Abu-Donia and Hosny *Journal of the Egyptian Mathematical Society* (2020) 28:24
https://doi.org/10.1186/s42787-020-00084-6

Journal of the Egyptian Mathematical Society

ORIGINAL RESEARCH **Open Access**

Generalized *w*closed sets in biweak structure spaces

H. M. Abu-Donia* and Rodyna A. Hosny

*Correspondence:
donia_1000@yahoo.com
Department of Mathematics, Faculty of Science, Zagazig University, Zagazig, Egypt

Abstract

As a generalization of the classes of *gw*closed (resp. *gw*open, *sgw*closed) sets in a weak structure space (X, w), the notions of *ij*-generalized *w*closed (resp. *ij*-generalized *w*open, *ij*-strongly generalized *w*closed) sets in a biweak structure space (X, w_1, w_2) are introduced. In terms of these concepts, new forms of continuous function between biweak spaces are constructed. Additionally, the concepts of *ij*-*w*normal, *ij*-*gw*normal, *ij*-$wT_{\frac{1}{2}}$, and *ij*-$w^{\sigma}T_{\frac{1}{2}}$ spaces are studied and several characterizations of them are acquired.

Keywords: Biweak structures, *ij*-*gw*closed sets, *ij*-$g(w, w^{\star})$-continuous functions, *ij*-*w*normal spaces

AMS classification: 54A05– 54C08– 54C10–54C20

Introduction

In recent years, many researchers studied bitopological, bigeneralized, biminimal, and biweak spaces due to the richness of their structure and potential for doing a generous area for the generalization of topological results in bitopological environment. The concept of a bitopological space was built by Kelly [1], and thereafter, an abundant number of manuscripts was done to generalize the topological notions to bitopological setting. Fukutake [2] presented the concept of generalized closed sets and in bitopological spaces. The notion has been studied extensively in recent years by many topologists. Csaszar and Makai Jr. proposed the concept of bigeneralized topology [3]. In 2010, Boonpok [4, 5] provided the concept of bigeneralized topological spaces and biminimal structure spaces, respectively. Csaszar [6] defined the concept of weak structure which is weaker than a supra topology, a generalized topology, and a minimal structure and then offered various properties of it. Ekici [7] have investigated further properties and the main rules of the weak structure space. In order to extend many of the important properties of *w*closed sets to a larger family, Zahran et al. [8] characterized the concepts of generalized closed and generalized open sets in weak structures and achieved a number of properties of these concepts. As a generalization of bitopological spaces, bigeneralized topological spaces, and biminimal structure spaces, Puiwong et al. [9] in 2017 defined a new space, which is known as biweak structure. The concept of biweak structure can substitute in many situations, biminimal structures and bigeneralized topology. A new space consists of a nonempty set X equipped with two arbitrary weak structures w_1, w_2 on X. A triple (X, w_1, w_2) is called a biweak structure space (in short, bi*w*ss).

The interior (resp. closure) of a subset A with respect to w_j are denoted by $int_{w_j}(A)$ (resp. $cl_{w_j}(A)$), for ($j = 1, 2$). A subset A of a bi*w*ss (X, w_1, w_2) is called *ij*-*w*closed if $cl_{w_i}(cl_{w_j}(A))$=A, where i, j= 1 or 2 and $i \neq j$. The complement of an *ij*-*w*closed set is called *ij*-*w*open.

The concepts of generalized closed sets in weak structures [8] and biweak structure spaces [9] motivated us to define a new class of sets which is called generalized *w*closed sets in a biweak structure space which are found to be effective in the study of digital topology. The purpose of this article is introducing the notions of *ij*-generalized *w*closed (written henceforth as *ij*-*gw*closed), *ij*-generalized *w*open (written henceforth as *ij*-*gw*open), and *ij*-strongly generalized *w*closed (*ij*-σgwclosed, for short) sets in a bi*w*ss (X, w_1, w_2) as a generalization of the concept of *gw*closed, *gw*open, and *sgw*closed sets, respectively, in a weak structure space (X, w) which presented in [8] and determining some of their behaviors. In terms of *ij*-*gw*closed and *ij*-*gw*open sets, new forms of continuous function between biweak spaces are constructed. Additionally, we try to extend the concepts of separation axioms on weak structures [8] to bi*w*ss and study some of their features. Some considerable results in articles [2, 8, 10] can be treated as particular cases of our outcomes.

Preliminaries

To prepare this article as self-contained as possible, we recollect the next definitions and results which are due to various references [8, 9, 11].

Definition 1 [8] *Let w be a weak structure on X. Then,*

(1) A subset A is called generalized wclosed (gwclosed, for short) if $cl_w(A) \subseteq U$*, whenever* $A \subseteq U$ *and* U *is wopen.*
(2) The complement of a generalized wclosed set is called generalized wopen (gwopen, for short), i.e, a subset A is gwopen if and only if $int_w(A) \supseteq F$*, whenever* $A \supseteq F$ *and F is wclosed.*

The family of all gwclosed (resp. gwopen) sets in a weak structure X will be denoted by $GWC(X)$ *(resp.* $GWO(X)$*).*

Definition 2 [11] *Let w and* $w^\star$ *be weak structures on X and Y, respectively. A function* $f : (X, w) \longrightarrow (Y, w^\star)$ *is called* $(w, w^\star)$*-continuous if for* $x \in X$ *and* $w^\star$*open set V containing* $f(x)$*, there is wopen set U containing x s.t.* $f(U) \subseteq V$*.*

Theorem 1 [11] *Let w and* $w^\star$ *be weak structures on X and Y, respectively. For a function* $f : (X, w) \longrightarrow (Y, w^\star)$*, the following statements are equivalent:*

(1) f is $(w, w^\star)$*-continuous,*
(2) $f^{-1}(B)$=$int_w(f^{-1}(B))$*, for every* $w^\star$*open set B in Y,*
(3) $f(cl_w(A)) \subseteq cl_{w^\star}(f(A))$*, for every set A in X,*
(4) $cl_w(f^{-1}(B)) \subseteq (f^{-1}(cl_{w^\star}(B))$*, for every set B in Y,*
(5) $f^{-1}(int_{w^\star}(B)) \subseteq int_w(f^{-1}(B))$*, for every set B in Y,*

(6) $cl_w(f^{-1}(F))=f^{-1}(F)$, *for every* $w^{\star}$*closed set F in Y.*

Theorem 2 [9] *Let* (X, w_1, w_2) *be a biwss and A be a subset of X. Then, the following are equivalent:*

(1) *A is ij-wclosed,*
(2) $A=cl_{w_i}(A)$ and $A=cl_{w_j}(A)$,
(3) $A=cl_{w_j}(cl_{w_i}(A))$, *where* $i, j=$ *1 or 2 and* $i\neq j$.

Proposition 1 [9] *Let* (X, w_1, w_2) *be a biwss and* $A\subseteq X$. *Then, A is a ij-wclosed set, if A is both* w_i*closed and* w_j*closed, where* $i, j=$ *1 or 2 and* $i\neq j$.

