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ORIGINAL ARTICLE

On positive braids motivated by Rossler dynamical system

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Abstract Rossler positive braids are defined and denoted by R_m^+. We represented a general form of Rossler positive braid, and its associated permutation. For the m period Rossler positive braid R_m^+, the number of crossing equals to $[2(m-2)+(m-1)]$, and the genus $g\ (R_m^+)=(m-2)$, it also has a braid index equals to m, $m>1$.

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1. Introduction

A flow is a family of maps $\Phi_t : R^3 \rightarrow R^3$, parameterized by $t \in R$. The solutions to three dimensional systems of differential equations are flows. For a given $x \in R^3$, the flow sweeps out a trajectory as t is varied. (Strictly speaking, dynamics primarily deals with semiflows, where t represents time and is restricted to be non-negative.) Knots and links arise in flows when the trajectory returns to its starting point in a finite time, and this is called a periodic orbit. Thus, the search for periodic orbits of a dynamic system is equivalent to the search for knots and links traced out by flow trajectories.

The periodic orbits that arise in the system introduced by Rossler system [1]:

$$\dot{x} = -(y+z),$$
$$\dot{y} = x + ay,$$
$$\dot{z} = b + z(x-c),$$

are chaotic by continuously increasing the parameter a from nonzero values. First an ordinary single limit cycle or "zero path" appears, then a double looped or "two path" one, then a spiral-type horseshoe chaos or "two path" one, then horseshoe chaos with a multiply folded underlying horse map.

The graphs for parameter values $a=b=0.2$ and four different values of c are shown in Fig. 1.

Definition 1 [2]. The braid group B_n can be defined via the following presentation, known as the braid presentation or Artin presentation:

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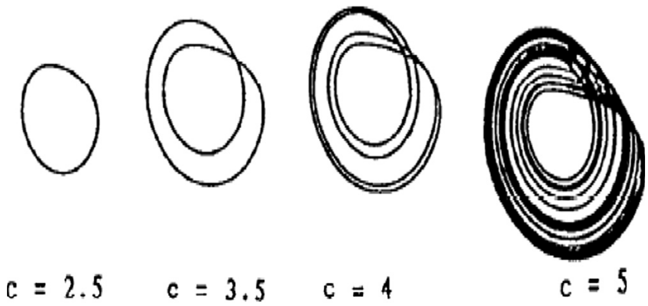

Figure 1 Zero, first, second, and third order Rossler knots.

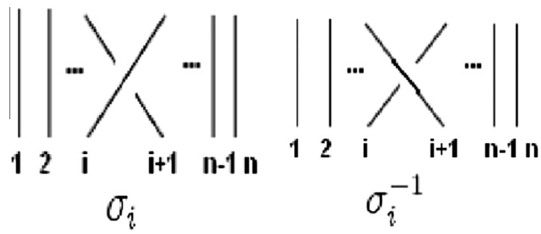

Figure 2 Generators of Braid group σ_i and σ_i^{-1}.

$$B_n = \left\{ \begin{array}{ll} \sigma_i, \ i = 1,2,\ldots,n-1: \ \sigma_i\sigma_j = \sigma_j\sigma_i & \text{if } |i-j| > 1, \\ \sigma_i\sigma_{i+1}\sigma_i = \sigma_{i+1}\sigma_i\sigma_{i+1} & \text{if } i = 1,2,\ldots,n-2 \end{array} \right\},$$

where σ_i and σ_i^{-1} as in Fig. 2.

Definition 2 [3]. A braid β consisting of an ordered sequence of the generators only, in which no inverse of any generator occurs will be called a positive braid and denoted by B_n^+.

Definition 3 [3]. A positive permutation braid is a positive braid where each pair of its strings crosses at most once.

The set of these braids in B_n is denoted by S_n^+, which was first introduced by Elrifai in [4].

Positive permutation braids are important and very useful in many fields, such as braid groups, knot theory, representation theory, cryptography and dynamical systems.

Definition 4 [4]. Lorenz link is a closed braid $\beta \in B_n$ for some integer n, where in β the strands have a natural ordering from left to right. Number them $1,2,\ldots,n$ on the top and on the bottom. These strings fall into two groups of parallel strands, a left group of k strands and a right group of r strands, $k+r=n$. The strands in the right group always pass over (not under) those in the left group, but strands in the same group never cross one another. These braids β are called a Lorenz braid of type (k,r) and denoted by $\beta(k,r)$.

Definition 5 [5]. An $n \times n$ matrix each of whose rows and columns has exactly two 1's and 0's elsewhere is called a Cromwell matrix. By joining two 1's in each column of a Cromwell matrix with a vertical line segment and two 1's in each row with a horizontal line segment which underpasses any vertical line segments that it crosses, we obtain its Cromwell diagram. Conversely, given a Cromwell diagram with n horizontal lines and n vertical lines, we place 1's at each corner and 0's at other points where the lines and their extensions cross, to construct its Cromwell matrix (see Fig. 3).

