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REVIEW PAPER

Stochastic amplitude equation for the stochastic generalized Swift–Hohenberg equation

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Abstract In this paper we derive rigorously the amplitude equation, using the natural separation of time-scales near a change of stability, for the stochastic generalized Swift–Hohenberg equation with quadratic and cubic nonlinearity in this form

$$du = \left[-(1+\partial_x^2)^2 u + \nu_\varepsilon u + \gamma u^2 - u^3\right] dt + \sigma_\varepsilon dW,$$

where $W(t)$ is a Wiener process. For deterministic PDE it is known that the quadratic term generates an additional cubic term, which is unstable. We consider two cases depending on γ^2. If $\gamma^2 < \frac{27}{38}$, then we have amplitude equation with cubic nonlinearities. In the other case $\gamma^2 = \frac{27}{38}$ the cubic term in the amplitude equation vanishes. Therefore we consider larger solutions to obtain an amplitude equation with quintic nonlinearities.

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Contents

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1. Introduction

Swift–Hohenberg equation was first used as a toy model for the convective instability in the Rayleigh–Bénard problem (see [1] or [2]). Today it is one of the celebrated equations for the examination of the dynamics of pattern formation.

Near the bifurcation the equation exhibits two widely separated characteristic time-scales and it is desirable to obtain a simplified equation which governs the evolution of the dominant modes. These equations are referred to as amplitude equations. The approximation of SPDEs on bounded domains via amplitude equations was first rigorously verified in [3] for a simple Swift–Hohenberg model, and later extended in [4–8]. In all these publications the amplitude equation for the dominant modes is given by an ODE or a SDE.

Mohammed et al. [9,10] studied the Equation

$$du = \left[-(1+\partial_x^2)^2 u + \nu_\varepsilon u + \gamma u^2 - u^3\right]dt + \sigma_\varepsilon dW, \tag{1}$$

in case the noise-strength is $\sigma_\varepsilon = \varepsilon$, and the noise does not act directly on the dominant modes. Here additional deterministic terms appear, due to the presence of noise, that change the stability of the system. In this paper, we will study two cases $\sigma_\varepsilon = \varepsilon^2$ and $\sigma_\varepsilon = \varepsilon^3$, and suppose that the noise acts directly on the dominant modes.

The main result of this paper is to show that near a change of stability on a time-scale of order ε^{-n} ($n = 2$ or 4) the solution of (1) with respect to Neumann boundary conditions on the interval $[0, \pi]$ is of the type

$$u(t,x) = \varepsilon b(\varepsilon^n t)\cos(x) + \text{error}, \tag{2}$$

where b is the solution of the amplitude equation on the slow time-scale $T = \varepsilon^n t$ given by

$$\partial_T b = \nu b + \mathcal{G}(b) + \alpha_1 \partial_T \tilde{\beta}_1, \tag{3}$$

where $\tilde{\beta}_1(T) := \varepsilon^{\frac{n}{2}}\beta_1(\varepsilon^{-n}T)$ is a rescaled version of the Brownian motion, and $\mathcal{G}(b)$ is given by

$$\mathcal{G}(b) := -\frac{3}{4}\left(1 - \frac{38}{27}\gamma^2\right)b^3, \tag{4}$$

in the case of $n = 2, \sigma_\varepsilon = \varepsilon^2$ and $\gamma^2 < \frac{27}{38}$, while in the case of $n = 4, \sigma_\varepsilon = \varepsilon^3$ and $\gamma^2 = \frac{27}{38}, \mathcal{G}(b)$ is quintic and given by

$$\mathcal{G}(b) := -C_0 b^5, \tag{5}$$

with $C_0 \simeq 1.8$.

The remainder of this paper is organized as follows. In the next section we formulate the assumptions that we need in this paper. In Section 3 we derive the amplitude equation with error term and state without proof the approximation theorem. In Section 4 we give bounds for high modes. Finally, we give the proof of the main results.

2. Preliminaries

We work in some Hilbert space $\mathcal{H}$ equipped with scalar product $\langle\cdot,\cdot\rangle$ and norm $\|\cdot\|$. We denote by $\{e_k\}_{k=1}^\infty$ and $\{\lambda_k\}_{k=1}^\infty$ an orthonormal basis of eigenfunctions and the corresponding eigenvalues such that $-(1+\partial_x^2)^2 e_k = \lambda_k e_k$, (cf. Courant and Hilbert [11]). In our case

$$e_k(x) = \begin{cases} \frac{1}{\sqrt{\pi}} & \text{if } k = 0, \\ \sqrt{\frac{2}{\pi}}\cos(kx) & \text{if } k > 0, \end{cases} \text{and } \lambda_k = (1-k^2)^2.$$

Suppose that $\mathcal{N} := \ker\mathcal{A} = span\{\cos\}$, where $\mathcal{A} = -(1+\partial_x^2)^2$. Define by $S = \mathcal{N}^\perp$ the orthogonal complement of $\mathcal{N}$ in $\mathcal{H}$ and by P_c the projection $P_c : \mathcal{H} \to \mathcal{N}$. Define $P_s := \mathcal{I} - P_c$, where $\mathcal{I}$ is the identity operator on $\mathcal{H}$. As the dimension of $\mathcal{N}$ is finite, it is well known that both P_c and P_s are bounded linear operators on $\mathcal{H}$.

Let us define the space H^1 by Fourier series:

$$\mathcal{H}^1 = \left\{\sum_{k=1}^\infty \gamma_k e_k : \sum_{k=1}^\infty k^2\gamma_k^2 < \infty\right\} \text{with norm} \left\|\sum_{k=1}^\infty \gamma_k e_k\right\|_{\mathcal{H}^1} = \sum_{k=1}^\infty k^2\gamma_k^2.$$

The operator $\mathcal{A}$ generates an analytic semigroup $\{e^{t\mathcal{A}}\}_{t\geqslant 0}$ defined by

$$e^{\mathcal{A}t}\left(\sum_{k=1}^\infty \gamma_k e_k\right) = \sum_{k=1}^\infty e^{-\lambda_k t}\gamma_k e_k \quad \forall\ t \geqslant 0.$$

Also, it has the following property that for all $t > 0, \omega = \lambda_1$ and all $u \in \mathcal{H}^1$

$$\|e^{t\mathcal{A}}P_s u\|_{\mathcal{H}^1} \leqslant e^{-\omega t}\|P_s u\|_{\mathcal{H}^1}. \tag{6}$$

In an abstract setting we need the following assumption, which is trivial to check in the concrete examples.

Assumption 1. Define the nonlinear term $\mathcal{G}(b) : \mathbb{R} \to \mathbb{R}$ via

$$\mathcal{G}(b) = -Cb^{2n+1}, \quad \text{for } n = 1, 2.$$

Assume there exists a constant $\delta_1 \geqslant 0$ such that for $u \in \mathbb{R}$ the following inequality is satisfied

$$\langle\mathcal{G}(u), u\rangle \leqslant -\delta_1|u|^{2n+2}, \quad \text{for } n = 1, 2.$$

For the noise we suppose the following:

Assumption 2. Let W be a Wiener process on an abstract probability space $(\Omega, F, \mathbb{P})$. For $t \geqslant 0$, we can write $W(t)$ (cf. Da Prato and Zabczyk [12]) as

$$W(t) = \sum_{k=0}^\infty \alpha_k\beta_k(t)e_k,$$

where $(\beta_k)_{k\in\mathbb{N}_0}$ are independent, standard Brownian motions in $\mathbb{R}$ and $(\alpha_k)_{k\in\mathbb{N}_0}$ are real numbers. We assume

$$\sum_{k\neq 1}^{\infty} k^2\alpha_k^2\lambda_k^{2\gamma-1} < \infty, \quad \text{for some } \gamma \in \left(0, \frac{1}{2}\right).$$

For our result we rely on a cutoff argument. We consider only solutions that are not too large, as given by the next definition.

