Existence of cooperative search technique to find a Brownian target

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Abstract

This article aims to study the existence of the cooperative search technique to find a Brownian target. We have 2n cooperative searchers coordinate their search to find a Brownian target that moves on one of n disjoint real lines. Each line has two searchers. All of these searchers start the searching process from the origin. Rather than finding the conditions that make the expected value of the first interviewing time between one of the searchers and the target is finite, we compute the approximate value of this expected value.

Keywords: Search theory, Coordinated search technique, Probability theory, Brownian motion

Mathematics subject classification: 37A50, 60 K30

Introduction

The searching problem for the lost targets that move with random movement is one of the most important contemporary problems that the mathematicians seek to solve it in different ways to reduce the cost of the search. One of these random movements is the Brownian motion. In an earlier work, this problem has been discussed on the real line, plan, and space by studying different kinds of search plans such as the linear search plan as in El-Hadidy et al [1, 2]., El-Rayes et al. [3], and Corwin [4]. They presented the finiteness of the search strategy which was used by founding the conditions that make $E(\tau)$ (the first interviewing time expected value for one of the searchers and the target where τ denote to this time) finite. In addition, they showed the existence of this optimal search plan and El-Hadidy [1] found it.

One of the famous search techniques is the coordinated search technique. Reyniers [5, 6] was the first one who investigated this technique on the line. This technique considered two searchers starting together at the origin of the real line with unit speed. These searchers aimed to find the expected time for detecting a hidden target after one of them return to the origin to tell the other. In addition, they aimed to minimize this expected time. But, the returning time increases this expected time. Here, we will treat this defect by not repeating the searching process on the searched parts.

In recent works, El-Hadidy and Abou-Gabal [7] presented a coordinated search technique to find a random Walk target, which moves randomly on the real line. Rather than showing the existence of the optimal search strategy that minimizes this first interviewing time, El-Hadidy and Abou-Gabal [7] found the conditions that make $E(\tau)$

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finite. El-Hadidy et al. [8–10] also studied this technique on the plan and space when the target is hidden at any random point. They found the expected value of the time to detect the target by one of the searchers. Besides that, they found the optimal search plan that minimizes this expected value. For different kinds of search plans on the line, plan, and space, the reader can see, El-Hadidy et al. [11–20], Kagan and Ben-Gal [21], Guerrier and Holcman [22], Palyulin et al. [23], Radmard and Croft [24], Stone et al. [25], and Jia et al. [26].

In this work, we use the technique which was presented in Reyniers [5, 6] and El-Hadidy and Abou-Gabal [7] but without repeating the searching process on the searched parts of n disjoint real lines. This technique is the generalization of the technique which was used in El-Hadidy and Alzulaibani [27]. We use n cooperative searchers to seek a Brownian target which moves in one of the n disjoint cylinders (real lines). This idea is supported through modern communication methods. Each line has two searchers starting at the same time from the origin. One of these searchers goes towards $+\infty$ to search the right part of the real line and the other goes towards to $-\infty$ to search the left part of the real line. This will reduce $E(\tau)$ (cost). Thus, we aim to show the existence of a finite search plan and compute this expected value.

The organization of this paper appears as follows: the "Problem formulation" section describes our search problem. We show the existence and finiteness of our search plan in the "Existence of the finite search plan" section. The computation of $E(\tau)$ appears in "Computational expected value for τ" section. Finally, we discuss the results and future works which were obtained in the conclusion part.

Problem formulation

We always seek for reducing the cost to detect the Brownian moving target on one of the n real lines. To do this, we use n cooperative searchers as in El-Hadidy et al. [1, 2, 20, 28]. But, the technique which was used in El-Hadidy et al. [1, 2, 20, 28] wasted a lot of costs because the searching process is repeated in the same part more than once. Our model does not waste this cost because the searchers do not repeat the searching process by using the connection technology methods. Besides that, we calculate the approximated expected value of the Brownian target detection.

The searching framework
The space of search
The search space is the n axis of n cylinders (n real lines L_i, $i = 1, 2, ..., n$), see Fig. 1.