Proposition 2 [9] *Let* (X, w_1, w_2) *be a biwss. If* A_α *is ij-wclosed for all* $\alpha\in\Lambda\neq\emptyset$, *then* $\cap_{\alpha\in\Lambda}A_\alpha$ *is ij-wclosed and the union of two ij-wclosed sets is not a ij-wclosed set, where* $i, j=$ *1 or 2 and* $i\neq j$.

In the rest of this article i, j will stand for fixed integers in the set $\{1, 2\}$ and $i\neq j$.

On *ij-gw*closed sets

In this part, a new family of sets called *ij*-generalized *w*closed (briefly, *ij-gw*closed) is presented and its properties are investigated.

Definition 3 *A subset A of a biwss* (X, w_1, w_2) *is called ij-generalized wclosed (ij-gwclosed, for short) if* $cl_{w_j}(A)\subseteq U$, *whenever* $A\subseteq U$ *and U is* w_i*open. The complement of ij-gwclosed set is called ij-gwopen.*

The family of all ij-gwclosed (resp. ij-gwopen) sets in a biwss (X, w_1, w_2) *will be denoted by ij-*$GWC(X)$ *(resp. ij-*$GWO(X)$*).*

Remark 1 *If* $A\in$*ij-*$GWC(X)\cap$*ji-*$GWC(X)$, *then a subset A of a biwss* (X, w_1, w_2) *is called pairwise gwclosed and its complement is pairwise gwopen.*

Example 1 *Let* $X=\{1, 2, 3\}$, $w_1=\{\emptyset, \{1\}, \{1, 2\}\}$, *and* $w_2=\{\emptyset, \{3\}\}$. *A set* $\{3\}$ *is pairwise gwclosed.*

Certainly, the next theorems are obtained:

Theorem 3 *A subset A of a biwss* (X, w_1, w_2) *is ij-gwopen iff* $int_{w_j}(A)\supseteq F$, *whenever* $A\supseteq F$ *and F is* w_i*closed.*

Theorem 4 *If A is an ij-gwclosed and* w_i*open set in* (X, w_1, w_2), *then* $A=cl_{w_j}(A)$.

Theorem 5 *Every* w_j*closed set in a biwss* (X, w_1, w_2) *is ij-gwclosed.*

Proof Let A be a $w_j$closed set and U be a $w_i$open set in X s.t. $A\subseteq U$. Then, $cl_{w_j}(A)=A$. It implies that $A\in$*ij-*$GWC(X)$. □

Corollary 1 *If A is a* w_j*open set in a biwss* (X, w_1, w_2), *then* $A\in$*ij-*$GWO(X)$.

Remark 2 *By the following example, we have a tendency to show that the converse of Theorem 5 is not always true.*

Example 2 *In Example 1, a set* $\{2\}$ *is 12-gwclosed and not* w_2*closed.*

Proposition 3 *Let* (X, w_1, w_2) *be a biwss. Then,*

(1) If $X \in w_j$ *and each* w_i*open set is* w_j*closed, then,* $A \in ij\text{-}GWC(X)$*, for each* $A \subset X$.
(2) $A \in ij\text{-}GWC(X)$*, for each* $A \subset X$ *iff* $cl_{w_j} U = U$ *for each* w_i *open set* U.

Proof We prove only (2) and the rest of the proof is simple. Suppose that $A \in ij\text{-}GWC(X)$, for each $A \subset X$. Then, every $w_i$open set U, $A \in ij\text{-}GWC(X)$. If $U \subseteq U$, hence $cl_{w_j}(U) \subseteq U$. Thus, $cl_{w_j}(U) = U$, for each $w_i$open set U. Conversely, suppose that $A \subseteq U$ and U be a $w_i$open set. Then, $cl_{w_j}(A) \subseteq cl_{w_j}(U)$. From assumption, $cl_{w_j}(A) \subseteq U$ and so $A \in ij\text{-}GWC(X)$. □

Remark 3 *In the biwss* (X, w_1, w_2)*, the converse of the Proposition 3(1) need not be true in general as shown by the next example.*

Example 3 *Let* $X = \{1, 2, 3\}$*,* $w_1 = \{\emptyset, \{2\}, \{1, 3\}\}$*, and* $w_2 = \{\emptyset, X, \{1\}, \{2\}, \{3\}, \{1, 2\}, \{2, 3\}\}$*. One may notice that every subset of X is 12-gwclosed, but A=*$\{2\}$ *is a* w_1*open set in X and it is not* w_2*closed.*

Remark 4 *In general,* $21\text{-}GWC(X) \neq 12\text{-}GWC(X)$ *as in Example 3.*

Proposition 4 *Let* (X, w_1, w_2) *be a biwss. If* $w_1 \subseteq w_2$*, then* $21\text{-}GWC(X) \subseteq 12\text{-}GWC(X)$.

Proof Straightforward. □

The converse of the Proposition 4 is not true as seen from the next example.

Example 4 *In Example 3, then* $21\text{-}GWC(X) \subseteq 12\text{-}GWC(X)$*, but* $w_1 \not\subseteq w_2$.

Now, one can conclude attitudes relative to the union as well as the intersection of two *ij-gw*closed sets in a biwss (X, w_1, w_2).

Example 5 *Let* $X = \{1, 2, 3, 4\}$*,* $w_1 = \{\emptyset, \{3\}, \{1, 3\}, \{1, 3, 4\}, \{1, 2, 4\}\}$ *and* $w_2 = \{\emptyset, \{2\}, \{3\}, \{2, 3, 4\}\}$*. Let us consider* $A = \{2\}$ *and* $B = \{3\}$*. Note that A and B are 21-gwclosed sets but its union is not 21-gwclosed.*

Example 6 *Let* $X = \{1, 2, 3\}$*,* $w_1 = \{\emptyset, \{1\}, \{3\}\}$ *and* $w_2 = \{\emptyset, \{1\}\}$*. Consider two 21-gwclosed sets* $A = \{1, 2\}$ *and* $B = \{1, 3\}$*, then* $A \cap B = \{1\}$ *is not 21-gwclosed.*

Theorem 6 *Let* (X, w_1, w_2) *be a biwss and* $cl_{w_j}(\emptyset) = \emptyset$*. Then, the family of all ij-gwclosed sets is a biminimal structure in X.*

Proof Obvious. □

Theorem 7 *Suppose* $X \in w_i$*. Then,* $\{x\}$ *is* w_i*closed or* $X \setminus \{x\} \in ij\text{-}GWC(X)$*, for each* $x \in X$.