Definition 3 (*Stopping time*). For the $\mathcal{N}\times S$-valued stochastic process (a,ψ) defined later in (10) we define, for some $T_0>0$ and $\kappa\in(0,\frac{1}{3n+11})$ for $n=1$ or 2, the stopping time τ^* as

$$\tau^* := T_0 \wedge \inf\left\{T>0 : \|a(T)\|_{\mathcal{H}^1} > \varepsilon^{-\kappa} \text{ or} \|\psi(T)\|_{\mathcal{H}^1} > \varepsilon^{-3\kappa}\right\}. \tag{7}$$

Definition 4. For a real-valued family of processes $\{X_\varepsilon(t)\}_{t\geqslant 0}$ we say $X_\varepsilon = \mathcal{O}(f_\varepsilon)$, if for every $p\geqslant 1$ there exists a constant C_p such that

$$\mathbb{E}\sup_{t\in[0,\tau^*]}|X_\varepsilon(t)|^p \leqslant C_p f_\varepsilon^p. \tag{8}$$

We use also the analogous notation for time-independent random variables.

3. Formal derivation and main result

In this section, we derive the amplitude equation with error term and state without proof the approximation theorem. For short, let $\mathcal{A}=-(1+\partial_x^2)^2$, $B(u)=B(u,u)=u^2$, and $\mathcal{F}(u)=\mathcal{F}(u,u,u)=u^3$. So, we can rewrite the Eq. (1) as follows

$$du = [\mathcal{A}u + \nu_\varepsilon u + \gamma B(u) - \mathcal{F}(u)]dt + \sigma_\varepsilon dW. \tag{9}$$

We are interested here in studying the behavior of solutions to (9) on time-scales of order ε^{-n}, for $n=2$ or 4. So, we split the solution u into

$$u(t) = \varepsilon a(\varepsilon^n t) + \varepsilon^2\psi(\varepsilon^n t), \tag{10}$$

where $a\in\mathcal{N}$ and $\psi\in S$. After rescaling to the slow time-scale $T=\varepsilon^n t$, we obtain the following system of equations:

$$da = \left[\nu a + 2\gamma\varepsilon^{-n+2}B_c(a,\psi) + \gamma\varepsilon^{-n+3}B_c(\psi,\psi) - \varepsilon^{-n+2}\mathcal{F}_c(a+\varepsilon\psi)\right]dT + d\tilde{W}_c, \tag{11}$$

and

$$d\psi = \left[\varepsilon^{-n}\mathcal{A}_s\psi + \nu\psi + \varepsilon^{-n}\gamma B_s(a+\varepsilon\psi) - \varepsilon^{-n+1}\mathcal{F}_s(a+\varepsilon\psi)\right]dT + \varepsilon^{-1}d\tilde{W}_s, \tag{12}$$

where $\tilde{W}(T) := \varepsilon^{n/2}W(\varepsilon^{-n}T)$ is a rescaled version of the Wiener process and $\sigma_\varepsilon=\varepsilon^2$ if $n=2$ or $\sigma_\varepsilon=\varepsilon^3$ if $n=4$. We denoted the projections by indices. This means $\mathcal{F}_c=P_c\mathcal{F}$ or $\mathcal{F}_s=P_s\mathcal{F}$. We define $B_c, B_s, \tilde{W}_c, \tilde{W}_s$ and $\mathcal{A}_s$ in a similar way.

Integrating Eq. (11) from 0 to T, we obtain

$$a(T) = a(0) + \int_0^T [\nu a + 2\gamma\varepsilon^{-n+2}B_c(a,\psi) + \gamma\varepsilon^{-n+3}B_c(\psi,\psi) - \varepsilon^{-n+2}\mathcal{F}_c(a+\varepsilon\psi)]ds + \tilde{W}_c(T), \tag{13}$$

Applying Itô's formula to $B_c(a,\mathcal{A}_s^{-1}\psi)$, yields

$$\begin{aligned}2\gamma\varepsilon^{-n+2}\int_0^T B_c(a,\psi)ds &= -4\gamma^2\varepsilon^{-n+4}\int_0^T B_c(B_c(a,\psi),\mathcal{A}_s^{-1}\psi)ds \\ &+ 2\gamma\varepsilon^{-n+4}\int_0^T B_c(\mathcal{F}_c(a),\mathcal{A}_s^{-1}\psi)ds - 2\gamma^2\varepsilon^{-n+2} \\ &\times\int_0^T B_c(a,\mathcal{A}_s^{-1}B_s(a,a))ds - 4\gamma^2\varepsilon^{-n+3}\int_0^T B_c(a,\mathcal{A}_s^{-1}B_s(a,\psi))ds \\ &- 2\gamma^2\varepsilon^{-n+4}\int_0^T B_c(a,\mathcal{A}_s^{-1}B_s(\psi))ds + 6\gamma\varepsilon^{-n+4} \\ &\times\int_0^T B_c(a,\mathcal{A}_s^{-1}\mathcal{F}_s(a,a,\psi))ds + 2\gamma\varepsilon^{-n+3}\int_0^T B_c(a,\mathcal{A}_s^{-1}\mathcal{F}_s(a))ds + R_1,\end{aligned} \tag{14}$$

where R_1 is given by

$$\begin{aligned}R_1(T) &= 2\varepsilon^2\gamma B(a(T),\mathcal{A}_s^{-1}\psi(T)) - 2\varepsilon^2\gamma B(a(0),\mathcal{A}_s^{-1}\psi(0)) \\ &- 4\gamma\nu\varepsilon^2\int_0^T B(a,\mathcal{A}_s^{-1}\psi)ds - 2\gamma^2\varepsilon^{-n+5}\int_0^T B(B_c(\psi,\psi), \\ &\times\mathcal{A}_s^{-1}\psi)ds + 6\gamma\varepsilon^{-n+5}\int_0^T B_c(\mathcal{F}_c(a,a,\psi),\mathcal{A}_s^{-1}\psi)ds \\ &+ 6\gamma\varepsilon^{-n+6}\int_0^T B_c(\mathcal{F}_c(a,\psi,\psi),\mathcal{A}_s^{-1}\psi)ds + 2\gamma\varepsilon^{-n+7} \\ &\times\int_0^T B_c(\mathcal{F}_c(\psi),\mathcal{A}_s^{-1}\psi)ds - 2\gamma\varepsilon^2\int_0^T B_c(d\tilde{W}_c,\mathcal{A}_s^{-1}\psi) \\ &+ 6\gamma\varepsilon^{-n+5}\int_0^T B_c(a,\mathcal{A}_s^{-1}\mathcal{F}_s(a,\psi,\psi))ds + 2\gamma\varepsilon^{-n+6} \\ &\times\int_0^T B(a,\mathcal{A}_s^{-1}\mathcal{F}_s(\psi))ds - 2\gamma\varepsilon\int_0^T B_c(a,\mathcal{A}_s^{-1}d\tilde{W}_s).\end{aligned} \tag{15}$$

By direct estimates we show that all terms in R_1 are small.

Now, let us consider two cases depending on the value of n and γ^2.

