1+\beta)\frac{(a+1)(b+1)}{(c+1)}\sum_{n=0}^{\infty}\frac{(a+2)_n(b+2)_n}{(c+2)_n(1)_n}+(1-\alpha)\frac{c}{ab}\left[\sum_{n=0}^{\infty}\frac{(a)_n(b)_n}{(c)_n(1)_n}-1\right]\\
&=[(1+\beta)-\gamma(\alpha-\beta-2)]\frac{\Gamma(c+1)\Gamma(c-a-b-1)}{\Gamma(c-a)\Gamma(c-b)}\\
&\quad+\gamma(1+\beta)\frac{(a+1)(b+1)}{(c+1)}\frac{\Gamma(c+2)\Gamma(c-a-b-2)}{\Gamma(c-a)\Gamma(c-b)}\\
&\quad+(1-\alpha)\frac{c}{ab}\left[\frac{\Gamma(c)\Gamma(c-a-b)}{\Gamma(c-a)\Gamma(c-b)}-1\right]\\
&=\frac{\Gamma(c+1)\Gamma(c-a-b-2)}{\Gamma(c-a)\Gamma(c-b)}\Big[[(1+\beta)-\gamma(\alpha-\beta-2)](c-a-b-2)\\
&\quad+\gamma(1+\beta)(a+1)(b+1)+\frac{(1-\alpha)}{ab}(c-a-b-2)_2\Big]-(1-\alpha)\frac{c}{ab}. \qquad (2.9)
\end{aligned}
$$

Hence (2.8) is equivalent to

$$
\begin{aligned}
&\frac{\Gamma(c+1)\Gamma(c-a-b-2)}{\Gamma(c-a)\Gamma(c-b)}\Big[[(1+\beta)-\gamma(\alpha-\beta-2)](c-a-b-2)\\
&\quad+\gamma(1+\beta)(a+1)(b+1)+\frac{(1-\alpha)}{ab}(c-a-b-2)_2\Big]\\
&\quad\leqslant(1-\alpha)\frac{c}{ab}-(1-\alpha)\frac{c}{ab}=0. \qquad (2.10)
\end{aligned}
$$

Thus, from (2.10), we have

$$
\begin{aligned}
&[(1+\beta)-\gamma(\alpha-\beta-2)](c-a-b-2)+\gamma(1+\beta)(a+1)(b+1)\\
&\quad+\frac{(1-\alpha)}{ab}(c-a-b-2)_2\leqslant 0.
\end{aligned}
$$

or, equivalently,

$$
\begin{aligned}
&\gamma(1+\beta)(a)_2(b)_2+[(1+\beta)-\gamma(\alpha-\beta-2)]ab(c-a-b-2)\\
&\quad+(1-\alpha)(c-a-b-2)_2\geqslant 0.
\end{aligned}
$$

This completes the proof of Theorem 2.3. □

**Remark 2.3.**

(i) Putting $\beta=\gamma=0$ in Theorem 2.3, we obtain the result obtained by Silverman [15, Theorem 2].
(ii) Putting $\beta=1$ and $\gamma=0$ in Theorem 2.3, we obtain the result obtained by Cho et al. [16, Theorem 2.2].
(iii) Putting $\gamma=0$ in Theorem 2.3, we obtain the result obtained by Swaminathan [17, Theorem 2.2] (see also Kwon and Cho [18, (i) of Theorem 2.3]).

Putting $\beta = 0$ and $\gamma = 1$ in Theorem 2.3, we obtain the following corollary which corrects the result obtained by Silverman [15, Theorem 4].

**Corollary 2.2.** *Let $a, b > -1$, $ab < 0$ and $c > a+b+2$, then the necessary and sufficient condition for $g(a,b;\ c;\ z)$ to be in the class $\mathcal{C}(\alpha)$ is that*

$$(a)_2(b)_2 + (3-\alpha)ab(c-a-b-2) + (1-\alpha)(c-a-b-2)_2 \geqslant 0.$$

*Putting $\beta = \gamma = 1$ in Theorem 2.3, we obtain the following corollary which corrects the result obtained by Cho et al. [16, Theorem 2.4].*

**Corollary 2.3.** *Let $a, b > -1$, $ab < 0$ and $c > a+b+2$, then the necessary and sufficient condition for $g(a,b;\ c;\ z)$ to be in the class $\mathcal{UCT}(\alpha)$ is that*

$$2(a)_2(b)_2 + (5-\alpha)ab(c-a-b-2) + (1-\alpha)(c-a-b-2)_2 \geqslant 0.$$

*Putting $\gamma = 1$ in Theorem 2.3, we obtain the following corollary which corrects the result obtained by Swaminathan [17, Theorem 2.4] and the result obtained by Kwon and Cho [18, (i) of Theorem 2.4]).*