Definition 6 [5]. Cromwell showed that every link diagram is isotopic to a diagram which is a finite union of the following local diagrams in such a way that no more than two corners exist in any vertical line and any horizontal line. Such a diagram is called a Cromwell diagram (see Fig. 4).

Elrifai and Ahmed studied the periodic orbit which arise by Rossler dynamical system as follows:

Definition 7 [6], Rossler Links. Begin from the inner most trajectory and follow it until it becomes an outer path. As it rejoins the inner trajectories it joins the next one. The crossings are alternating and the just one is an over crossing. This construction is shown in Fig. 5 for one path R_1, two path R_2,

Figure 4 Cromwell diagram of a trefoil knot.

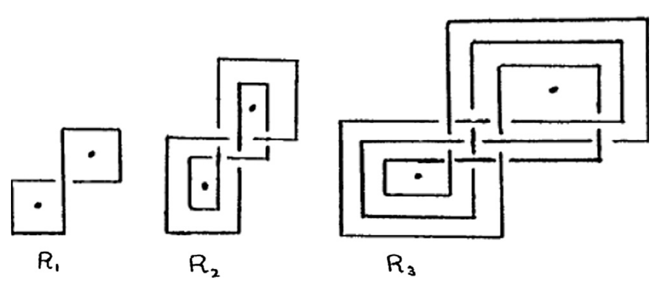

Figure 5 The Rossler link of one path R_1, two path R_2, and three path R_3, respectively.

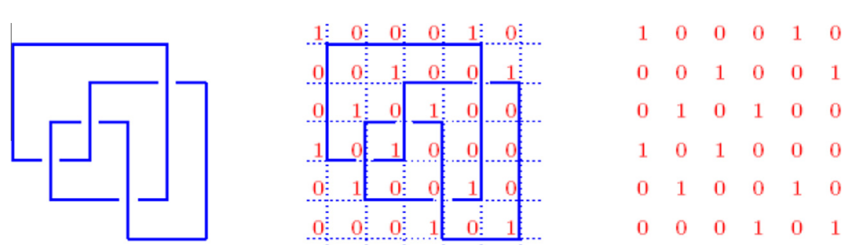

Figure 3 Construction of Cromwell matrix from Cromwell diagram and its inverse.

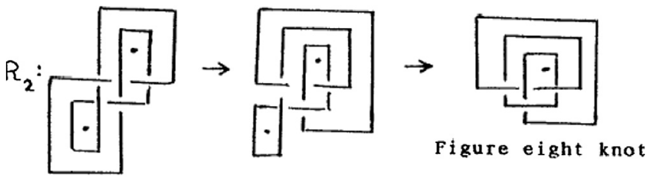

Figure 6 R_2, as a dosed three-braid.

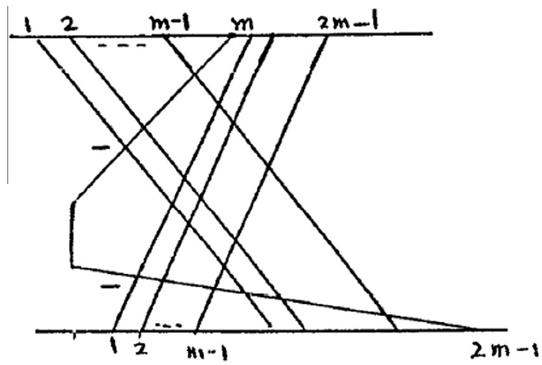

Figure 7 General pattern for R_m.

and three path R_3, respectively. Let R_m, be the Rossler link of the m path, where the order of the chaotic path is the number of inner trajectories.

Definition 8 (*[6], Rossler braids*). Two types of moves can be applied, which do not change the knot type, on the diagram of R_m, to put it into a nice form such as a closed $(2m-l)$-braid, which we call the Rossler braid. First turn over those arcs of R_m, which run around the trivial loop and then remove the trivial loop, see Fig. 6 for R_2, as a closed three-braid (see Fig. 7).

Theorem 9 [5]. *The m Rossler knot R_m, has genus $\frac{m(m-1)}{2}$. For any regular alternating diagram, the number of crossings is invariant and equals to $(m+2)(m-1)$. It also has braid index equals to $(2m-1)$.*

2. Main results

In this paper we introduced the Rossler positive braid and denoted it by R_m^+. This construction is shown in Fig. 8 for one path R_1^+, two path R_2^+, and three path R_3^+, respectively. And R_m^+, be the Rossler link of the m path.

Corollary 10. *A General form of Rossler positive knot has a braid representation*

$$\alpha = \prod_{i=2}^{m-1}\sigma_i \prod_{i=1}^{m-2}\sigma_i \prod_{i=m-1}^{1}\sigma_i.$$

The associated permutation of Rossler positive braid R_m^+ is one cycle

$\alpha = (1 \quad m \quad m-1 \quad m-2 \quad \ldots \quad 2).$

As in Fig. 9.

