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Further results on Parity Combination Cordial Labeling

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Abstract

Let G be a (p, q)-graph. Let f be an injective mapping from $V(G)$ to $\{1, 2, \ldots, p\}$. For each edge xy, assign the label $\binom{x}{y}$ or $\binom{y}{x}$ according as $x > y$ or $y > x$. Call f a parity combination cordial labeling if $|e_f(0) - e_f(1)| \leq 1$, where $e_f(0)$ and $e_f(1)$ denote the number of edges labeled with an even number and an odd number, respectively. In this paper we make a survey on all graphs of order at most six and find out whether they satisfy a parity combination cordial labeling or not and get an upper bound for the number of edges q of any graph to satisfy this condition and describe the parity combination cordial labeling for two families of graphs.

Math Subject Classification: 05C78, 05C85, 05C30

Introduction

In this paper we will deal with finite simple undirected graphs. By $G(V, E)$ we mean a graph with p vertices and q edges, where $p = |V|$ and $q = |E|$. We follow Harary [1] for standard terminology and notations, and see Gallian [2] for more details on graph labelings.

Definition 1.1 [3] For a graph $G(p, q)$, let f be an injective mapping from $V(G)$ to $\{1, 2, \ldots, p\}$. For each edge xy, assign the label $\binom{x}{y}$ or $\binom{y}{x}$ according as $x > y$ or $y > x$. Call f a parity combination cordial labeling if $|e_f(0) - e_f(1)| \leq 1$, where $e_f(0)$ and $e_f(1)$ denote the number of edges labeled with an even number and an odd number, respectively. A graph with a parity combination cordial labeling is called a parity combination cordial graph (PCCG).

Ponraj, Sathish Narayanan, and Ramasamy [3, 4] proved that the following are parity combination cordial graphs: paths, cycles, stars, triangular snakes, alternate triangular snakes, olive trees, combs, crowns, fans, umbrellas, P_n^2, helms, dragons, bistars, butterfly graphs, and graphs obtained from C_n and $K_{1,m}$ by unifying a vertex of C_n and a pendent vertex of $K_{1,m}$. They also proved that W_n admits a parity combination cordial labeling if and only if $n \geq 4$, and conjectured that for $n \geq 4$, K_n is not a parity combination cordial graph. They also proved that if G is a parity combination cordial graph, then $G \cup P_n$ is also parity combination cordial if $n \neq 2, 4$.

In this paper we try to present some further results, we give the parity combination cordial labeling (PCCL) of all graphs of order at most six, make an algorithm that identify whether any graph of order p and size q can be a PCCG or not, give an upper bound to the number of edges of any graph which satisfy this condition and finally describe the PCCL function of the two graphs $K_{2,\ n}$ and $P_n^{(t)}$.

General results

Proposition 2.1 For a simple graph G with q-edges, if G is PCCL and q is even then $G \pm e$ is PCCG.

Proof Since G is PCCL and q is even, then $e_f(0) = e_f(1)$. Adding a new edge e will lead to $|e_f(0) - e_f(1)| = 1$, satisfying the PCCL condition.

Proposition 2.2 [2] $\binom{n}{2}$ is even if $n \equiv 0, 1 \pmod 4$ and odd if $n \equiv 2, 3 \pmod 4$.

Proposition 2.3 $\binom{n}{3}$ is even if $n \equiv 0, 1, 2 \pmod 4$ and odd if $n \equiv 3 \pmod 4$.

Proof For $n \equiv 1 \pmod 4$, we have three cases.

Case 1: $n = 12t + 1 \Longrightarrow \binom{n}{3} = \frac{(12t+1)(12t)(12t-1)}{6} = 2t(12t + 1)(12t-1) \Longrightarrow$ even.

Case 2: $n = 12t + 5 \Longrightarrow \binom{n}{3} = \frac{(12t+5)(12t+4)(12t+3)}{6} = 2(12t + 5)(3t + 1)(4t + 1) \Longrightarrow$ even.