**Corollary 2.4.** *Let $a, b > -1$, $ab < 0$ and $c > a+b+2$, then the necessary and sufficient condition for $g(a,b;\ c;\ z)$ to be in the class $\mathcal{UCT}(\alpha,\beta)$ is that*

$$\begin{aligned}&(1+\beta)(a)_2(b)_2 + (3+2\beta-\alpha)ab(c-a-b-2)\\&\quad + (1-\alpha)(c-a-b-2)_2 \geqslant 0.\end{aligned}$$

**Theorem 2.4.** *Let $a, b > -1$, $ab < 0$ and $c > a+b+3$, then the necessary and sufficient condition for $h_\mu(a,b;\ c;\ z)$ to be in the class $\mathcal{TS}_\gamma(h_\mu;\ \alpha,\beta)$ is that*

$$\begin{aligned}&\gamma\mu(1+\beta)(a)_3(b)_3 + [(1+\beta)(\gamma+\mu+4\gamma\mu)\\&\quad - \gamma\mu(\alpha+\beta)](a)_2(b)_2(c-a-b-3)\\&\quad + [(1+\beta)(1+2\gamma+2\mu+2\gamma\mu)\\&\quad - (\alpha+\beta)(\gamma+\mu+\gamma\mu)]ab(c-a-b-3)_2\\&\quad + (1-\alpha)(c-a-b-3)_3 \geqslant 0. \qquad (2.11)\end{aligned}$$

**Proof.** Since

$$\begin{aligned}h_\mu(a,b;\ c;\ z) &= z + \frac{ab}{c}\sum_{n=2}^{\infty}[1+\mu(n-1)]\frac{(a+1)_{n-2}(b+1)_{n-2}}{(c+1)_{n-2}(1)_{n-1}}z^n\\&= z - \left|\frac{ab}{c}\right|\sum_{n=2}^{\infty}[1+\mu(n-1)]\frac{(a+1)_{n-2}(b+1)_{n-2}}{(c+1)_{n-2}(1)_{n-1}}z^n,\end{aligned}$$

then, according to Lemma 2.2, we need only to prove that

$$\begin{aligned}&\sum_{n=2}^{\infty}[n(1+\beta)-(\alpha+\beta)][1+\gamma(n-1)][1+\mu(n-1)]\\&\quad\times\frac{(a+1)_{n-2}(b+1)_{n-2}}{(c+1)_{n-2}(1)_{n-1}} \leqslant \left|\frac{c}{ab}\right|(1-\alpha). \qquad (2.12)\end{aligned}$$