3.1. First Case: $n=2$ and $\gamma^2<\frac{27}{38}$

In this case, by substituting from (14) into (13) we obtain the following amplitude equation with error term

$$a(T) = a(0) + \nu\int_0^T a(\tau)d\tau + \int_0^T \tilde{\mathcal{G}}(a(\tau))d\tau + \tilde{W}_c(T) + \tilde{R}_1(T), \tag{16}$$

where the cubic term $\tilde{\mathcal{G}}(a)$ and the remainder $\tilde{R}_1$ are given by

$$\tilde{\mathcal{G}}(a) = -2\gamma^2 B_c(a,\mathcal{A}_s^{-1}B_s(a,a)) + \mathcal{F}_c(a) = -\frac{3}{4}\left(1-\frac{38}{27}\gamma^2\right)\langle a,e_1\rangle^3 e_1, \tag{17}$$

and

$$\begin{aligned}\tilde{R}_1(T) &= R_1(T) - 4\gamma^2\varepsilon^2\int_0^T B_c(B_c(a,\psi),\mathcal{A}_s^{-1}\psi)ds + 2\gamma\varepsilon^2 \\ &\times\int_0^T B_c(\mathcal{F}_c(a),\mathcal{A}_s^{-1}\psi)ds - 4\gamma^2\varepsilon\int_0^T B_c(a,\mathcal{A}_s^{-1}B_s(a,\psi)) \\ &\times ds - 2\gamma^2\varepsilon^2\int_0^T B_c(a,\mathcal{A}_s^{-1}B_s(\psi))ds + 2\gamma\varepsilon\int_0^T B_c(a,\mathcal{A}_s^{-1} \\ &\times\mathcal{F}_s(a))ds + \gamma\varepsilon\int_0^T B_c(\psi,\psi)d\tau - \varepsilon^3\int_0^T \mathcal{F}_c(\psi)d\tau - 3\varepsilon \\ &\times\int_0^T \mathcal{F}_c(a,a,\psi)d\tau - 3\varepsilon^2\int_0^T \mathcal{F}_c(a,\psi,\psi)d\tau,\end{aligned} \tag{18}$$

with R_1 defined in (15).

We fix $\nu_\varepsilon=\nu\varepsilon^2$ and $\sigma_\varepsilon=\varepsilon^2$. Then the main result in this case is given in the following theorem:

Theorem 5 (*Approximation 1*). *Under Assumptions* 1 *and* 2 *let* u *be a solution of* (1) *with the splitting introduced in* (10) *and initial condition* $u(0)=\varepsilon a(0)+\varepsilon^2\psi(0)$ *such that* $a(0)\in\mathcal{N}$ *and* $\psi(0)\in S$ *where* $a(0)$ *and* $\psi(0)$ *are of order one. Let* b *be a solution of* (3) *with* $b(0)=\langle a(0),e_1\rangle$. *Then for all* $p>1$ *and* $T_0>0$ *and all* $\kappa\in(0,\frac{1}{14})$, $\varepsilon\in(0,1)$, *there exists a constant* $C>0$ *such that*

$$\mathbb{P}\left(\sup_{t\in[0,\varepsilon^{-2}T_0]}\|u(t)-\varepsilon b(\varepsilon^2 t)\cos\|_{\mathcal{H}^1}>\varepsilon^{2-28\kappa}\right)\leqslant C\varepsilon^p. \tag{19}$$

3.2. Second case: $n=4$ and $\gamma^2=\frac{27}{38}$

The second case is slightly more stable, as we loose the cubic in the amplitude equation. Thus we need a different scaling. In this case, we apply Itô's formula to $B_c(\psi_k e_k,\psi_\ell e_\ell)$ in order to obtain

$$\begin{aligned}\frac{\gamma}{\varepsilon}\int_0^T B_c(\psi,\psi)ds&=\frac{1}{\varepsilon}\gamma\sum_{k,\ell}\int_0^T B_c(\psi_k e_k,\psi_\ell e_\ell)ds\\&=\frac{1}{\varepsilon}\sum_{k,\ell}\frac{2\gamma^2}{(\lambda_k+\lambda_\ell)}\int_0^T B_c(B_k(a)e_k,\psi_\ell e_\ell)ds+\sum_{k,\ell}\frac{4\gamma^2}{(\lambda_k+\lambda_\ell)}\\&\quad\times\int_0^T B_c(B_k(a,\psi)e_k,\psi_\ell e_\ell)ds-\sum_{k,\ell}\frac{2\gamma}{(\lambda_k+\lambda_\ell)}\\&\quad\times\int_0^T B_c(\mathcal{F}_k(a)e_k,\psi_\ell e_\ell)ds+R_2,\end{aligned}\tag{20}$$

where we used $B_k(w)=\langle B(w),e_k\rangle$ and $\mathcal{F}_k(w)=\langle\mathcal{F}(w),e_k\rangle$ for shorthand notation. The error R_2 in Eq. (20) is defined by

$$\begin{aligned}R_2(T)&=\sum_{k,\ell}\frac{-\varepsilon^3\gamma}{(\lambda_k+\lambda_\ell)}[B_c(\psi_k(0)e_k,\psi_\ell(0)e_\ell)-B_c(\psi_k(T)e_k,\psi_\ell(T)e_\ell)]\\&\quad+2\varepsilon^3\gamma\nu\sum_{k,\ell}\int_0^T B_c(\psi_k e_k,\psi_\ell e_\ell)ds+\varepsilon\sum_{k,\ell}\frac{2\gamma^2}{(\lambda_k+\lambda_\ell)}\\&\quad\times\int_0^T B_c(B_k(\psi)e_k,\psi_\ell e_\ell)ds-\varepsilon\sum_{k,\ell}\frac{3\gamma}{(\lambda_k+\lambda_\ell)}\\&\quad\times\int_0^T B_c(\mathcal{F}_k(a,a,\psi)e_k,\psi_\ell e_\ell)ds-\varepsilon^2\sum_{k,\ell}\frac{\alpha_\ell\gamma}{(\lambda_k+\lambda_\ell)}\\&\quad\times\int_0^T\left[2B_c(\psi_k e_k,e_\ell)-\frac{\gamma}{2}\varepsilon^2\sum_{k,\ell}\frac{\alpha_\ell\alpha_k}{(\lambda_k+\lambda_\ell)}B_c(e_k,e_\ell)d\tilde\beta_k\right]d\tilde\beta_\ell\\&\quad-\varepsilon^2\sum_{k,\ell}\frac{\gamma}{(\lambda_k+\lambda_\ell)}\int_0^T[3B_c(\mathcal{F}_k(a,\psi,\psi)e_k,\psi_\ell e_\ell)+\varepsilon B_c(\mathcal{F}_k(\psi)e_k,\psi_\ell e_\ell)]ds\end{aligned}\tag{21}$$