Proof Suppose that the singleton $\{x\}$ is not $w_i$closed for some $x \in X$. Then, $X\backslash\{x\}$ is not $w_i$open. Since X is $w_i$open set and $X\backslash\{x\}\subseteq X$. Hence, $X\backslash\{x\}\in ij\text{-}GWC(X)$. □

Theorem 8 *If $A\in ij\text{-}GWC(X)$, then $cl_{w_j}(A)\backslash A$ contains no nonempty $w_i$closed.*

Proof For an *ij-gw*closed set A, let S be a nonempty $w_i$closed set s.t. $S\subseteq cl_{w_j}(A)\backslash A$. Then, $S\subseteq cl_{w_j}(A)$ and $S\subseteq X\backslash A$. Since $X\backslash S$ is $w_i$open and A is *ij-gw*closed, then $cl_{w_j}(A)\subseteq X\setminus S$ or $S\subseteq X\backslash cl_{w_j}(A)$. Thus, $S=\emptyset$. Therefore, $cl_{w_j}(A)\backslash A$ does not contain nonempty $w_i$closed. □

Remark 5 *In general, the converse of Theorem 8 is not true as shown in the next example.*

Example 7 *In Example 6, if $A=\{1\}$, then $c_{w_1}(A)\backslash A=\{2\}$. So we know that there is no any nonempty $w_2$closed contained in $c_{w_1}(A)\backslash A$. But $A\notin 21\text{-}GWC(X)$.*

It thus follows from Theorem 8 that

Corollary 2 *If $A\in ij\text{-}GWC(X)$ and $cl_{w_j}(A)\backslash A$ is a $w_i$closed set, then $cl_{w_j}(A)=A$.*

Remark 6 *If A is an ij-gwclosed set in a biwss (X, w_1, w_2) and $cl_{w_j}(A)=A$, then $cl_{w_j}(A)\backslash A$ need not to be $w_i$closed as shown by the following example.*

Example 8 *Let $X=\{1, 2, 3\}$, $w_1=\{\emptyset, \{2\}\}$, and $w_2=\{\emptyset, \{1\}, \{3\}, \{1, 2\}\}$. If $A=\{2\}$, one may notice that $c_{w_2}(A)=A$ and hence $c_{w_2}(A)\backslash A=\emptyset$, which is not $w_1$closed.*

Theorem 9 *If $A\in ij\text{-}GWC(X)$, then $cl_{w_j}(A)\backslash A\in ij\text{-}GWO(X)$.*

Proof Let $A\in ij\text{-}GWC(X)$ and F be a $w_i$closed set s.t. $F\subseteq cl_{w_j}(A)\backslash A$. Then, by Theorem 8, we have $F=\emptyset$ and hence $F\subseteq int_{w_j}(cl_{w_j}(A)\backslash A)$. So by Theorem 3, we have $cl_{w_j}(A)\backslash A\in ij\text{-}GWO(X)$. □

Remark 7 *The converse of the Theorem 9 need not to be true in general as shown by the following example.*

Example 9 *In Example 6. If $A=\{1\}$, one may notice that $cl_{w_1}(A)\backslash A\in 21\text{-}GWO(X)$, but $A\notin 21\text{-}GWC(X)$.*

Theorem 10 *If $A\in ij\text{-}GWC(X)$ and $A\subseteq B\subseteq cl_{w_j}(A)$, then $B\in ij\text{-}GWC(X)$.*

Proof Let U be any $w_i$open set s.t. $B\subseteq U$. Since $A\subseteq B$ and $A\in ij\text{-}GWC(X)$, then $cl_{w_j}(A)\subseteq U$. Since $B\subseteq cl_{w_j}(A)$, then we have $cl_{w_j}(B)\subseteq cl_{w_j}cl_{w_j}(A)=cl_{w_j}(A)\subseteq U$. Consequently $B\in ij\text{-}GWC(X)$. □

Corollary 3 *Let (X, w_1, w_2) be a biwss. Then,*

(1) If $A\in ij\text{-}GWO(X)$ and $int_{w_j}(A)\subseteq B\subseteq A$, then, $B\in ij\text{-}GWO(X)$.
(2) $cl_{w_j}(A)\in ij\text{-}GWC(X)$ if $A\in ij\text{-}GWC(X)$.
(3) $int_{w_j}(A)\in ij\text{-}GWO(X)$ if $A\in ij\text{-}GWO(X)$.

In view of Theorems 8 and 10, the next theorem is valid.

Theorem 11 *Let A be an ij-gwclosed set with $A \subseteq B \subseteq cl_{w_j}(A)$, then, $cl_{w_j}(B) \setminus B$ does not contain nonempty $w_i$closed.*

Theorem 12 *If A is an ij-gwopen set in X, then $U = X$ whenever U is $w_i$open and $int_{w_j}(A) \cup (X \setminus A) \subseteq U$.*

Proof Let U be a $w_i$open set in X and $int_{w_j}(A) \cup (X \setminus A) \subseteq U$ for any *ij-gw*open set A. Then, $X \setminus U \subseteq (X - int_{w_j}(A)) \cap A$ and so $X \setminus U \subseteq cl_{w_j}(X \setminus A) \setminus (X \setminus A)$. Since $X \setminus A$ is *ij-gw*closed, then by Theorem 8, we have $X \setminus U = \emptyset$ and hence $U = X$. □

Definition 4 *If $cl_{w_j}(\cup_\alpha A_\alpha) = \cup_\alpha cl_{w_j}(A_\alpha)$, for $(j = 1, 2)$, then a family $\{A_\alpha \mid \alpha \in \Delta\}$ is called w_j-locally finite.*

Theorem 13 *Let (X, w_1, w_2) be a biwss. If the family $\{A_\alpha \mid \alpha \in \Delta\}$ is w_j-locally finite, then the arbitrary union of ij-gwclosed sets A_α, $\alpha \in \Delta$ is an ij-gwclosed set.*

Proof Direct to prove. □

In the next definition, as an application of *ji-gw*open sets, we offer a new type of sets namely *ij-σgw*closed sets.

Definition 5 *A subset A of a biwss (X, w_1, w_2) is called ij-strongly generalized wclosed (briefly, ij-σgwclosed), if $cl_{w_j}(A) \subseteq U$, whenever $A \subseteq U$ and U is ji-gwopen. The complement of ij-σgwclosed set is called ij-σgwopen.*

The family of all ij-σgwclosed (resp. ij-σgwopen) sets in a biwss (X, w_1, w_2) will be denoted by ij-$\sigma GWC(X)$ (resp. ij-$\sigma GWO(X)$).

Remark 8 *If $A \in$ ij-$\sigma GWC(X) \cap$ ji-$\sigma GWC(X)$, then a subset A of a biwss (X, w_1, w_2) is called pairwise σgwclosed and its complement is called pairwise σgwopen.*

For brevity the proof of the next proposition is omitted.

Proposition 5 *In a biwss (X, w_1, w_2), we have the following relation:*

$w_j$closed set $\Rightarrow$ ij-σgwclosed set $\Rightarrow$ ij-gwclosed set.