Thus

$$\begin{aligned}&\sum_{n=2}^{\infty}[n(1+\beta)-(\alpha+\beta)][1+\gamma(n-1)][1+\mu(n-1)]\frac{(a+1)_{n-2}(b+1)_{n-2}}{(c+1)_{n-2}(1)_{n-1}}\\
&=[(1+\beta)(1+2\gamma+2\mu+2\gamma\mu)-(\alpha+\beta)(\gamma+\mu+\gamma\mu)]\sum_{n=2}^{\infty}\frac{(a+1)_{n-2}(b+1)_{n-2}}{(c+1)_{n-2}(1)_{n-2}}\\
&\quad+[(1+\beta)(\gamma+\mu+4\gamma\mu)-\gamma\mu(\alpha+\beta)]\sum_{n=3}^{\infty}\frac{(a+1)_{n-2}(b+1)_{n-2}}{(c+1)_{n-2}(1)_{n-3}}\\
&\quad+\gamma\mu(1+\beta)\sum_{n=4}^{\infty}\frac{(a+1)_{n-2}(b+1)_{n-2}}{(c+1)_{n-2}(1)_{n-4}}+(1-\alpha)\sum_{n=2}^{\infty}\frac{(a+1)_{n-2}(b+1)_{n-2}}{(c+1)_{n-2}(1)_{n-1}}\\
&=[(1+\beta)(1+2\gamma+2\mu+2\gamma\mu)-(\alpha+\beta)(\gamma+\mu+\gamma\mu)]\sum_{n=0}^{\infty}\frac{(a+1)_n(b+1)_n}{(c+1)_n(1)_n}\\
&\quad+[(1+\beta)(\gamma+\mu+4\gamma\mu)-\gamma\mu(\alpha+\beta)]\frac{(a+1)(b+1)}{(c+1)}\sum_{n=0}^{\infty}\frac{(a+2)_n(b+2)_n}{(c+2)_n(1)_n}\\
&\quad+\gamma\mu(1+\beta)\frac{(a+1)_2(b+1)_2}{(c+1)_2}\sum_{n=0}^{\infty}\frac{(a+3)_{n-2}(b+3)_{n-2}}{(c+3)_{n-2}(1)_n}+(1-\alpha)\frac{c}{ab}\left[\sum_{n=0}^{\infty}\frac{(a)_n(b)_n}{(c)_n(1)_n}-1\right]\\
&=[(1+\beta)(1+2\gamma+2\mu+2\gamma\mu)-(\alpha+\beta)(\gamma+\mu+\gamma\mu)]\frac{\Gamma(c+1)\Gamma(c-a-b-1)}{\Gamma(c-a)\Gamma(c-b)}\\
&\quad+[(1+\beta)(\gamma+\mu+4\gamma\mu)-\gamma\mu(\alpha+\beta)]\frac{(a+1)(b+1)}{(c+1)}\frac{\Gamma(c+2)\Gamma(c-a-b-2)}{\Gamma(c-a)\Gamma(c-b)}\\
&\quad+\gamma\mu(1+\beta)\frac{(a+1)_2(b+1)_2}{(c+1)_2}\frac{\Gamma(c+3)\Gamma(c-a-b-3)}{\Gamma(c-a)\Gamma(c-b)}\\
&\quad+(1-\alpha)\frac{c}{ab}\left[\frac{\Gamma(c)\Gamma(c-a-b)}{\Gamma(c-a)\Gamma(c-b)}-1\right]\\
&=\frac{\Gamma(c+1)\Gamma(c-a-b-3)}{\Gamma(c-a)\Gamma(c-b)}\Big[\{(1+\beta)(1+2\gamma+2\mu+2\gamma\mu)-(\alpha+\beta)(\gamma+\mu+\gamma\mu)\}\\
&\quad\times(c-a-b-3)_2+[(1+\beta)(\gamma+\mu+4\gamma\mu)-\gamma\mu(\alpha+\beta)](a+1)(b+1)(c-a-b-3)\\
&\quad+\gamma\mu(1+\beta)(a+1)_2(b+1)_2+\frac{(1-\alpha)}{ab}(c-a-b-3)_3\Big]-(1-\alpha)\frac{c}{ab}.\end{aligned}$$

Hence (2.12) is equivalent to

$$\begin{aligned}&\frac{\Gamma(c+1)\Gamma(c-a-b-3)}{\Gamma(c-a)\Gamma(c-b)}\Big[\{(1+\beta)(1+2\gamma+2\mu+2\gamma\mu)\\
&\quad-(\alpha+\beta)(\gamma+\mu+\gamma\mu)\}(c-a-b-3)_2\\
&\quad+[(1+\beta)(\gamma+\mu+4\gamma\mu)-\gamma\mu(\alpha+\beta)](a+1)(b+1)\\
&\quad\times(c-a-b-3)+\gamma\mu(1+\beta)(a+1)_2(b+1)_2\\
&\quad+\frac{(1-\alpha)}{ab}(c-a-b-3)_3\Big]\leqslant(1-\alpha)\frac{c}{ab}-(1-\alpha)\frac{c}{ab}=0.\end{aligned} \qquad (2.13)$$

Thus, from (2.13), we have

$$\begin{aligned}&[(1+\beta)(1+2\gamma+2\mu+2\gamma\mu)-(\alpha+\beta)(\gamma+\mu+\gamma\mu)](c-a-b-3)_2\\
&\quad+[(1+\beta)(\gamma+\mu+4\gamma\mu)-\gamma\mu(\alpha+\beta)](a+1)(b+1)(c-a-b-3)\\
&\quad+\gamma\mu(1+\beta)(a+1)_2(b+1)_2+\frac{(1-\alpha)}{ab}(c-a-b-3)_3\leqslant 0,\end{aligned}$$

which implies to (2.11). This completes the proof of Theorem 2.4. □

Putting $\alpha = \gamma = 0$ in Theorem 2.4, we obtain the following corollary which corrects the result obtained by Ramachandran et al. [19, Theorem 2.2, with $p = 2$ and $q = 1$].