Case 3: $n = 12t + 9 \Longrightarrow \binom{n}{3} = \frac{(12t+9)(12t+8)(12t+7)}{6} = 2(4t + 3)(3t + 2)(12t + 7) \Longrightarrow$ even.

Similarly for $n \equiv 0, 2 \pmod 4$.

For $n \equiv 3 \pmod 4$, we have also three cases.

Case 1: $n = 12t + 3 \Longrightarrow \binom{n}{3} = \frac{(12t+3)(12t+2)(12t+1)}{6} = (4t + 1)(6t + 1)(12t + 1) \Longrightarrow$ odd.

Case 2: $n = 12t + 7 \Longrightarrow \binom{n}{3} = \frac{(12t+7)(12t+6)(12t+5)}{6} = (12t + 7)(2t + 1)(12t + 5) \Longrightarrow$ odd.

Case 3: $n = 12t + 11 \Longrightarrow \binom{n}{3} = \frac{(12t+11)(12t+10)(12t+9)}{6} = (12t + 11)(6t + 5)(4t + 3) \Longrightarrow$ odd.

Proposition 2.4 All simple connected graphs of order at most five are PCCG except K_4, K_5, which is proved later using Algorithms 2.5 and 2.7, as shown in Fig. 1.

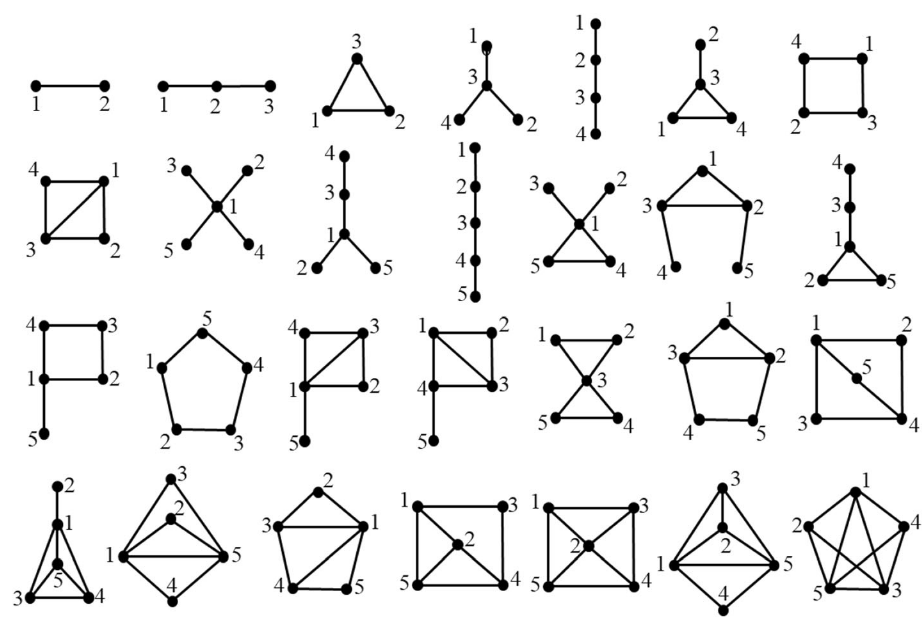


Fig. 1 PCCL for graphs with order at most 5

Algorithm 2.5 We made an algorithm to test any graph whether it is PCCG or not.

INPUT The number of vertices p and edges q of the graph G and the adjacent vertices to each edge.

OUTPUT Print out the set of vertex labels of G (if exist) satisfying a PCCL.

Step 1: Set v; (array with length n stores the labels of the vertices)
adj1; (array with length m stores the labels of the first vertex adjacent to each edge)
adj2; (array with length m stores the labels of the second vertex adjacent to each edge)
edgelabel; (array with length m stores the calculated labels of all edges)

Step 2: Enter the adjacent vertices to each edge (adj1 and adj2);

Step 3: Initialize $v = [1\ 2\ \dots\ n]$;

Step 4: Initialize Even = 0; (count even edge labels)
Odd = 0; (count odd edge labels)

Step 5: Define the function ${}^{n}C_{m}$

Step 6: Calculate the label of each edge: edgelabel $= {}^{n}C_{m}(v[\text{adj1}], v[\text{adj2}])$;

Step 7: If edgelabel is even, then Even++
Else Odd++

Step 8: If $|\text{Even} - \text{Odd}| \leq 1$, then the graph is PCCG.