Remark 9 *The converse of Proposition 5 is not true as can be seen from the next example.*

Example 10 *In Example 6, one may notice that $\{4\}$ is 21-gwclosed set, but it is not 21-σgwclosed.*

Example 11 *In Example 8. One may notice that, $\{2\}$ is 12-σgwclosed set, but it is not $w_2$closed.*

Theorem 14 *If $A \in$ ji-$GWO(X) \cap$ ij-$\sigma GWC(X)$, then $cl_{w_j}(A) = A$*

Proof Straightforward. □

Theorem 15 *Let $cl_{w_i}\emptyset = \emptyset$. Then, $\{x\} \in$ ji-$GWC(X)$ or $X \setminus \{x\} \in$ ij-$\sigma GWC(X)$, for each $x \in X$.*

Proof Similar to Theorem 7. □

Theorem 16 *If* $A \in ij\text{-}\sigma GWC(X)$*, then* $cl_{w_j}(A) \setminus A$ *contains no nonempty ji-gwclosed.*

Proof Similar to Theorem 8. □

Separation axioms in biweak spaces

By using *ij-gw*closed, *ij-gw*open and *ij-σgw*closed sets, we introduce and study the notions of $ij\text{-}wT_{\frac{1}{2}}$, $ij\text{-}wT^{\sigma}_{\frac{1}{2}}$, $ij\text{-}w^{\sigma}T_{\frac{1}{2}}$, *ij-w*normal, and *ij-gw*normal spaces.

Definition 6 *Let* $cl_{w_j}(\emptyset)=\emptyset$*. A biwss* (X, w_1, w_2) *is called*

(1) *ij-*wT_1 *if for each distinct points* $x, y \in X$*, there exist a* w_i*-open set* U *and* w_j*-open set* V *s.t.* $x \in U$*,* $y \notin U$ *and* $y \in V$*,* $x \notin V$*.*
(2) *ij-*$wT_{\frac{1}{2}}$ *if each ij-gwclosed set A of X,* $cl_{w_j}(A)=A$*.*
(3) *ij-*$wT^{\sigma}_{\frac{1}{2}}$ *if each ij-σgwclosed set A of X,* $cl_{w_j}(A)=A$*.*

Theorem 17 *A biwss* (X, w_1, w_2) *is ij-*wT_1 *if every singleton in X is ij-wclosed.*

Proof Let $x, y \in X$ and $x \neq y$. Then, $\{x\}$, $\{y\}$ are *ij-w*closed sets. From Theorem 1, we have $x \notin cl_{w_i}(\{y\})$ and $y \notin cl_{w_j}(\{x\})$. Hence, there exist w_i-open set U containing x and w_j-open set V s.t. $x \in U$, $y \notin U$, and $y \in V$, $x \notin V$. Consequently, (X, w_1, w_2) is a *ij-*wT_1 space. □

In view of Proposition 5, the class of $ij\text{-}wT^{\sigma}_{\frac{1}{2}}$ spaces properly contains the class of $ij\text{-}wT_{\frac{1}{2}}$ spaces.

Proposition 6 *Every* $ij\text{-}wT_{\frac{1}{2}}$ *space is* $ij\text{-}wT^{\sigma}_{\frac{1}{2}}$*.*

The following example supports that the converse of the Proposition 6 is not true in general.

Example 12 *In Example 5,* (X, w_1, w_2) *is a 21-*$wT^{\sigma}\frac{1}{2}$ *space but not 21-*$wT_{\frac{1}{2}}$*.*

Theorem 18 *Let X be a* w_i*open set and* $int_{w_j}\{x\}$ *is* w_j*open. A biwss* (X, w_1, w_2) *is* $ij\text{-}wT_{\frac{1}{2}}$ *iff* $\{x\}$ *is* w_i*closed or* $\{x\}=int_{w_j}\{x\}$ *for each* $x \in X$*.*

Proof Suppose that $\{x\}$ is not $w_i$closed for some $x \in X$. Then, by using Theorem 7, $X \setminus \{x\}$ is *ij-gw*closed. Since (X, w_1, w_2) is $ij\text{-}wT_{\frac{1}{2}}$, then $\{x\}=int_{w_j}\{x\}$. On the other hand, let B be an *ij-gw*closed set. By assumption, $\{x\}$ is $w_i$closed or $\{x\}=int_{w_j}\{x\}$ for any $x \in cl_{w_j}B$.
Case (I): Suppose $\{x\}$ is $w_i$closed. If $x \notin B$, then $\{x\} \subseteq cl_{w_j}B \setminus B$, which is a contradiction to Theorem 8. Hence $x \in B$.
Case (II): Suppose $\{x\}=int_{w_j}\{x\}$ and $x \in cl_{w_j}B$. Since $\{x\} \cap B \neq \emptyset$, we have $x \in B$. Thus, in both cases, we conclude that $cl_{w_j}B=B$. Therefore, (X, w_1, w_2) is $ij\text{-}wT_{\frac{1}{2}}$ space. □

Theorem 19 *Suppose* $cl_{w_i}\emptyset=\emptyset$*. If* (X, w_1, w_2) *is an* $ij\text{-}wT^{\sigma}_{\frac{1}{2}}$ *space, then* $\{x\}$ *is ji-gwclosed or* $\{x\}=int_{w_j}\{x\}$*, for each* $x \in X$*.*

Proof Follows directly from Theorem 15 and Definition 6. □

Lemma 1 *If* $\{x\}$ *is ji-gwclosed, then* (X, w_1, w_2) *is an ij-w-*$T^{\sigma}_{\frac{1}{2}}$ *space, for each* $x \in X$.

Proof Straightforward. □

Definition 7 *A biwss* (X, w_1, w_2) *is called*

(1) Pairwise $wT_{\frac{1}{2}}$ if it is both ij-$wT_{\frac{1}{2}}$ and ji-$wT_{\frac{1}{2}}$.
(2) Pairwise $wT^{\sigma}\frac{1}{2}$ if it is both ij-$wT^{\sigma}_{\frac{1}{2}}$ and ji-$wT^{\sigma}_{\frac{1}{2}}$.

Proposition 7 *If* (X, w_1, w_2) *is a pairwise* $wT_{\frac{1}{2}}$ *space, then it is pairwise* $wT^{\sigma}_{\frac{1}{2}}$.

Proof Uncomplicated. □

Remark 10 *The converse of Proposition 7 is not true as can be seen from the next example.*

Example 13 *Let* X, w_1, w_2 *be as in Example 12. Then,* (X, w_1, w_2) *is also a 21-*$wT^{\sigma}_{\frac{1}{2}}$ *space, and therefore, it is a pairwise* $wT^{\sigma}_{\frac{1}{2}}$ *space. But* (X, w_1, w_2) *is not a pairwise* $wT_{\frac{1}{2}}$ *space.*

Definition 8 *A biwss* (X, w_1, w_2) *is called an ij-*$w^{\sigma}T_{\frac{1}{2}}$ *if ij-*$GWC(X)$*=ij-*$\sigma GWC(X)$.

Proposition 8 *Every ij-*$wT_{\frac{1}{2}}$ *space is ij-*$w^{\sigma}T_{\frac{1}{2}}$.

Proof Obvious. □

Remark 11 *The converse of Proposition 8 may not be applicable as we see in the next example.*

Example 14 *Let* $X=\{1, 2, 3, 4\}$. *Define weak structures* w_1, w_2 *on* X *as follows:* $w_1=\{\emptyset, \{1, 3\}, \{1, 4\}, \{2, 3, 4\}\}$ *and* $w_2=\{\emptyset, \{2\}, \{1, 2\}, \{3, 4\}, \{1, 3, 4\}\}$. *Then,* (X, w_1, w_2) *is an 12-*$w^{\sigma}T_{\frac{1}{2}}$ *space but not 12-*$wT_{\frac{1}{2}}$.