Again, we show later that all terms in R_2 are of order ε. By substituting (14) and (20) into (13) we obtain

$$\begin{aligned}a(T)&=a(0)+\nu\int_0^T ads-4\gamma^2\int_0^T B_c(B_c(a,\psi),\mathcal{A}_s^{-1}\psi)ds+2\gamma\\&\quad\times\int_0^T B_c(\mathcal{F}_c(a),\mathcal{A}_s^{-1}\psi)ds-\frac{4\gamma^2}{\varepsilon}\int_0^T B_c(a,\mathcal{A}_s^{-1}B_s(a,\psi))ds\\&\quad-2\gamma^2\int_0^T B_c(a,\mathcal{A}_s^{-1}B_s(\psi))ds+\frac{2\gamma}{\varepsilon}\int_0^T B_c(a,\mathcal{A}_s^{-1}\mathcal{F}_s(a))ds\\&\quad+6\gamma\int_0^T B_c(a,\mathcal{A}_s^{-1}\mathcal{F}_s(a,a,\psi))ds+\frac{1}{\varepsilon}\sum_{k,\ell}\frac{2\gamma^2B_k(a)}{(\lambda_k+\lambda_\ell)}\\&\quad\times\int_0^T B_c(e_k,\psi_\ell e_\ell)ds+\sum_{k,\ell}\frac{2\gamma}{(\lambda_k+\lambda_\ell)}\\&\quad\times\int_0^T[2\gamma B_c(B_k(a,\psi)e_k,\psi_\ell e_\ell)-B_c(\mathcal{F}_k(a)e_k,\psi_\ell e_\ell]ds\\&\quad-\frac{1}{\varepsilon}\int_0^T\mathcal{F}_c(a,a,\psi)ds-\int_0^T\mathcal{F}_c(a,\psi,\psi)ds+\tilde W_c(T)+R_3,\end{aligned}\tag{22}$$

where we used

$$-\frac{1}{\varepsilon^2}\mathcal{F}_c(a)-\frac{2\gamma^2}{\varepsilon^2}B_c(a,\mathcal{A}_s^{-1}B_s(a,a))=0,$$

when $\gamma^2=\frac{27}{38}$, and

$$R_3=R_1+R_2-\varepsilon\int_0^T\mathcal{F}_c(\psi)ds,\tag{23}$$

where R_1 and R_2 are defined in (15) and (21), respectively. Now, we need to remove ψ from the right hand side of (22). To do this, we explicitly average all terms by applying Itô formula to every term containing ψ on the right hand side. For the first term containing ψ in (22) we apply Itô formula to $B_c(B_c(a,\psi_k e_k),\mathcal{A}_s^{-1}\psi_\ell e_\ell)$ and obtain

$$\begin{aligned}-4\gamma^2\int_0^T B_c(B_c(a,\psi),\mathcal{A}_s^{-1}\psi)ds&=\sum_{k,\ell}\frac{8\gamma^3B_k(a)}{\lambda_\ell(\lambda_k+\lambda_\ell)}\\&\quad\times\int_0^T B_c(B_c(a,e_k),\psi_\ell e_\ell)ds+\mathcal{O}(\varepsilon^{1-15\kappa})=\sum_{k,\ell}\frac{8\gamma^4B_k(a)B_\ell(a)}{\lambda_\ell^2(\lambda_k+\lambda_\ell)}\\&\quad\times\int_0^T B_c(B_c(a,e_k),e_\ell)ds+\mathcal{O}(\varepsilon^{1-15\kappa}).\end{aligned}\tag{24}$$

For the second term containing ψ in (22) we consider $B_c(\mathcal{F}_c(a),\mathcal{A}_s^{-1}\mathcal{A}_s^{-1}\psi)$ to get

$$2\gamma\int_0^T B_c(\mathcal{F}_c(a),\mathcal{A}_s^{-1}\psi)ds=-2\gamma^2\int_0^T B_c(\mathcal{F}_c(a),\mathcal{A}_s^{-1}\mathcal{A}_s^{-1}B_s(a))ds+\mathcal{O}(\varepsilon^{1-14\kappa}).\tag{25}$$

For the third term containing ψ in (22) we apply Itô formula to $B_c(a,\mathcal{A}_s^{-1}B_s(a,\mathcal{A}_s^{-1}\psi))$. This yields

$$\begin{aligned}-\frac{4\gamma^2}{\varepsilon}\int_0^T B_c(a,\mathcal{A}_s^{-1}B_s(a,\psi))ds&=\frac{4\gamma^3}{\varepsilon}\\&\quad\times\int_0^T B_c(a,\mathcal{A}_s^{-1}B_s(a,\mathcal{A}_s^{-1}B_s(a)))ds+8\gamma^3\\&\quad\times\int_0^T B_c(a,\mathcal{A}_s^{-1}B_s(a,\mathcal{A}_s^{-1}B_s(a,\psi)))ds-4\gamma^2\\&\quad\times\int_0^T B_c(a,\mathcal{A}_s^{-1}B_s(a,\mathcal{A}_s^{-1}\mathcal{F}_s(a)))ds+\mathcal{O}(\varepsilon^{1-13\kappa})=-8\gamma^4\\&\quad\times\int_0^T B_c(a,\mathcal{A}_s^{-1}B_s(a,\mathcal{A}_s^{-1}B_s(a,\mathcal{A}_s^{-1}B_s(a))))ds-4\gamma^2\\&\quad\times\int_0^T B_c(a,\mathcal{A}_s^{-1}B_s(a,\mathcal{A}_s^{-1}\mathcal{F}_s(a)))ds+\mathcal{O}(\varepsilon^{1-13\kappa}).\end{aligned}\tag{26}$$

For the fourth term containing ψ in (22) we work with $B_c(a,\mathcal{A}_s^{-1}B_s(\psi_k e_k,\psi_\ell e_\ell))$ to obtain

$$\begin{aligned}&-2\gamma^2\int_0^T B_c(a,\mathcal{A}_s^{-1}B_s(\psi))ds\\&\quad=\sum_{k,\ell}\frac{-4\gamma^3B_k(a)}{(\lambda_k+\lambda_\ell)}\int_0^T B_c(a,\mathcal{A}_s^{-1}B_s(e_k,\psi_\ell e_\ell))ds+\mathcal{O}(\varepsilon^{1-15\kappa})\\&\quad=\sum_{k,\ell}\frac{-4\gamma^4B_k(a)B_\ell(a)}{\lambda_\ell(\lambda_k+\lambda_\ell)}\int_0^T B_c(a,\mathcal{A}_s^{-1}B_s(e_k,e_\ell))ds+\mathcal{O}(\varepsilon^{1-15\kappa}).\end{aligned}\tag{27}$$

For the fifth term containing ψ in (22) we apply Itô formula to $B_c(a,\mathcal{A}_s^{-1}\mathcal{F}_s(a,a,\mathcal{A}_s^{-1}\psi))$ in order to obtain

$$6\gamma\int_0^T B_c(a,\mathcal{A}_s^{-1}\mathcal{F}_s(a,a,\psi))ds = -6\gamma^2\int_0^T B_c(a,\mathcal{A}_s^{-1}\mathcal{F}_s(a,a,\mathcal{A}_s^{-1}\times B_s(a)))ds + \mathcal{O}(\varepsilon^{1-14\kappa}). \quad (28)$$

For the sixth term containing ψ in (22) we consider $B_c(B_k(a)e_k,\psi_\ell e_\ell)$.

$$\begin{aligned}\frac{1}{\varepsilon}\sum_{k,\ell}\frac{2\gamma^2}{(\lambda_k+\lambda_\ell)}\int_0^T B_c(B_k(a)e_k,\psi_\ell e_\ell)ds &= \frac{2}{\varepsilon}\sum_{k,\ell}\frac{\gamma^3 B_k(a)B_\ell(a)}{\lambda_\ell(\lambda_k+\lambda_\ell)}\\ &\times\int_0^T B_c(e_k,e_\ell)ds + \sum_{k,\ell}\frac{4\gamma^3 B_k(a)}{\lambda_\ell(\lambda_k+\lambda_\ell)}\int_0^T B_c(e_k,B_\ell(a,\psi)e_\ell)ds\\ &-\sum_{k,\ell}\frac{2\gamma^2 B_k(a)}{\lambda_\ell(\lambda_k+\lambda_\ell)}\int_0^T B_c(e_k,\mathcal{F}_\ell(a)e_\ell)ds + \mathcal{O}(\varepsilon^{1-13\kappa})\\ &= -\sum_{k,\ell}\frac{4\gamma^4 B_k(a)}{\lambda_\ell(\lambda_k+\lambda_\ell)}\int_0^T B_c(e_k,B_\ell(a,\mathcal{A}_s^{-1}B_s(a))e_\ell)ds\\ &-\sum_{k,\ell}\frac{2\gamma^2 B_k(a)}{\lambda_\ell(\lambda_k+\lambda_\ell)}\int_0^T B_c(e_k,\mathcal{F}_\ell(a)e_\ell)ds + \mathcal{O}(\varepsilon^{1-13\kappa}). \quad (29)\end{aligned}$$