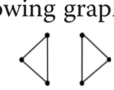
OUTPUT (v); (display the labels of the vertices)

Else, permute v; (make another permutation of the vector v)

Go to step 4

Step 9: STOP.

Proposition 2.6 All simple graphs of order six are PCCG, as shown in Appendix 1, except K_6, $K_5 \cup P_1$, $K_6 - e$ and the following graph (using Algorithm 2.5).



The upper bound for the number of edges q of any graph with $n \leq 100$ vertices to satisfy a PCCL is computed using the following algorithm and shown in Table 1.

Algorithm 2.7 In this algorithm we count the number of even entries and odd entries that are greater than one in the Pascal's triangle and compute twice the smaller number plus one.

INPUT The number of vertices p.

OUTPUT Print out the number of even and odd combinations up to the p^{th} row in Pascal's triangle and consequently the upper bound of number of edges satisfying a PCCL.

Step 1: Set $p - 1$; (number of rows in Pascal's triangle)

Step 2: Construct Pascal's triangle as
For ($i = 1$ to $p - 1$)
Entry (row i)(column 1) = Entry (row i)(last column)
For ($j = 2$ to $i - 1$)
Entry (row i)(column j) = Entry (row $i - 1$)(column $j - 1$) + Entry (row $i - 1$)(column j)

Step 3: Set Even = 0 and Odd = 0

Step 4: Check each Entry
If Entry (row i)(column j) is even, Then Even ++
Else, Then Odd++

Step 5: If Even > Odd
Output (Upper bound = 2*Odd + 1)
Else
Output (Upper bound = 2*Even + 1)

Step 6: STOP.

Based on the upper bound listed in Table 1, the following conjecture gives an upper bound for the number of edges of any graph to be PCCG.

Conjecture 2.8 The upper bound for the number of edges of any graph with n vertices ($\forall n \in \mathbb{N} - \{3, 7\}$ and $n \geq 2$) to satisfy a PCCL is given by

$$1 + 2\sum_{i=2}^{n}\left[2^{sb(i)} - 2\right]$$

where $sb(i)$ is number of times the digit 1 occurs in the binary representation of i.

Table 1 The upper bound for the number of edges of a graph with *n* vertices

n	Upper bound	$\|K_n\|$	*n*	Upper bound	$\|K_n\|$	*n*	Upper bound	$\|K_n\|$
2	1	1	35	381	595	68	1229	2278
3	3	3	36	385	630	69	1241	2346
4	5	6	37	397	666	70	1253	2415
5	9	10	38	409	703	71	1281	2485
6	13	15	39	437	741	72	1285	2556
7	19	21	40	441	780	73	1297	2628
8	25	28	41	453	820	74	1309	2701
9	29	36	42	465	861	75	1337	2775
10	33	45	43	493	903	76	1349	2850
11	45	55	44	505	946	77	1377	2926
12	49	66	45	533	990	78	1405	3003
13	61	78	46	561	1035	79	1465	3081
14	73	91	47	621	1081	80	1469	3160
15	101	105	48	625	1128	81	1481	3240
16	101	120	49	637	1176	82	1493	3321
17	105	136	50	649	1225	83	1521	3403
18	109	153	51	677	1275	84	1533	3486
19	121	171	52	689	1326	85	1561	3570
20	125	190	53	717	1378	86	1589	3655
21	137	210	54	745	1431	87	1649	3741
22	149	231	55	805	1485	88	1661	3828
23	177	253	56	817	1540	89	1689	3916
24	181	276	57	845	1596	90	1717	4005
25	193	300	58	873	1653	91	1777	4095
26	205	325	59	933	1711	92	1805	4186
27	233	351	60	961	1770	93	1865	4278
28	245	378	61	1021	1830	94	1925	4371
29	273	406	62	1081	1891	95	2049	4465
30	301	435	63	1205	1953	96	2053	4560
31	361	465	64	1205	2016	97	2065	4656
32	361	496	65	1209	2080	98	2077	4753
33	365	528	66	1213	2145	99	2105	4851
34	369	561	67	1225	2211	100	2117	4950