Remark 12 *ij-*$w^{\sigma}T_{\frac{1}{2}}$ *and ij-*$wT^{\sigma}_{\frac{1}{2}}$ *spaces are independent as may be seen from Example 15 and Example 16.*

Example 15 *Let* $w_1=\{\emptyset, \{1\}, \{1, 2\}\}$, $w_2=\{\emptyset, \{3\}, X\}$ *be weak structures on* $X=\{1, 2, 3\}$, *then* (X, w_1, w_2) *is a 12-*$wT^{\sigma}\frac{1}{2}$ *space but not 12-*$w^{\sigma}T_{\frac{1}{2}}$.

Example 16 *In Example 14,* (X, w_1, w_2) *is an 12-*$w^{\sigma}T_{\frac{1}{2}}$*, but it is not 12-w-*$T^{\sigma}_{\frac{1}{2}}$.

Theorem 20 *Let* $cl_{w_j}(\emptyset)=\emptyset$. *A biwss* (X, w_1, w_2) *is ij-*$wT_{\frac{1}{2}}$ *if and only if it is both ij-*$wT^{\sigma}_{\frac{1}{2}}$ *and ij-*$w^{\sigma}T_{\frac{1}{2}}$ *space.*

Proof Suppose that (X, w_1, w_2) is an *ij-*$wT_{\frac{1}{2}}$ space. Then, by Propositions 6 and 8, (X, w_1, w_2) is both *ij-*$wT^{\sigma}_{\frac{1}{2}}$ and *ij-*$w^{\sigma}T_{\frac{1}{2}}$ space. Conversely, suppose that (X, w_1, w_2) is both *ij*-$wT^{\sigma}_{\frac{1}{2}}$ and *ij*-$w^{\sigma}T_{\frac{1}{2}}$. Let $A \in$ *ij*-$GWC(X)$. Since (X, w_1, w_2) is an *ij*-$w^{\sigma}T_{\frac{1}{2}}$ space, $A\in$*ij*-$\sigma GWC(X)$. Since (X, w_1, w_2) is an *ij*-$wT^{\sigma}_{\frac{1}{2}}$ space, then $cl_{w_j}(A)$=A. Therefore, (X, w_1, w_2) is *ij*-$wT_{\frac{1}{2}}$. □

Definition 9 *A biwss (X, w_1, w_2) is called ij-wnormal if for each $w_i$closed set A and $w_j$closed set B s.t. $A\cap B$=∅, there are $w_j$open set U and $w_i$open set V s.t. $A\subseteq U$, $B\subseteq V$, and $U\cap V$=∅.*

Theorem 21 *Let (X, w_1, w_2) be a biwss. Consider the following statements:*

(1) (X, w_1, w_2) *is ij-wnormal,*
(2) *For each $w_i$closed set A and $w_j$open set N with $A\subseteq N$, there exists $w_j$open set U s.t. $A\subseteq U\subseteq cl_{w_i}(U)\subseteq N$,*
(3) *For each $w_i$closed set A and each ij-gwclosed set H with $A\cap H$=∅, there exist $w_j$open set U and $w_i$open set V s.t. $A\subseteq U$, $H\subseteq V$ and $U\cap V$=∅,*
(4) *For each $w_i$closed set A and ij-gwopen N with $A\subseteq N$, there exists $w_j$open set U s.t. $A\subseteq U\subseteq cl_{w_i}(U)\subseteq N$.*

Then, the implications $(1) \Rightarrow (2)$ *and* $(3) \Rightarrow (4) \Rightarrow (2)$ *are hold.*

Proof Obvious. □

Theorem 22 *Let (X, w_1, w_2) be a biwss. If $cl_{w_i}(A)$ is $w_i$closed for each $w_j$open or ij-gwclosed, then the statements in Theorem 21 are equivalent.*

Proof According to Theorem *21*, we need to prove $(2) \Rightarrow (1)$ and $(1) \Rightarrow (3)$ only.
$(2) \Rightarrow (1)$: Let A be a $w_i$closed set and B be a $w_j$closed set with $A\cap B$=∅. Then, $X\backslash B$ is a $w_j$open set with $A\subseteq X\backslash B$. Thus, by (2) there exists w_j open set U s.t. $A\subseteq U\subseteq cl_{w_i}(U)\subseteq X\backslash B$. Hence $A\subseteq U$ and $B\subseteq X\backslash cl_{w_i}(U)$. Since $cl_{w_i}(U)$ is $w_i$closed for each $w_j$open U, then $X\backslash cl_{w_i}(U)$=V is $w_i$open and $U\cap V$=∅. Hence (X, w_1, w_2) is *ij*-$w$normal.
$(1) \Rightarrow (3)$: Let A be a $w_i$closed set and H be an *ij*-*gw*closed set with $A\cap H$=∅. Then, $H\subseteq X\backslash A$. From Definition 3, we have $cl_{w_j}(H)\subseteq X\backslash A$. Since H is *ij*-*gw*closed, then $cl_{w_j}(H)$ is $w_j$closed. Since $A\cap cl_{w_j}(H)$=∅, then from (1) there exist $w_j$open set U and $w_i$open set V s.t. $A\subseteq U$, $H\subseteq cl_{w_j}(H)\subseteq V$ and $U\cap V$=∅. □

Theorem 23 *Let (X, w_1, w_2) be a biwss. Consider the following statements:*

(1) (X, w_1, w_2) *is ij-wnormal,*
(2) *For each $w_i$closed set A and $w_j$closed set B s.t. $A\cap B$=∅, there exist ij-gwopen U and ji-gwopen V s.t. $A\subseteq U$, $B\subseteq V$ and $U\cap V$=∅,*
(3) *For each $w_i$closed set A and $w_j$open N with $A\subseteq N$, there exists ij-gwopen U s.t. $A\subseteq U\subseteq cl_{w_i}(U)\subseteq N$.*

Then, the implication $(1) \Rightarrow (2) \Rightarrow (3)$ *is hold.*

Proof $(1) \Rightarrow (2)$: Let A be a $w_i$closed set and B be a $w_j$closed set with $A\cap B$=∅. Since (X, w_1, w_2) is *ij*-$w$normal, then there exist $w_j$open set U and $w_i$open set V s.t. $A\subseteq U$, $B\subseteq V$ and $U\cap V$=∅. From Corollary 1, there exist *ij*-*gw*open U and *ji*-*gw*open V s.t. $A\subseteq U$, $B\subseteq V$ and $U\cap V$=∅.