**Corollary 2.5.** *Let $a, b > -1$, $ab < 0$ and $c > a+b+2$, then the necessary and sufficient condition for $h_\mu(a,b;\ c;\ z)$ to be in the class $\mathcal{S}_p\mathcal{T}(\beta)$ is that*

$$\begin{aligned}&\mu(1+\beta)\frac{(a+1)(b+1)}{(c+1)}F(a+2,b+2;c+2;1)\\&\quad+[\mu(\beta+2)+\beta+1]F(a+1,b+1;\ c+1;\ 1)\\&\quad+\frac{c}{ab}F(a,b;\ c;\ 1)\leqslant 0.\end{aligned}$$

*Putting $\alpha = 0$ and $\gamma = 1$ in Theorem 2.4, we obtain the following corollary which corrects the result obtained by Ramachandran et al. [19, Theorem 2.4, with $p = 2$ and $q = 1$].*

**Corollary 2.6.** *Let $a,\ b > 0$ and $c > a + b + 3$, then the sufficient condition for $h_\mu(a, b;\ c;\ z)$ to be in the class $\mathcal{UCT}(\beta)$ is that*

$$\mu(1+\beta)\frac{(a+1)_2(b+1)_2}{(c+1)_2}F(a+3, b+3;\ c+3;\ 1) + (4\mu\beta + 5\mu + \beta + 1)\frac{(a+1)(b+1)}{(c+1)}F(a+2, b+2;\ c+2;\ 1) + (2\mu\beta + 4\mu + 2\beta + 3)F(a+1, b+1;\ c+1;\ 1) + \frac{c}{ab}F(a, b;\ c;\ 1) \leqslant 0.$$

*Using similar arguments to the proof of the above theorems, we obtain the following theorems.*

**Theorem 2.5.** *Let $a,\ b > 0$ and $c > a + b + 2$. If the inequality (2.3) is satisfied, then $[\mathcal{M}_{a,b,c}(f)](z)$ maps the class $\mathcal{S}$ (or $\mathcal{S}^*$) to the class $\mathcal{S}_\gamma(f;\ \alpha, \beta)$.*

**Theorem 2.6.** *Let $a > 1,\ b > 1$ and $c > a + b - 1$. If the following inequality*

$$\frac{\Gamma(c)\Gamma(c-a-b)}{\Gamma(c-a)\Gamma(c-b)}\left[[(1+\beta) - \gamma(\alpha+\beta)] + \gamma(1+\beta)\frac{ab}{(c-a-b-1)} - (1-\gamma)(\alpha+\beta)\frac{(c-a-b)}{(a-1)(b-1)}\right] + (1-\gamma)(\alpha+\beta)\frac{(c-1)}{(a-1)(b-1)} \leqslant 2(1-\alpha), \tag{2.14}$$

*is true, then $[\mathcal{M}_{a,b,c}(f)](z)$ maps the class $\mathcal{K}$ to the class $\mathcal{S}_\gamma(f;\ \alpha, \beta)$.*

**Theorem 2.7.** *Let $a,\ b > 0$ and $c > a + b + 3$. If the inequality (2.5) is satisified, then $[\mathcal{N}_\mu(f)](z)$ maps the class $\mathcal{S}$ (or $\mathcal{S}^*$) to the class $\mathcal{S}_\gamma(f;\ \alpha, \beta)$.*

**Theorem 2.8.** *Let $a > 1,\ b > 1$ and $c > a + b + 2$. If the following inequality*

$$\frac{\Gamma(c)\Gamma(c-a-b)}{\Gamma(c-a)\Gamma(c-b)}\left[\{(1+\beta) - (\alpha+\beta)(\gamma+\mu-\gamma\mu)\} + \frac{ab[(1+\beta)(\gamma+\mu+\gamma\mu) - \gamma\mu(\alpha+\beta)]}{(c-a-b-1)} + \gamma\mu(1+\beta)\frac{(a)_2(b)_2}{(c-a-b-2)_2} + (\alpha+\beta)(\gamma+\mu-\gamma\mu-1)\frac{(c-a-b)}{(a-1)(b-1)}\right] \leqslant 2(1-\alpha) + (\alpha+\beta)(\gamma+\mu-\gamma\mu-1)\frac{(c-1)}{(a-1)(b-1)}, \tag{2.15}$$

*holds, then $[\mathcal{N}_\mu(f)](z)$ maps the class $\mathcal{K}$ to the class $\mathcal{S}_\gamma(f;\ \alpha, \beta)$.*

**Remark 2.4.** By specializing $\alpha,\ \beta$ and $\gamma$ in Theorems from 2.5 to 2.8, we will obtain new results for different classes mentioned in the introduction.


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