In Table 2 we compare the upper bound calculated from Algorithm 2.7 and that calculated from Conjecture 2.8 for $n = 100$ vertices and found that they match in all cases except for $n = 3$ and $n = 7$.

A PCCL of two graphs

In this section we present a PCCL of two families of graphs, the graph $K_{2,\,n}$ and the graph $P_n^{(t)}$which is the one point union of t copies of P_n.

Proposition 3.1: The graph $K_{2,\,n}$ is PCCG for $n \equiv 0, 2, 3 \pmod 4$.

Table 2 The upper bound for the number of edges of a graph with *n* vertices

n	Odd	upper bound	odd in row	$\|K_n\|$	bin(n)	sb(n)	$2^{sb(n))}-2$	upper bound
		Algorithm 2.7				Conjecture 2.8		
2	0	1	0	1	10	1	0	1
3	2	3	2	3	11	2	2	5
4	2	5	0	6	100	1	0	5
5	4	9	2	10	101	2	2	9
6	6	13	2	15	110	2	2	13
7	12	19	6	21	111	3	6	25
8	12	25	0	28	1000	1	0	25
9	14	29	2	36	1001	2	2	29
10	16	33	2	45	1010	2	2	33
11	22	45	6	55	1011	3	6	45
12	24	49	2	66	1100	2	2	49
13	30	61	6	78	1101	3	6	61
14	36	73	6	91	1110	3	6	73
15	50	101	14	105	1111	4	14	101
16	50	101	0	120	10000	1	0	101
17	52	105	2	136	10001	2	2	105
18	54	109	2	153	10010	2	2	109
19	60	121	6	171	10011	3	6	121
20	62	125	2	190	10100	2	2	125
21	68	137	6	210	10101	3	6	137
22	74	149	6	231	10110	3	6	149
23	88	177	14	253	10111	4	14	177
24	90	181	2	276	11000	2	2	181
25	96	193	6	300	11001	3	6	193
26	102	205	6	325	11010	3	6	205
27	116	233	14	351	11011	4	14	233
28	122	245	6	378	11100	3	6	245
29	136	273	14	406	11101	4	14	273
30	150	301	14	435	11110	4	14	301
31	180	361	30	465	11111	5	30	361
32	180	361	0	496	100000	1	0	361
33	182	365	2	528	100001	2	2	365
34	184	369	2	561	100010	2	2	369
35	190	381	6	595	100011	3	6	381
36	192	385	2	630	100100	2	2	385
37	198	397	6	666	100101	3	6	397
38	204	409	6	703	100110	3	6	409
39	218	437	14	741	100111	4	14	437
40	220	441	2	780	101000	2	2	441
41	226	453	6	820	101001	3	6	453
42	232	465	6	861	101010	3	6	465
43	246	493	14	903	101011	4	14	493
44	252	505	6	946	101100	3	6	505

Table 2 The upper bound for the number of edges of a graph with n vertices *(Continued)*