(2) $\Rightarrow$ (3): Let A be a $w_i$closed set and N be a $w_j$open set with $A\subseteq N$. Then, $A\cap X\backslash N=\emptyset$. From (2), there exist *ij-gw*open U and *ji-gw*open V s.t. $A\subseteq U$, $X\backslash N\subseteq V$, and $U\cap V=\emptyset$. Since $X\backslash V$ is *ji-gw*closed, N is $w_j$open, and $X\backslash V\subseteq N$, then from Definition 3, we have $cl_{w_i}(X\backslash V)\subseteq N$. Since $U\subseteq X\backslash V$, hence $U\subseteq cl_{w_i}(U)\subseteq cl_{w_i}(X\backslash V)$. Consequently, $A\subseteq U\subseteq cl_{w_i}(U)\subseteq N$. □

Theorem 24 *Let* (X, w_1, w_2) *be an ij-w*$T_{\frac{1}{2}}$. *If* $cl_{w_i}(U)$ *is* w_i*closed for each ij-gwclosed and* $int_{w_j}(U)$ *is* w_j*open for each ij-gwclosed U, then the statements in Theorem 23 are equivalent.*

Proof According to Theorem 23, we need to prove (3) $\Rightarrow$ (1).
(3) $\Rightarrow$ (1): Let A be a $w_i$closed set and B be a $w_j$closed set with $A \cap B=\emptyset$. Take $N=X \setminus B$, then by using (3) there exists *ij-gw*open U s.t. $A \subseteq U \subseteq cl_{w_i}(U) \subseteq N$. Since (X, w_1, w_2) is an *ij-w*$T_{\frac{1}{2}}$ space, then, $int_{w_j}(U)=U$. By assumption U is $w_j$open. Also, $X\backslash cl_{w_i}(U)$ is $w_i$open and $B\subseteq X\backslash cl_{w_i}(U)$. □

Definition 10 *A biwss* (X, w_1, w_2) *is called ij-gwnormal if for each ji-gwclosed set A and ij-gwclosed set B s.t.* $A\cap B=\emptyset$, *there are* w_j*open set U and* w_i*open set V s.t.* $A\subseteq U$, $B\subseteq V$ *and* $U\cap V=\emptyset$.

Remark 13 *It is clear that every ij-gwnormal space is ij-wnormal. It can be checked that the converse is not true by the following example.*

Theorem 25 *Let* (X, w_1, w_2) *be a biwss. Consider the following statements:*

(1) (X, w_1, w_2) *is ij-gwnormal,*
(2) For each ji-gwclosed set A and ij-gwopen set N with $A\subseteq N$, *there exists* w_j*open set U s.t.* $A\subseteq U\subseteq cl_{w_i}(U)\subseteq N$,
(3) For each ji-gwclosed set A and ij-gwclosed set B s.t. $A\cap B=\emptyset$, *there exist* w_j*open set U s.t.* $A\subseteq U$ *and* $cl_{w_i}(U)\cap B=\emptyset$.

Then, the implication (1) $\Rightarrow$ (2) $\Rightarrow$ (3) *is hold.*

Proof Obvious. □

Remark 14 *If* $cl_{w_i}(U)$ *is* w_i*closed for each* w_i*open set U, then the statements in Theorem 25 are equivalent.*

Theorem 26 *Let* (X, w_1, w_2) *be a biwss. Consider the following statements:*

(1) (X, w_1, w_2) *is ij-gwnormal,*
(2) For each ji-gwclosed set A and ij-gwclosed set B s.t. $A\cap B=\emptyset$, *there exist ij-σgwopen set U, ji-σgwopen set V s.t.* $A\subseteq U$, $B\subseteq V$ *and* $U\cap V=\emptyset$,
(3) For each ji-gwclosed set A and ij-gwopen set N with $A\subseteq N$, *there exists ij-σgwopen set U s.t.* $A\subseteq U\subseteq cl_{w_i}(U)\subseteq N$.

Then, the implication (1) $\Rightarrow$ (2) $\Rightarrow$ (3) *is hold.*

Proof (1) $\Rightarrow$ (2) Follows directly from Proposition 5.
(2) $\Rightarrow$ (3) Let A be a *ji-gw*closed set and N be an *ij-gw*open set with $A \subseteq N$. Take $B = X \setminus N$. Then, by assumption, there exist *ij-σgw*open set U, *ji-σgw*open set V s.t. $A \subseteq U$, $B \subseteq V$ and $U \cap V = \emptyset$. Hence, $U \subseteq X \setminus V$, $X \setminus V \subseteq N$. Since $X \setminus V$ is *ji-σgw*closed, then $cl_{w_i}(X \setminus V) \subseteq N$ and so $A \subseteq U \subseteq cl_{w_i}(U) \subseteq N$. □

The question that comes to our mind, under what conditions can be achieved parity in Theorem 26.

Theorem 27 *Let (X, w_1, w_2) be an ij-w-$T^{\sigma}_{\frac{1}{2}}$ space. If $int_{w_j}(U)$ is $w_j$open and $int_{w_i}(U)$ is $w_i$open for each ij-σgwopen set U, then the statements in Theorem 26 are equivalent.*

Proof Straightforward. □

Corollary 4 *If a biwss (X, w_1, w_2) is ij-gwnormal, then for each ji-gwclosed set A and ij-σgwopen set N with $A \subseteq N$, there exists ij-σgwopen set U s.t. $A \subseteq U \subseteq cl_{w_i}(U) \subseteq N$.*

Proof Obvious from Proposition 5. □

Theorem 28 *If a biwss (X, w_1, w_2) is ij-gwnormal, then for each ji-gwclosed set A and ij-gwclosed set B s.t. $A \cap B = \emptyset$, there exist ij-gwopen set U and ji-gwopen set V s.t. $A \subseteq U$, $B \subseteq V$ and $U \cap V = \emptyset$.*

Proof Clear. □

Theorem 29 *If a biwss (X, w_1, w_2) is ji-w-$T^{\sigma}_{\frac{1}{2}}$ and $cl_{w_i}(\emptyset) = \emptyset$. Consider the following statements:*

(1) *(X, w_1, w_2) is ij-gwnormal,*
(2) *For each ji-gwclosed set A and ij-gwopen set N with $A \subseteq N$, there exists ij-gwopen set U s.t. $A \subseteq U \subseteq cl_{w_i}(U) \subseteq N$.*

Then, the implication (1) $\Rightarrow$ (2) is hold.

Proof Obvious. □

Some types of *ij*-$(w, w^{\star})$ continuous functions

In this section, types of continuous functions between biweak spaces are defined and some of their features are established.

Definition 11 *A function $f : (X, w_1, w_2) \longrightarrow (Y, w_1^{\star}, w_2^{\star})$ is called:*

(1) *j-$(w, w^{\star})$-continuous if for $x \in X$ and $w_j^{\star}$open set V containing $f(x)$, there is a w_j open set U containing x s.t. $f(U) \subseteq V$.*
(2) *ij-g$(w, w^{\star})$-continuous if for $x \in X$ and $w_j^{\star}$open set V containing $f(x)$, there is an ij-gwopen set U containing x s.t. $f(U) \subseteq V$.*
(3) *ij-g$(w, w^{\star})$closed if for each $w_j$closed set B, $f(B)$ is ji-g$w^{\star}$closed set.*

We describe *ij-g*$(w, w^{\star})$-continuous function in the following part.