For the seventh term containing ψ in (22) we work with $B_c(B_k(a,\psi_j e_j)e_k,\psi_\ell e_\ell)$ to obtain

$$\begin{aligned}\sum_{k,\ell}\frac{4\gamma^2}{(\lambda_k+\lambda_\ell)}\int_0^T B_c(B_k(a,\psi)e_k,\psi_\ell e_\ell)ds &= \sum_{k,\ell,j}\frac{4\gamma^3 B_\ell(a)}{(\lambda_k+\lambda_\ell)(\lambda_j+\lambda_\ell)}\\ &\times\int_0^T B_c(B_k(a,\psi_j e_j)e_k,e_\ell)ds + \sum_{k,\ell,j}\frac{4\gamma^3 B_j(a)}{(\lambda_k+\lambda_\ell)(\lambda_j+\lambda_\ell)}\\ &\times\int_0^T B_c(B_k(a,e_j)e_k,\psi_\ell e_\ell)ds + \mathcal{O}(\varepsilon^{1-15})\\ &= \sum_{k,\ell,j}\frac{8\gamma^4 B_j(a)B_\ell(a)}{\lambda_\ell(\lambda_k+\lambda_\ell)(\lambda_j+\lambda_\ell)}\int_0^T B_c(B_k(a,e_j)e_k,e_\ell)ds + \mathcal{O}(\varepsilon^{1-15\kappa}).\end{aligned} \quad (30)$$

For the eighth term. we apply Itô formula to $B_c(\mathcal{F}_k(a)e_k,\psi_\ell e_\ell)$.

$$\sum_{k,\ell}\frac{-2\gamma}{(\lambda_k+\lambda_\ell)}\int_0^T B_c(\mathcal{F}_k(a)e_k,\psi_\ell e_\ell)ds = \sum_{k,\ell}\frac{-2\gamma^2\mathcal{F}_k(a)B_\ell(a)}{\lambda_\ell(\lambda_k+\lambda_\ell)}\times\int_0^T B_c(e_k,e_\ell)ds + \mathcal{O}(\varepsilon^{1-14\kappa}). \quad (31)$$

For the ninth term containing ψ in (22) we apply Itô formula to $\mathcal{F}_c(a,a,\mathcal{A}_s^{-1}\psi)$.

$$\begin{aligned}\frac{-3}{\varepsilon}\int_0^T \mathcal{F}_c(a,a,\psi)ds &= \frac{-3\gamma}{\varepsilon}\int_0^T \mathcal{F}_c(a,a,\mathcal{A}_s^{-1}B_s(a))ds - 3\\ &\times\int_0^T \mathcal{F}_c(a,a,\mathcal{A}_s^{-1}\mathcal{F}_s(a))ds + 6\gamma\int_0^T \mathcal{F}_c(a,a,\mathcal{A}_s^{-1}B_s(a,\psi))ds\\ &+\mathcal{O}(\varepsilon^{1-13\kappa}) = -3\int_0^T \mathcal{F}_c(a,a,\mathcal{A}_s^{-1}\mathcal{F}_s(a))ds - 6\gamma^2\\ &\times\int_0^T \mathcal{F}_c(a,a,\mathcal{A}_s^{-1}B_s(a,\mathcal{A}_s^{-1}B_s(a)))ds + \mathcal{O}(\varepsilon^{1-13\kappa}), \quad (32)\end{aligned}$$

where we used that $\mathcal{F}_c(a,a,\mathcal{A}_s^{-1}B_s(a)) = 0$. For the last term containing ψ in (22). Consider $\mathcal{F}_c(a,\psi_k e_k,\psi_\ell e_\ell)$ in order to obtain

$$\begin{aligned}-3\int_0^T \mathcal{F}_c(a,\psi,\psi)ds &= \sum_{k,\ell}\frac{-6\gamma}{(\lambda_k+\lambda_\ell)}\int_0^T \mathcal{F}_c(a,B_k(a)e_k,\psi_\ell e_\ell)ds\\ &+\mathcal{O}(\varepsilon^{1-15\kappa}) = -\sum_{k,\ell}\frac{6\gamma^2 B_k(a)B_\ell(a)}{\lambda_\ell(\lambda_k+\lambda_\ell)}\int_0^T \mathcal{F}_c(a,e_k,e_\ell)ds\\ &+\mathcal{O}(\varepsilon^{1-15\kappa}). \quad (33)\end{aligned}$$

By substituting from (24)–(33) into (22) we obtain the following amplitude equation with error

$$a(T) = a(0) + \nu\int_0^T a(\tau)d\tau + \int_0^T \tilde{\mathcal{G}}(a(\tau))d\tau + \tilde{W}_c(T) + \tilde{R}_2(T), \quad (34)$$

where the quintic term $\tilde{\mathcal{G}}(a)$ is given by

$$\begin{aligned}\tilde{\mathcal{G}}(a) &= \sum_{k,\ell}\frac{8\gamma^4 B_k(a)B_\ell(a)}{\lambda_\ell^2(\lambda_k+\lambda_\ell)}B_c(B_c(a,e_k),e_\ell) - 2\gamma^2 B_c(\mathcal{F}_c(a),\mathcal{A}_s^{-1}\\ &\times\mathcal{A}_s^{-1}B_s(a)) - 8\gamma^4 B_c(a,\mathcal{A}_s^{-1}B_s(a,\mathcal{A}_s^{-1}B_s(a,\mathcal{A}_s^{-1}B_s(a))))\\ &-4\gamma^2 B_c(a,\mathcal{A}_s^{-1}B_s(a,\mathcal{A}_s^{-1}\mathcal{F}_s(a))) - \sum_{k,\ell}\frac{4\gamma^4 B_k(a)B_\ell(a)}{\lambda_\ell(\lambda_k+\lambda_\ell)}\\ &\times B_c(a,\mathcal{A}_s^{-1}B_s(e_k,e_\ell)) - 6\gamma^2 B_c(a,\mathcal{A}_s^{-1}\mathcal{F}_s(a,a,\mathcal{A}_s^{-1}B_s(a)))\\ &-\sum_{k,\ell}\frac{4\gamma^4 B_k(a)}{\lambda_\ell(\lambda_k+\lambda_\ell)}B_c(e_k,B_\ell(a,\mathcal{A}_s^{-1}B_s(a))e_\ell)\\ &-\sum_{k,\ell}\frac{2\gamma^2\mathcal{F}_k(a)B_\ell(a)}{\lambda_k\lambda_\ell}B_c(e_k,e_\ell) + \sum_{k,\ell,j}\frac{8\gamma^4 B_j(a)B_\ell(a)}{\lambda_\ell(\lambda_k+\lambda_\ell)(\lambda_j+\lambda_\ell)}\\ &\times B_c(B_k(a,e_j)e_k,e_\ell) - 3\mathcal{F}_c(a,a,\mathcal{A}_s^{-1}\mathcal{F}_s(a))\\ &-6\gamma^2\mathcal{F}_c(a,a,\mathcal{A}_s^{-1}B_s(a,\mathcal{A}_s^{-1}B_s(a))) - \sum_{k,\ell}\frac{6\gamma^2 B_k(a)B_\ell(a)}{\lambda_\ell(\lambda_k+\lambda_\ell)}\\ &\times\mathcal{F}_c(a,e_k,e_\ell) = -C_0\langle a,e_1\rangle^5 e_1,\end{aligned}$$

with $C_0 \simeq 1.8$ and the error term $\tilde{R}_2(T)$ is defined by

$$\tilde{R}_2 = R_3 + \mathcal{O}(\varepsilon^{1-15k}), \quad (35)$$

where R_3 was defined in (23).