n	Odd	upper bound	odd in row	$\|K_n\|$	bin(n)	sb(n)	$2^{sb(n))}-2$	upper bound
		Algorithm 2.7				Conjecture 2.8		
45	266	533	14	990	101101	4	14	533
46	280	561	14	1035	101110	4	14	561
47	310	621	30	1081	101111	5	30	621
48	312	625	2	1128	110000	2	2	625
49	318	637	6	1176	110001	3	6	637
50	324	649	6	1225	110010	3	6	649
51	338	677	14	1275	110011	4	14	677
52	344	689	6	1326	110100	3	6	689
53	358	717	14	1378	110101	4	14	717
54	372	745	14	1431	110110	4	14	745
55	402	805	30	1485	110111	5	30	805
56	408	817	6	1540	111000	3	6	817
57	422	845	14	1596	111001	4	14	845
58	436	873	14	1653	111010	4	14	873
59	466	933	30	1711	111011	5	30	933
60	480	961	14	1770	111100	4	14	961
61	510	1021	30	1830	111101	5	30	1021
62	540	1081	30	1891	111110	5	30	1081
63	602	1205	62	1953	111111	6	62	1205
64	602	1205	0	2016	1000000	1	0	1205
65	604	1209	2	2080	1000001	2	2	1209
66	606	1213	2	2145	1000010	2	2	1213
67	612	1225	6	2211	1000011	3	6	1225
68	614	1229	2	2278	1000100	2	2	1229
69	620	1241	6	2346	1000101	3	6	1241
70	626	1253	6	2415	1000110	3	6	1253
71	640	1281	14	2485	1000111	4	14	1281
72	642	1285	2	2556	1001000	2	2	1285
73	648	1297	6	2628	1001001	3	6	1297
74	654	1309	6	2701	1001010	3	6	1309
75	668	1337	14	2775	1001011	4	14	1337
76	674	1349	6	2850	1001100	3	6	1349
77	688	1377	14	2926	1001101	4	14	1377
78	702	1405	14	3003	1001110	4	14	1405
79	732	1465	30	3081	1001111	5	30	1465
80	734	1469	2	3160	1010000	2	2	1469
81	740	1481	6	3240	1010001	3	6	1481
82	746	1493	6	3321	1010010	3	6	1493
83	760	1521	14	3403	1010011	4	14	1521
84	766	1533	6	3486	1010100	3	6	1533
85	780	1561	14	3570	1010101	4	14	1561
86	794	1589	14	3655	1010110	4	14	1589
87	824	1649	30	3741	1010111	5	30	1649

Table 2 The upper bound for the number of edges of a graph with n vertices *(Continued)*

n	Odd	upper bound	odd in row	$\|K_n\|$	bin(n)	sb(n)	$2^{sb(n))}-2$	upper bound
		Algorithm 2.7				Conjecture 2.8		
88	830	1661	6	3828	1011000	3	6	1661
89	844	1689	14	3916	1011001	4	14	1689
90	858	1717	14	4005	1011010	4	14	1717
91	888	1777	30	4095	1011011	5	30	1777
92	902	1805	14	4186	1011100	4	14	1805
93	932	1865	30	4278	1011101	5	30	1865
94	962	1925	30	4371	1011110	5	30	1925
95	1024	2049	62	4465	1011111	6	62	2049
96	1026	2053	2	4560	1100000	2	2	2053
97	1032	2065	6	4656	1100001	3	6	2065
98	1038	2077	6	4753	1100010	3	6	2077
99	1052	2105	14	4851	1100011	4	14	2105
100	1058	2117	6	4950	1100100	3	6	2117

Proof. Let the set of vertices be $V(K_{2,\,n}) = \{u_1, u_2; v_1, v_2, \dots, v_n\}$ and the set of edges be $E(K_{2,\,n}) = \{u_1v_1, u_1v_2, \dots, u_1v_n; u_2v_1, u_2v_2, \dots, u_2v_n\}$. It's clear that $|V(K_{2,\,n})| = n + 2$ and $|E(K_{2,\,n})| = 2n$. Then for $n \equiv 0, 2, 3 \pmod 4$, define the labeling function $f: V(K_{2,\,n}) \rightarrow \{1, 2, \dots, p = n + 2\}$ by

$$f(u_1) = 1, f(u_2) = 2,$$

and

$$f(v_i) = i + 2, 1 \le i \le n.$$

Which will lead to $e_f(0) = e_f(1) = n$ depending on proposition 2.2.