Theorem 30 *Let (X, w_1, w_2) be an ij-$wT_{\frac{1}{2}}$ space. If $int_{w_j}(U)$ is $w_j$open for each ij-gwopen set U, then, a function $f : (X, w_1, w_2) \longrightarrow (Y, w^\star_1, w^\star_2)$ is ij-$g(w, w^\star)$-continuous iff $f^{-1}(V)$=$int_{w_j}f^{-1}(V)$ for each $w^\star_j$open set V.*

Proof (⇒): Let V be a $w^\star_j$open set and $x \in f^{-1}(V)$. Since f is *ij-g*$(w, w^\star)$-continuous, then there is an *ij-gw*open set U containing x s.t. $f(U) \subseteq V$. Hence, $U \subseteq f^{-1}(V)$. Since (X, w_1, w_2) is an *ij-w*$T_{\frac{1}{2}}$ space, then $int_{w_j}(U)$=U. From assumptions, U is a $w_j$open set s.t. $x \in U \subseteq f^{-1}(V)$ and so $x \in int_{w_j}f^{-1}(V)$. Therefore, $f^{-1}(V)$=$int_{w_j}f^{-1}(V)$.
(⇐): Let $x \in X$ and V be a $w^\star_j$open set in Y with $f(x) \in V$, then $x \in f^{-1}(V)$. Since $f^{-1}(V)$=$int_{w_j}f^{-1}(V)$, then there exists $w_j$open set U s.t. $x \in U \subseteq f^{-1}(V)$. From Corollary 1, U is an *ij-gw*open set containing x s.t. $f(U) \subseteq V$. Consequently, f is *ij-g*$(w, w^\star)$-continuous. □

Theorem 31 *For a function $f : (X, w_1, w_2) \longrightarrow (Y, w^\star_1, w^\star_2)$, the following are equivalent:*

(1) $f^{-1}(V)$=$int_{w_j}(f^{-1}(V))$, for every $w^\star_j$open set V in Y,
(2) $f(cl_{w_j}(A)) \subseteq cl_{w^\star_j}(f(A))$, for every set A in X,
(3) $cl_{w_j}(f^{-1}(V)) \subseteq (f^{-1}(cl_{w^\star_j}(V))$, for every set V in Y,
(4) $f^{-1}(int_{w^\star_j}(V)) \subseteq int_{w_j}(f^{-1}(V))$, for every set V in Y,
(5) $cl_{w_j}(f^{-1}(F))$=$f^{-1}(F)$, for every $w^\star_j$closed set F in Y.

Proof Obvious. □

Theorem 32 *For any function $f : (X, w_1, w_2) \longrightarrow (Y, w^\star_1, w^\star_2)$, every j-$(w, w^\star)$-continuous function is ij-$g(w, w^\star)$-continuous.*

Proof Obvious from Theorem 30. □

Remark 15 *The following example justifies the converse of the Theorem 32 need not to be true in general.*

Example 17 *Let X=$\{a, b, c, d\}$, Y=$\{1, 2, 3\}$, w_1=$\{\emptyset, \{a\}, \{a, d\}\}$, w_2=$\{\emptyset, \{a, b\}, \{c, d\}\}$, $w^\star_1$=$\{\emptyset, \{1\}, \{2, 3\}\}$, and $w^\star_2$=$\{\emptyset, \{2\}, \{1, 2\}\}$. If f is defined by $f(a)$=$f(b)$=2, $f(c)$=1, $f(d)$=3, we have f is 12-$g(w, w^\star)$-continuous, but it is not 2-$(w, w^\star)$-continuous.*

Proposition 9 *For any surjection function $f : (X, w_1, w_2) \longrightarrow (Y, w^\star_1, w^\star_2)$, the following are equivalent.*

(1) f is an ij-$g(w, w^\star)$closed function.
(2) For any set B in Y and each $w_i$open U s.t. $f^{-1}(B) \subseteq U$, there exists ij-$gw^\star$open set V of Y s.t. $B \subseteq V$ and $f^{-1}(V) \subseteq U$.

Proof (1) ⇒ (2): Let $B \subseteq Y$ and U be a $w_i$open set s.t. $f^{-1}(B) \subseteq U$. Since f is an *ij-g*$(w, w^\star)$closed function, then $f(U)$ is an *ij-gw**open set in Y. Take $f^{-1}(V)$=U. Since f is a surjection function and $f^{-1}(B) \subseteq U$, then B=$f(f^{-1}(B)) \subseteq f(U)$=$V$.
(2) ⇒ (1): Let U be a $w_i$open set, $F \subseteq f(U)$ s.t. F is a $w^\star_i$closed set, then $f^{-1}(F) \subseteq U$. This implies that there exists *ij-gw**open set V in Y s.t. $F \subseteq V$ and $f^{-1}(V) \subseteq U$. Consequently,

$F\subseteq int_{w_j^\star}(V)$ and so $F\subseteq int_{w_j^\star}(f(U))$. This implies that $f(U)$ is *ij-gw**open in Y. Therefore, f is an *ij-g*$(w, w^\star)$closed function. □

Theorem 33 *Let $(Y, w_1^\star, w_2^\star)$ be an ij-$w^\star T_{\frac{1}{2}}$ space. If $int_{w_j^\star}(A)$ is $w_i^\star$open for each ij-gw*open set A. If $f : (X, w_1, w_2) \longrightarrow (Y, w_1^\star, w_2^\star)$ is a surjection ij-g$(w, w^\star)$closed and ij-g$(w, w^\star)$-continuous function, then $f^{-1}(B)$ is ij-gwclosed set of X for every ij-gw*closed set B of Y.*

Proof Let $B\subseteq Y$ be an *ij-gw**closed set. Let U be a $w_i$open set of X s.t. $f^{-1}(B)\subseteq U$. Since f is a surjection *ij-g*$(w, w^\star)$closed function, then by Proposition 9, there exists *ij-gw**open set V of Y s.t. $B\subseteq V$ and $f^{-1}(V)\subseteq U$. Since $(Y, w_1^\star, w_2^\star)$ is an *ij-*$w^\star T_{\frac{1}{2}}$ space, then $int_{w_j^\star}(V)=V$. From assumptions, V is a $w_i^\star$open set. Since B is *ij-gw**closed, then $cl_{w_j^\star}(B) \subseteq V$. Hence, $f^{-1}(cl_{w_j^\star}(B)) \subseteq f^{-1}(V) \subseteq U$. By Theorems 30 and 31, $cl_{w_j}f^{-1}(B) \subseteq U$, and hence, $f^{-1}(B)$ is *ij-gw*closed set in X. □

Lemma 2 *Let $(Y, w_1^\star, w_2^\star)$ be an ji-$w^\star T_{\frac{1}{2}}$ space. If $f : (X, w_1, w_2) \longrightarrow (Y, w_1^\star, w_2^\star)$ is an ij-g$(w, w^\star)$closed function, then $cl_{w_i^\star}f(A)=f(cl_{w_j}(A))$, for every $w_j$closed set A in X.*

Proof Let A be a $w_j$closed set in X, then $A=cl_{w_j}(A)$. Since f is an *ij-g*$(w, w^\star)$closed function, then $f(A)$ is *ji-gw**closed set since $(Y, w_1^\star, w_2^\star)$ is a *ji-*$w^\star T_{\frac{1}{2}}$ space, then $cl_{w_i^\star}f(A)=f(A)$. Hence, $cl_{w_i^\star}f(A)=f(cl_{w_j}(A))$. □