The target
Our target moves with a Brownian motion $\{B(t), t \geq 0\}$, where the target's initial position is unknown, but its probability distribution is known at time 0.
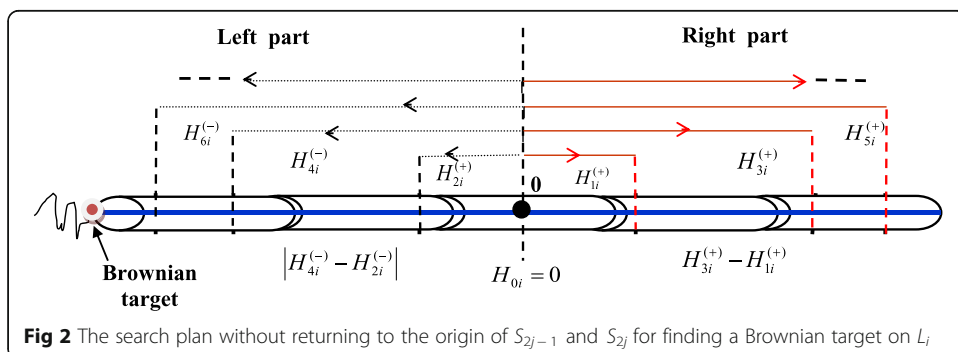
The means of search
All the searchers S_j, $j = 1, 2, ..., 2n$ start the searching process from the origin. Each line L_i, $i = 1, 2, ..., n$ needs two unit speed searchers S_{2j-1} and S_{2j}, where the searcher S_{2j-1} searches in the right part and S_{2j} in the left part of the real line. After searching successively common distances, they do not return to the origin of L_i. The searching process

Fig. 1 The search space (*n* disjoint cylinders)
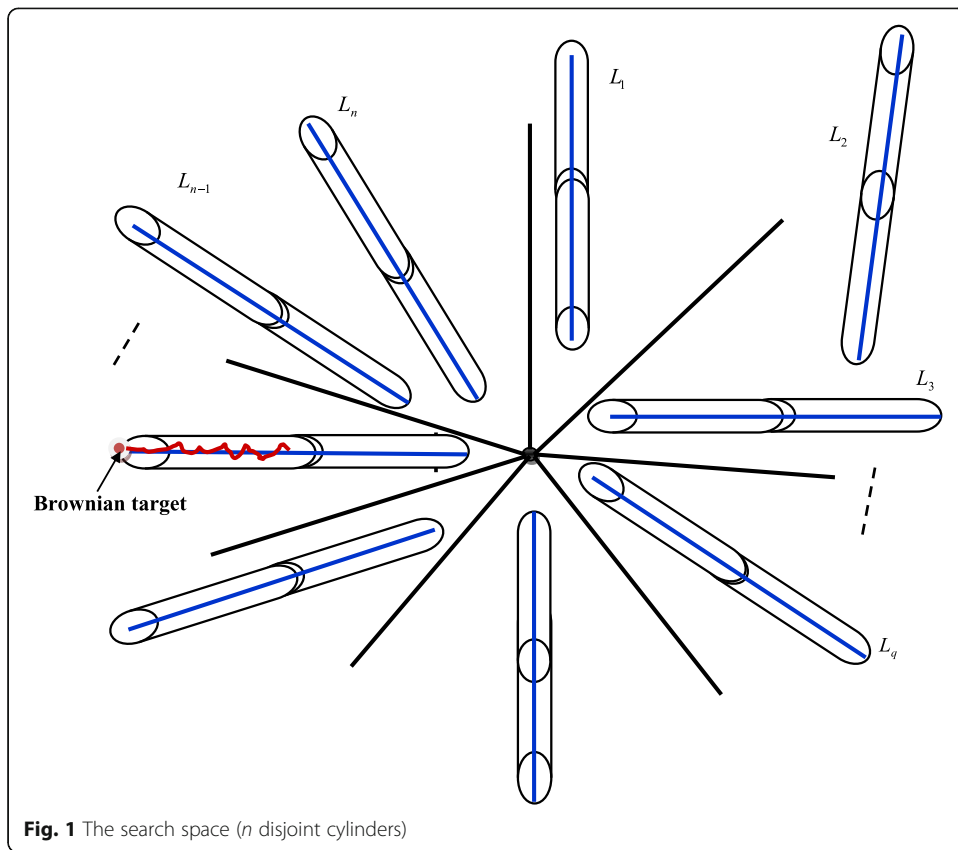
on L_i would conduct as follows: begin at $H_{0i} = 0$ and go to the right (left) as far as $H_{1i}^{(+)}$ ($H_{2i}^{(-)}$). Then, the first one of S_{2j-1} and S_{2j} which meets the target will tell the other by using the wireless communication methods. If the target is not found, then do the above step again at the point $H_{1i}^{(+)}(H_{2i}^{(-)})$ to search the right (left) part of $H_{1i}^{(+)}(H_{2i}^{(-)})$ as far as $[H_{3i}^{(+)} - H_{1i}^{(+)}]([|H_{4i}^{(-)} - H_{2i}^{(-)}|])$ and so on, see Fig. 2.