The main result in this case (with the scaling $\nu_\varepsilon = \nu\varepsilon^4$ and $\sigma_\varepsilon = \varepsilon^3$) is given in the following:

Theorem 6 (*Approximation 2*). *Under Assumptions* 1 *and* 2 *let* u *be a solution of (1) defined in (10) with initial condition* $u(0) = \varepsilon a(0) + \varepsilon^2\psi(0)$ *where* $a(0)\in\mathcal{N}$ *and* $\psi(0)\in S$ *such that* $a(0)$ *and* $\psi(0)$ *are of order one. Let* b *be a solution of (3) with* $b(0) = \langle a(0),e_1\rangle$. *Then for all* $p > 1, \varepsilon\in(0,1)$, *and* $T_0 > 0$ *and all* $\kappa\in(0,\frac{1}{17})$, *there exists* $C > 0$ *such that*

$$\mathbb{P}\left(\sup_{t\in[0,\varepsilon^{-4}T_0]}\left\|u(t) - \varepsilon b(\varepsilon^4 t)\cos\right\|_{\mathcal{H}^1} > \varepsilon^{2-34\kappa}\right) \leqslant C\varepsilon^p. \quad (36)$$

4. Bounds for the high modes

In the following lemma we show that in (10) the modes $\psi\in S$ are essentially an OU-process plus a quadratic term in the modes $a\in\mathcal{N}$.

Lemma 7. *Under Assumption 2 let* $\mathcal{Z}(T)$ *be the* $\mathcal{S}$*-valued process solving for* $n = 2, 4$ *the SDE*

$$d\mathcal{Z} = \varepsilon^{-n}A_s\mathcal{Z}dT + \varepsilon^{-1}d\tilde{W}_s, \quad \mathcal{Z}(0) = \psi(0). \tag{37}$$

Then for $\varepsilon \in (0,1)$ *and* $0 < T \leqslant \tau^*$

$$\left\| \psi(T) - \mathcal{Z}(T) - \gamma\varepsilon^{-n}\int_0^T e^{\varepsilon^{-n}A_s(T-\tau)}B_s(a(\tau))d\tau \right\|_{\mathcal{H}^1} \leqslant C\varepsilon^{1-9\kappa}. \tag{38}$$

Proof. The mild formulation of (12) is

$$\psi(T) = \mathcal{Z}(T) + \int_0^T e^{\varepsilon^{-n}A_s(T-\tau)} \times \left[\nu\psi + \varepsilon^{-n}\gamma B_s(a+\varepsilon\psi) - \varepsilon^{-n+1}\mathcal{F}_s(a+\varepsilon\psi)\right]d\tau. \tag{39}$$

Thus we obtain

$$\left\|\psi(T) - \mathcal{Z}(T) - \gamma\varepsilon^{-n}\int_0^T e^{\varepsilon^{-n}A_s(T-\tau)}B_s(a(\tau))d\tau\right\|_{\mathcal{H}^1} \leqslant C \left\|\int_0^T e^{\varepsilon^{-n}A_s(T-\tau)}\psi d\tau\right\|_{\mathcal{H}^1} + C\varepsilon^{-n+1}\left\|\int_0^T e^{\varepsilon^{-n}A_s(T-\tau)}B_s(a(\tau),\psi(\tau))d\tau\right\|_{\mathcal{H}^1} + \varepsilon^{-n+2}C\left\|\int_0^T e^{\varepsilon^{-n}A_s(T-\tau)}B_s(\psi)d\tau\right\|_{\mathcal{H}^1} + C\varepsilon^{-n+1} \left\|\int_0^T e^{\varepsilon^{-4}A_s(T-\tau)}\mathcal{F}_s(a+\varepsilon\psi)d\tau\right\|_{\mathcal{H}^1} := I_1 + I_2 + I_3 + I_4.$$

We now bound all four terms separately. For the first term, using (6), we obtain for all $T \leqslant \tau^*$

$$I_1 \leqslant C \sup_{\tau\in[0,\tau^*]} \|\psi(\tau)\|_{\mathcal{H}^1} \int_0^{\varepsilon^{-n}\omega T} e^{-\eta}d\eta \leqslant C\varepsilon^{n-3\kappa},$$

where we used the definition of τ^*. For the second term we obtain

$$I_2 \leqslant C\varepsilon^{-n+1}\int_0^T e^{-\varepsilon^{-n}\omega(T-\tau)}\|B_s(a(\tau),\psi(\tau))\|_{\mathcal{H}^1}d\tau$$
$$\leqslant C\varepsilon \sup_{\tau\in[0,\tau^*]}\{\|a(\tau)\|_{\mathcal{H}^1}\|\psi(\tau)\|_{\mathcal{H}^1}\} \cdot \int_0^{\varepsilon^{-n}\omega T} e^{-\eta}d\eta \leqslant C\varepsilon^{1-4\kappa},$$

where we used again the definition of τ^*. Analogously, we derive for the third term

$$I_3 \leqslant C\varepsilon^2 \sup_{\tau\in[0,\tau^*]}\|\psi(\tau)\|^2_{\mathcal{H}^1}\int_0^{\varepsilon^{-n}\omega T} e^{-\eta}d\eta \leqslant C\varepsilon^{2-6\kappa}.$$

For the fourth term we obtain by using (6) and the definition of τ^*, that

$$I_4 \leqslant C\varepsilon^{-n+1}\int_0^T e^{-\varepsilon^{-n}\omega(T-\tau)}\|\mathcal{F}_s(a(\tau)+\varepsilon\psi(\tau))\|_{\mathcal{H}^1}d\tau$$
$$\leqslant C\varepsilon\left(\sup_{[0,\tau^*]}\|a\|^3_{\mathcal{H}^1} + \varepsilon\sup_{[0,\tau^*]}\|\psi\|^3_{\mathcal{H}^1}\right)\int_0^{\varepsilon^{-n}\omega T} e^{-\eta}d\eta \leqslant C\varepsilon^{1-9\kappa}.$$

Combining all results, yields (38). The proof is complete. □

The next lemma provides bounds for the stochastic convolution $\mathcal{Z}(T)$ defined in (37).

Lemma 8. *Under Assumption 2, for every* $\kappa_0 > 0$ *and* $p \geqslant 1$, *there exists a constant* C*, depending on* p, α_k, λ_k, κ_0 *and* T_0, *such that*

$$\mathbb{E}\sup_{T\in[0,T_0]}\|\mathcal{Z}(T)\|^p_{\mathcal{H}^1} \leqslant C\varepsilon^{-\kappa_0}.$$

Proof. See the proof of Lemma 20 in [7]. □

We now need the following simple estimate.

Lemma 9. *Using* τ^* *defined in Definition 3, then for* $n = 2, 4$ *we obtain*

$$\mathbb{E}\left(\sup_{T\in[0,\tau^*]}\left\|\int_0^T e^{\varepsilon^{-n}A_s(T-\tau)}B_s(a,a)d\tau\right\|^p_{\mathcal{H}^1}\right) \leqslant C\varepsilon^{np-2p\kappa}, \tag{40}$$

for all $\varepsilon \in (0,1)$.