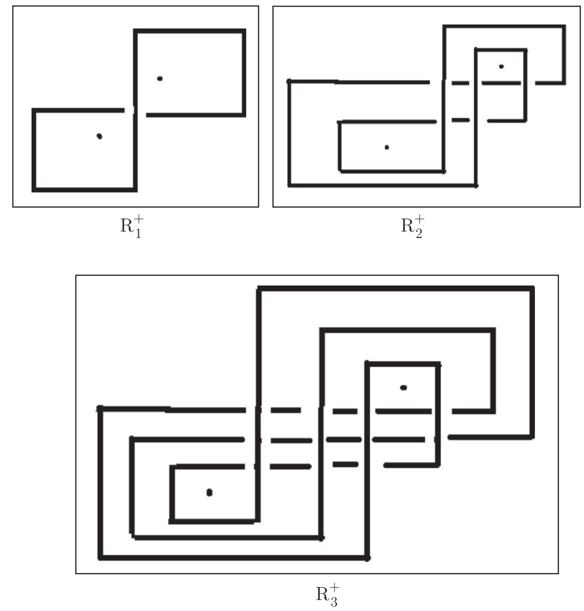

Figure 8 The Rossler positive braid for one path R_1^+, two path R_2^+, and three path R_3^+, respectively.

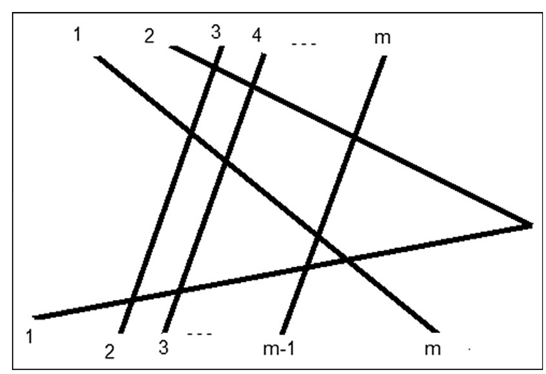

Figure 9 A General pattern form for R_m^+.

Theorem 11. *The m period Rossler positive braid R_m^+ has the number of crossing equals to $[2(m-2)+(m-1)]$, the genus g $(R_m^+)=(m-2)$. It also has braid index equals to m, $m>1$.*

Proof. From the definition of the associated permutation of Rossler positive braid R_m^+, and its general pattern form as in Fig. 9, we have $2(m-2)$ crossing from $\alpha(2)=1$, and $(m-1)$ crossing from $\alpha(1)=m$. So the number of crossing is

$c(R_m^+)=2(m-2)+(m-1).$

For any knot K,

$g(K)=[1+c(K)-s(K)]/2,$

where $s(K)$ is the number of Seifert circles, or the number of strings, so

$g(R_m^+)=[1+2(m-2)+(m-1)-m)]/2=(m-2).$

Corollary 12.

For Rossler positive braid R_m^+, the integer m is knot invariant. For all $m \geqslant 2$.

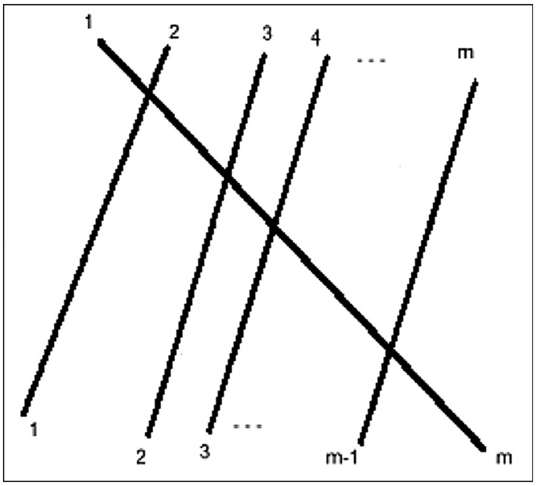

Figure 10 R_m^+ become Lorenz braid of type $(1, m-1)$ If $\alpha(2) = 1$.

Proposition 13. *If $\alpha(2) = 1$ as in Fig. 10, then R_m^+ become Lorenz braid of type $(1, m-1)$. Also it becomes positive permutation braid.*

Corollary 14. *The Rossler positive braid R_m^+, has Cromwell matrix of order $(3m \times 3m)$ with zero diagonal.*

Example 15. Cromwell matrix for R_1^+ is $\begin{bmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{bmatrix}$,

Cromwell matrix for R_2^+ is $\begin{bmatrix} 0 & 0 & 1 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 & 1 & 0 \\ 1 & 0 & 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 & 1 & 0 \\ 0 & 1 & 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 1 & 0 & 0 \end{bmatrix}$, and

Cromwell matrix for R_3^+ is $\begin{bmatrix} 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \end{bmatrix}$.

In this paper we introduced the Rossler positive braid and denoted it by R_m^+, a general form of Rossler positive braid and its associated permutation. We calculated the number of crossing which equals to $[2(m-2) + (m-1)]$, the genus $g(R_m^+) = (m-2)$, also its braid index equals to m, for $m > 1$.

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