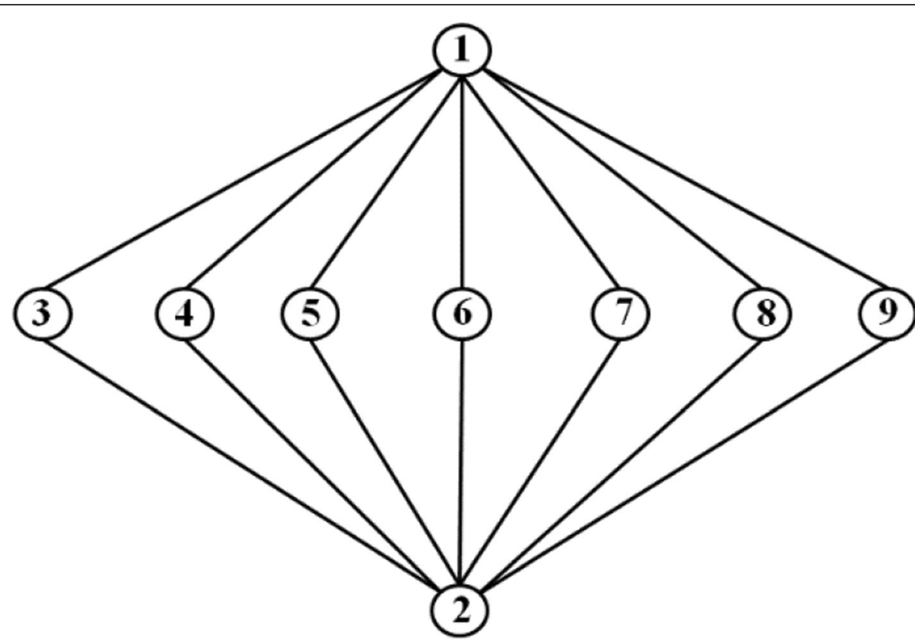


Fig. 2 A PCCL of the graph $K_{2,7}$

Example: The graph $K_{2,\,7}$ is PCCG as shown in Fig. 2.

Proposition 3.2: The graph $P_n^{(t)}$, the one point union of t copies of P_n, is PCCG.

Proof. Let the set of vertices be $V(P_n^{(t)}) = \{v_1^{(1)}, v_2^{(1)}, \ldots, v_n^{(1)}; v_2^{(2)}, \ldots, v_n^{(2)}; \ldots; v_2^{(t)}, \ldots, v_n^{(t)}\}$. It's clear that $|V(P_n^{(t)})| = (n-1)t + 1$ and $|E(P_n^{(t)})| = (n-1)t$. Then define the labeling function $f : V(P_n^{(t)}) \to \{1, 2, \ldots, (n-1)t + 1\}$ by

$$f\left(v_i^{(j)}\right) = \begin{cases} 1, & \mathit{if}\ i = j = 1 \\ (j-1)(n-1) + i, & 2 \le i \le n, \quad 1 \le j \le t \end{cases}$$

This function will lead to $0 \le |e_f(0) - e_f(1)| \le 1$ depending on proposition 2.2.

Example: The graph $P_7^{(4)}$ is PCCG as shown in Fig. 3.