Proof. Using (6) we obtain, for $T < \tau^*$, that

$$\left\|\int_0^T e^{\varepsilon^{-n}A_s(T-\tau)}B_s(a)d\tau\right\|_{\mathcal{H}^1} \leqslant C\varepsilon^n \sup_{\tau\in[0,\tau^*]}\|a(\tau)\|^2_{\mathcal{H}^1}\int_0^{\varepsilon^{-4}\omega T} e^{-\eta}d\eta \leqslant C\varepsilon^{n-2\kappa}. \qquad \square$$

The following corollary states that $\psi(T)$ is with high probability much smaller than $\varepsilon^{-\kappa}$ as asserted by the Definition 3 for $T \leqslant \tau^*$. We will show later $\tau^* \geqslant T_0$ with high probability (cf. proof of Theorem 5).

Corollary 10. *Under the assumptions of Lemmas 7 and 8, if* $\psi(0) = \mathcal{O}(1)$, *then for* $p > 0$ *and for all* $\kappa_0 > 0$ *there exists a constant* $C > 0$ *such that*

$$\mathbb{E}\left(\sup_{T\in[0,\tau^*]}\|\psi(T)\|^p_{\mathcal{H}^1}\right) \leqslant C\varepsilon^{-2\kappa}. \tag{41}$$

Proof. From (39), by triangle inequality and Lemmas 8 and 9, we obtain

$$\mathbb{E}\left(\sup_{T\in[0,\tau^*]}\|\psi(T)\|^p_{\mathcal{H}^1}\right) \leqslant C + C\varepsilon^{-\kappa_0} + C\varepsilon^{-2p\kappa} + C\varepsilon^{p-9p\kappa},$$

for $\kappa < \frac{1}{9}$ and $\kappa_0 \leqslant \kappa$. This yields (41). The proof is complete. □

Now the next step is to bound the remainder $\tilde{R}_1$ in the case $n = 2$ (or $\tilde{R}_2$ for $n = 4$). This was defined in (18) (or (35)) we use it in order to show the approximation result later.

Lemma 11. *We assume that Assumption 2 holds. Then for all* $p > 0$, *there exists a constant* $C > 0$ *such that*

$$\mathbb{E}\left(\sup_{T\in[0,\tau^*]}\|\tilde{R}_\mu(T)\|^p_{\mathcal{H}^1}\right) \leqslant C\varepsilon^{1-\delta_\mu\kappa}, \tag{42}$$

where $\delta_\mu = 3\mu + 9$ *with* $\mu = \frac{n}{2}$ *for* $n = 2, 4$.

Proof. We use similar arguments as in the proof of Lemma 7 to obtain (42).

5. Proof of the main result

In order to prove the approximation result, we first need the following a-priori estimate for solutions of the amplitude equation.

Lemma 12. *Let Assumption 1, holds. Define $b(t)$ in $\mathbb{R}$ as the solution of (3). If the initial condition satisfies $\mathbb{E}|b(0)|^p \leqslant C$ for some $p > 1$, then there exists another constant C such that*

$$\mathbb{E}\sup_{T\in[0,\tau^*]}|b(T)|^p \leqslant C. \tag{43}$$

Proof. The existence and uniqueness of solutions for Eq. (3) are standard. To verify the bound in (43), we define Y as

$$Y(T) = b(T) - \alpha_1\tilde{\beta}_1(T). \tag{44}$$

Substituting this into (3), we obtain

$$\partial_T Y = \nu(Y + \alpha_1\tilde{\beta}_1) + \mathcal{G}(Y + \alpha_1\tilde{\beta}_1). \tag{45}$$

Taking the scalar product $\langle\cdot, Y\rangle_{\mathbb{R}}$ on both sides of (45), yields

$$\frac{1}{2}\partial_T|Y|^2 = \langle\nu(Y + \alpha_1\tilde{\beta}_1), Y\rangle_{\mathbb{R}} + \langle\mathcal{G}(Y + \alpha_1\tilde{\beta}_1), Y\rangle_{\mathbb{R}}.$$

Using Young and Cauchy-Schwarz inequalities and Assumption 1, for $n = 2, 4$, we obtain that

$$\frac{1}{2}\partial_T|Y|^2 \leqslant C + C\left|\tilde{\beta}_1\right|^{n+2} - \delta|Y|^{n+2} \leqslant C + C\left|\tilde{\beta}_1\right|^{n+2}$$

Taking $\frac{p}{2}$-th power and expectation, we obtain, for $\mu = 1, 2$ where $\mu = \frac{n}{2}$,

$$\mathbb{E}\sup_{[0,T_0]}|Y|^p \leqslant CT_0^{\frac{1}{2}p} + CT_0^{\frac{1}{2}p}\mathbb{E}\sup_{[0,T_0]}\left|\tilde{\beta}_1\right|^{p(\mu+1)} \leqslant C.$$

Together with (44), this implies

$$\mathbb{E}\sup_{[0,T_0]}|b|^p \leqslant C\mathbb{E}\sup_{[0,T_0]}|Y|^p + C\mathbb{E}\sup_{[0,T_0]}\left|\tilde{\beta}_1\right|^p \leqslant C. \qquad \square$$

Definition 13. Fix $\mu = \frac{n}{2}$ and $\kappa \in (0, \frac{1}{3n+11})$ for $n = 2$ or 4 . Define the set $\Omega^* \subset \Omega$ such that the following three estimates

$$\sup_{[0,\tau^*]}\|\psi\|_{\mathcal{H}^1} < C\varepsilon^{-\frac{5}{2}\kappa}, \tag{46}$$

$$\sup_{[0,\tau^*]}\|\tilde{R}_\mu\|_{\mathcal{H}^1} < C\varepsilon^{1-\delta_\mu\kappa-\kappa}, \tag{47}$$

and

$$\sup_{[0,\tau^*]}|b| < C\varepsilon^{-\frac{\kappa}{\mu+1}}, \tag{48}$$

hold on Ω^*.

Proposition 14. *The set Ω^* has approximately probability 1.*

Proof.

$$\mathbb{P}(\Omega^*) \geqslant 1 - \mathbb{P}(\sup_{[0,\tau^*]}\|\psi\|_{\mathcal{H}^1} \geqslant C\varepsilon^{-\frac{5}{2}\kappa})$$

$$-\mathbb{P}(\sup_{[0,\tau^*]}\|\tilde{R}_\mu\|_{\mathcal{H}^1} \geqslant C\varepsilon^{1-\delta_\mu\kappa-\kappa}) - \mathbb{P}(\sup_{[0,\tau^*]}|b| \geqslant C\varepsilon^{-\frac{\kappa}{\mu+1}}).$$

for $\mu = 1$ (or $\mu = 2$). Using Chebychev inequality, Corollary 10 and Lemmas 11, 12, we obtain for sufficiently large $q > 0$ that

$$\mathbb{P}(\Omega^*) \geqslant 1 - C[\varepsilon^{\frac{1}{2}q\kappa} + \varepsilon^{q\kappa} + \varepsilon^{\frac{1}{\mu+1}q\kappa}] \geqslant 1 - C\varepsilon^{\frac{1}{\mu+1}q\kappa} \geqslant 1 - C\varepsilon^p. \ \square \tag{49}$$

In the following we identify $\mathcal{N}$ with $\mathbb{R}$ and rewrite the amplitude Eq. (16) (or (34)) as

$$a_1(T) = a_1(0) + \nu\int_0^T a_1(\tau)d\tau + \int_0^T \mathcal{G}(a_1(\tau))d\tau + \alpha_1\tilde{\beta}_1(T) + \mathcal{R}_\mu(T), \tag{50}$$

where $a_1 = \langle a, e_1\rangle, \mathcal{R}_\mu = \langle\tilde{R}_\mu, e_1\rangle$ and $\mathcal{G}(a_1) = -Ca_1^{2\mu+1}$ for $\mu = 1$ (or 2).