Lemma 3 *Let $(Y, w_1^\star, w_2^\star)$ be an ij-$w^\star T_{\frac{1}{2}}$ space. and $f : (X, w_1, w_2) \longrightarrow (Y, w_1^\star, w_2^\star)$ be a ji-g$(w, w^\star)$closed function. If $cl_{w_j}(A)$ is a $w_i$closed set for each set A in X, then $cl_{w_j^\star}f(A) \subseteq f(cl_{w_j}(A))$.*

Proof Suppose $cl_{w_j}(A)$ is a $w_i$closed set in X. Since f is an *ji-g*$(w, w^\star)$closed function, then $f(cl_{w_j}(A))$ is *ij-gw**closed set containing $f(A)$. Since $(Y, w_1^\star, w_2^\star)$ is an *ij-*$w^\star T_{\frac{1}{2}}$ space, then $cl_{w_j^\star}f(cl_{w_j}(A))=f(cl_{w_j}(A))$. Hence, $cl_{w_j^\star}f(A)\subseteq f(cl_{w_j}(A))$. □

Theorem 34 *Let $(Y, w_1^\star, w_2^\star)$ be an ij-$w^\star T_{\frac{1}{2}}$ space. If $int_{w_i}f^{-1}(U)$ is $w_i$open for each $w_i^\star$open set U in Y and $cl_{w_j}(A)$ is a $w_i$closed set for each set A in X. If $f : (X, w_1, w_2) \longrightarrow (Y, w_1^\star, w_2^\star)$ is a ji-g$(w, w^\star)$closed and ji-g$(w, w^\star)$-continuous function, then $f(A)$ is ij-gw*closed set of Y for every ij-gwclosed set A of X.*

Proof Follows directly from Theorem 30, Theorem 31, and Lemma 3. □

Theorem 35 *Let $(Y, w_1^\star, w_2^\star)$ be an ij-$w^\star T_{\frac{1}{2}}$ space. If $i_{w_j^\star}(A)$ is $w_j^\star$open for each ij-gw*open set A of Y. If $f : (X, w_1, w_2) \longrightarrow (Y, w_1^\star, w_2^\star)$ and $h : (Y, w_1^\star, w_2^\star) \longrightarrow (Z, \upsilon_1, \upsilon_2)$ are ij-g$(w, w^\star)$-continuous and ij-g$(w^\star, \upsilon)$-continuous functions, respectively, then $h \circ f : (X, w_1, w_2) \longrightarrow (Z, \upsilon_1, \upsilon_2)$ is ij-g(w, υ)-continuous.*

Proof Let $x \in X$ and V be a $\upsilon_j$open set of Z containing $h \circ f(x)$. Since h is *ij-g*$(w^\star, \upsilon)$-continuous, then there is an *ij-gw**open set U containing $h(x)$ s.t. $h(U) \subseteq V$. Since $(Y, w_1^\star, w_2^\star)$ is an *ij-*$w^\star T_{\frac{1}{2}}$ space, hence, $i_{w_j^\star}(U)=U$. From assumptions, U is a $w_j^\star$open for each *ij-gw**open set U of Y containing $h(x)$. Since f is an *ij-g*$(w, w^\star)$-continuous function, so there is an *ij-gw*open set G containing x s.t. $f(G)\subseteq U$. It follows that there exists an *ij-gw*open set G containing x s.t. $h\circ f(G)\subseteq V$. Consequently, $h\circ f$ is *ij*-$g(w,\upsilon)$-continuous. □

Theorem 36 *If* $f : (X, w_1, w_2)\longrightarrow(Y, w_1^\star, w_2^\star)$ *and* $h : (Y, w_1^\star, w_2^\star)\longrightarrow(Z, \upsilon_1, \upsilon_2)$ *are ij-*$g(w, w^\star)$*-continuous and j-*$(w^\star, \upsilon)$*-continuous respectively, then* $h\circ f : (X, w_1, w_2)\longrightarrow(Z, \upsilon_1, \upsilon_2)$ *is ij-*$g(w, \upsilon)$*-continuous.*

Proof Straightforward. □

Future work

In the future, we intend to introduce the bisoft weak structure spaces and study the notions *ij*-soft *gw* closed, *ij*-soft *gw*open, and *ij*-soft σgwclosed sets in it. Also, using these sets, diverse classes of mappings on soft biweak structures can be examined. Further, we suggest studying the properties of some kinds of *ij-gw*closed subsets with respect to a biweak structure modified by elements of an ideal or a hereditary class. Accordingly, we construct a kind of continuity depending on the new class of *ij-gw*closed subsets. Moreover, one may take research to find the suitable way of defining the biweak structure spaces associated to the digraphs by using *ij-gw*closed such that there is a one-to-one correspondence between them. It may also lead to the new properties of separation axioms on these spaces. It will be necessary to perform more research to strengthen a comprehensive framework for the practical applications.

Acknowledgements
The authors thank the reviewers for their useful comments that led to the improvement of the original manuscript.

Authors' contributions
All authors jointly worked on the results and they read and approved the final manuscript.

Funding
Not applicable.

Availability of data and materials
The datasets used and/or analyzed during the current study are available from the corresponding author on reasonable request.

Competing interests
The authors declare that they have no competing interests.

Received: 26 December 2019 Accepted: 16 April 2020
Published online: 19 May 2020

References

1. Kelly, J.: Bitopological spaces. Proc. London. Math. Soc. **13**, 71–89 (1963)
2. Fukutake, T.: On generalized closed sets in bitopological spaces. Bull. Fukuoka Univ. Ed. III. **35**, 19–28 (1985)
3. Csaszar, A., Makai Jr., E.: Further remarks on δ- and θ-modifications. Acta Math. Hungar. **123**, 223–228 (2009)
4. Boonpok, C.: Weakly open functions on bigeneralized topological spaces. Int. J. Math. Anal. **4**, 891–897 (2010)
5. Boonpok, C.: Biminimal structure spaces. Int. Math. Forum. **5**, 703–707 (2010)
6. Csaszar, A.: Weak structures. Acta Math. Hungar. **131**(1-2), 193–195 (2011)
7. Ekici, E: On weak structures due to Csaszar. Acta Math. Hungar. **134**(4), 565–570 (2012)
8. Zahran, A, Mousa, A, Ghareeb, A: generalized closed sets and some separation axioms on weak structure. Hacettepe J. Math. Stat. **44**(3), 669–677 (2015)
9. Puiwong, J, Viriyapong, C, Khampakdee, J: Weak separation axioms in biweak structure spaces. Burapha Sci. J. **22**(2), 110–117 (2017)
10. Al-Omari, A, Noiri, T: a unified theory of generalized closed sets in weak structure. Acta Math. Hungar. **135**(1–2), 174–183 (2012)
11. Al-Omari, A, Noiri, T: weak continuity between WSS and GTS due to Csaszar. Malays. J. Math. Sci. **7**(2), 297–313 (2013)

Publisher's Note