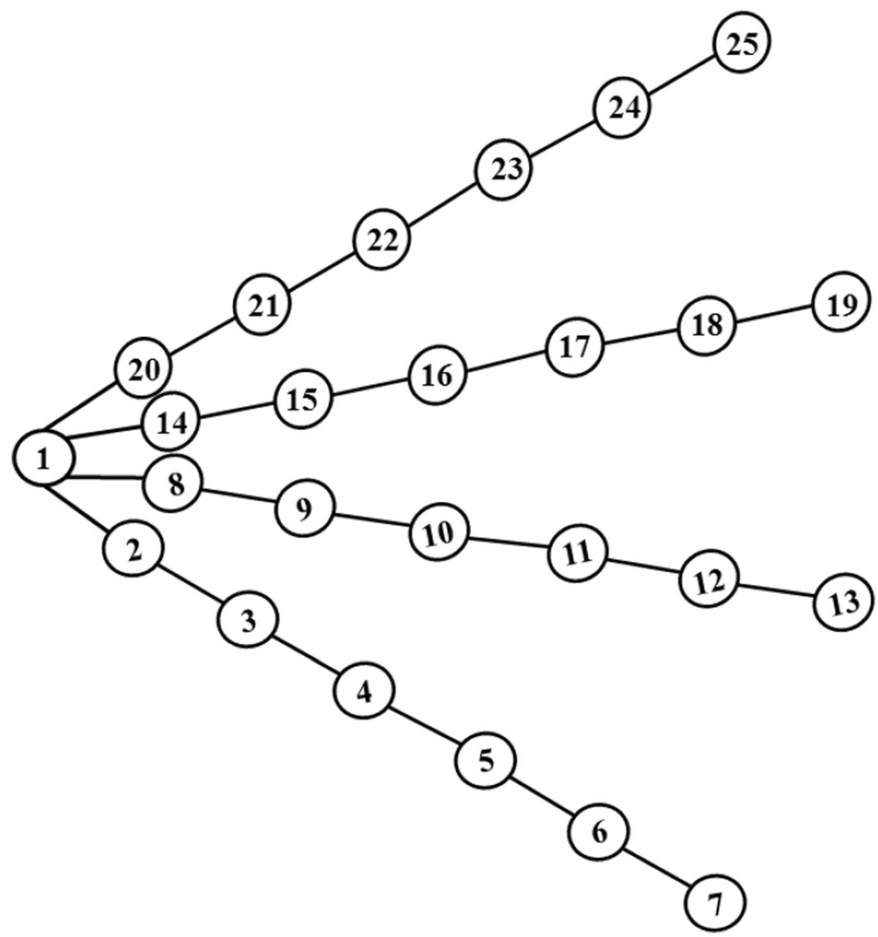
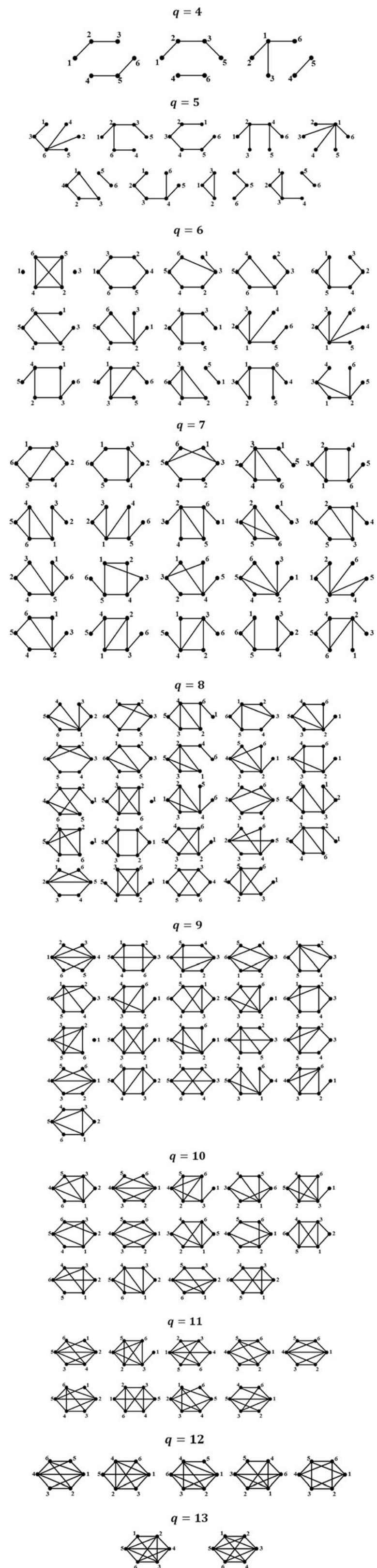


Fig. 3 A PCCL of the graph $P_7^{(4)}$

Appendix

Appendix 1: All graphs of order six according to Harary [2] that satisfy a PCCL



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Competing interests
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