Theorem 15. *Assume that Assumption 1 holds and suppose $a_1(0) = \mathcal{O}(1)$. Let $b(t)$ be a solution of (3) and a_1 is defined as in (50). If the initial condition satisfies $a_1(0) = b(0)$, then for $\kappa < \frac{2}{3n+22}$ with $n = 2$ (or $n = 4$),we obtain*

$$\sup_{T\in[0,\tau^*]}|a_1(T) - b(T)| \leqslant C\varepsilon^{2-(3n+22)\kappa} \text{ on } \Omega^*, \tag{51}$$

and

$$\sup_{T\in[0,\tau^*]}|a_1(T)| \leqslant C\varepsilon^{-\frac{2\kappa}{n+1}} \text{ on } \Omega^*. \tag{52}$$

Proof. Define $\varphi(T)$ as

$$\varphi(T) := a_1(T) - \mathcal{R}_\mu(T).$$

From (50) we obtain

$$\varphi(T) = a_1(0) + \nu\int_0^T(\varphi(\tau) + \mathcal{R}_\mu(\tau))d\tau + \int_0^T \mathcal{G}(\varphi(\tau) + \mathcal{R}_\mu(\tau))d\tau. \tag{53}$$

Define now $h(T)$ by

$$h(T) := \varphi(T) - b(T). \tag{54}$$

Subtracting (53) from (3), we obtain

$$h(T) = \nu\int_0^T h(\tau)d\tau + \nu\int_0^T \mathcal{R}_\mu(\tau)d\tau + \int_0^T[\mathcal{G}(h + b + \mathcal{R}_\mu) - \mathcal{G}(b)](\tau)d\tau.$$

Thus,

$$\partial_T h = \nu(h + \mathcal{R}_\mu) + \mathcal{G}(h + b + \mathcal{R}_\mu) - \mathcal{G}(b). \tag{55}$$

Taking the scalar product $\langle\cdot, h\rangle_{\mathbb{R}}$ on both sides of (55), yields

$$\frac{1}{2}\partial_T|h|^2 = \langle\partial_T h, h\rangle_{\mathbb{R}} = \nu\langle h, h\rangle_{\mathbb{R}} + \nu\langle\mathcal{R}_\mu, h\rangle_{\mathbb{R}} + \langle\mathcal{G}(b + h + \mathcal{R}_\mu) - \mathcal{G}(b), h\rangle_{\mathbb{R}},$$

where $\mathcal{G}(b) = -Cb^{2\mu+1}$ for $\mu = 1$ (or $\mu = 2$) where $\mu = \frac{n}{2}$. Using Young and Cauchy-Schwarz inequalities and Assumption 1, we obtain the following linear ordinary differential inequality

$$\partial_T|h|^2 \leqslant C[|h|^2 + |h|^{2\mu+2}] + C|\mathcal{R}_\mu|^2\left[1 + |\mathcal{R}_\mu|^{2\mu+2} + |b|^{2\mu+2}\right].$$

Using (47) and (48), we obtain, for $\mu = 1$ (or $\mu = 2$),

$$\partial_T|h|^2 \leqslant C[|h|^2 + |h|^{2\mu+2}] + C\varepsilon^{2-2(3\mu+11)\kappa} \text{ on } \Omega^*.$$

As long as $|h| < 1$, we obtain

$$\partial_T|h|^2 \leqslant 2C|h|^2 + C\varepsilon^{2-2(3\mu+11)\kappa} \text{ on } \Omega^*.$$

Integrating from 0 to T and using Gronwall's lemma, yields

$$|h|^2 \leqslant C\varepsilon^{2-2(3\mu+11)\kappa}.$$

Thus,

$$\sup_{[0,\tau^*]}|h| \leqslant C\varepsilon^{1-(3\mu+11)\kappa} \text{ on } \Omega^*. \tag{56}$$

We finish the first part by using (54) and (56) and

$$\sup_{[0,\tau^*]}|a_1 - b| = \sup_{[0,\tau^*]}|h + \mathcal{R}_\mu| \leqslant \sup_{[0,\tau^*]}|h| + \sup_{[0,\tau^*]}|\mathcal{R}_\mu|.$$

For the second part of the theorem we consider

$$\sup_{[0,\tau^*]}|a_1| \leqslant \sup_{[0,\tau^*]}|a_1 - b| + \sup_{[0,\tau^*]}|b|.$$

Using the first part and (48), we obtain the final result (52). □

Now, we can use the previous results to prove the main result of Theorem 5 in the case of $n = 2$ (or Theorem 6 in the case of $n = 4$) for the approximation of the solution (2) of the SPDE (1).

Proof of the Main Theorem. For the stopping time we note that

$$\Omega \supset \{\tau^* = T_0\} \supseteq \left\{ \sup_{[0,T_0]}\|a\|_{\mathcal{H}^1} < \varepsilon^{-\kappa}, \ \sup_{[0,T_0]}\|\psi\|_{\mathcal{H}^1} < \varepsilon^{-3\kappa} \right\} \supset \Omega^*.$$

Hence

$$\mathbb{P}\{\tau^* < T_0\} \leqslant \mathbb{P}\left\{ \sup_{[0,\tau^*]}\|a\|_{\mathcal{H}^1} > \varepsilon^{-\kappa}, \ \sup_{[0,\tau^*]}\|\psi\|_{\mathcal{H}^1} > \varepsilon^{-3\kappa} \right\} \leqslant C\varepsilon^{q\kappa}, \tag{57}$$

where we used Chebychev's inequality and (41). Now let us turn to the approximation result. Using (10) and triangle inequality, yields on Ω^* that

$$\begin{aligned} \sup_{T\in[0,\tau^*]} \|u(\varepsilon^{-n}T) - \varepsilon b(T)e_1\|_{\mathcal{H}^1} &\leqslant \varepsilon \sup_{[0,\tau^*]}\|a - be_1\|_{\mathcal{H}^1} + \varepsilon^2 \sup_{[0,\tau^*]}\|\psi\|_{\mathcal{H}^1} \\ &\leqslant \varepsilon \sup_{[0,\tau^*]}|a_1 - b| + \varepsilon^2 \sup_{[0,\tau^*]}\|\psi\|_{\mathcal{H}^1}. \end{aligned}$$

From (46) and (51) we obtain for $n = 2$ (or $n = 4$)

$$\sup_{t\in[0,\varepsilon^{-n}T_0]} \|u(t) - \varepsilon b(\varepsilon^n t)e_1\|_{\mathcal{H}^1} = \sup_{t\in[0,\varepsilon^{-n}\tau^*]} \|u(t) - \varepsilon b(\varepsilon^n t)\|_{\mathcal{H}^1} \leqslant C\varepsilon^{2-(3n+22)\kappa}.$$

Thus

$$\mathbb{P}\left(\sup_{t\in[0,\varepsilon^{-n}T_0]} \|u(t) - \varepsilon b(\varepsilon^n t)\|_{\mathcal{H}^1} > \varepsilon^{2-(3n+22)\kappa} \right) \leqslant 1 - \mathbb{P}(\Omega^*).$$

Using (49), yields (19) for $n = 2$ (or (19) for $n = 4$). The proof is complete. □

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