Existence of the finite search plan

In this section, we aim to get the conditions that make the search plan be finite. On the line L_i, $i = 1, 2, ..., n$ the two searchers S_{2j-1} and S_{2j} are coordinating their search

Fig 2 The search plan without returning to the origin of S_{2j-1} and S_{2j} for finding a Brownian target on L_i

and following the search paths $\phi_{2j-1}:\ R^+\to R$ and $\overline{\phi}_{2j}:\ R^+\to R$, respectively, to meet the target, where

$$|\ \phi_{2j-1}(t_1)-\phi_{2j-1}(t_2)\ |\ \le v^{(+)}\ |\ t_1-t_2\ |\ \&\ |\ \overline{\phi}_{2j}(t_1)-\overline{\phi}_{2j}(t_2)\ |\ \le v^{(-)}\ |\ t_1-t_2\ |\ \forall\ t_1,t_2\in R^+,\quad (1)$$

(i.e., the search paths should satisfy the Lipschitz condition), and $v^{(+)}$ and $v^{(-)}$ are the velocities of the searchers in the right and left part, respectively. Assuming that $\theta_{i(+)}$, $\theta_{i(-)}$, $\lambda_{i(+)}$, and $\lambda_{i(-)}$ are positive integer numbers greater than one. For any line, L_i, $i = 1$, 2, ..., n define the sequences $\{G_{ki}^{(+)}\}_{k\ge 1, i=1,2,...,n}$, $\{G_{ki}^{(-)}\}_{k\ge 1, i=1,2,...,n}$, $\{H_{ki}^{(+)}\}_{k\ge 1, i=1,2,...,n}$ and $\{H_{ki}^{(-)}\}_{k\ge 1, i=1,2,...,n}$ (see El-Rayes et al. [3]), respectively, to obtain the distances which the searchers should do them as a function of $\theta_{i(+)}$, $\theta_{i(-)}$, $\lambda_{i(+)}$, and $\lambda_{i(-)}$. The main idea of this model is all searchers do not return to the origin as in Fig. 1 that is reducing $E(\tau)$. Thus, we can define $G_{ik}^{(+)} = \lambda_{i(+)}(\theta_{i(+)}^k-1)$, $G_{ik}^{(-)} = \lambda_{i(-)}(\theta_{i(-)}^k-1)$. This leads to the traveled distances at the time step k, $k = 1, 2, ...$ are given by $H_{ki}^{(+)} = C_i^{(+)}(G_{ki}^{(+)}+2)$, $H_{ki}^{(-)} = C_i^{(-)}(G_{ki}^{(-)}+2)$, in the right and left part of L_i, $i = 1, 2, ..., n$, respectively, where $C_i^{(+)} = \frac{\varpi_i^{(+)}(\theta_{i(+)}-1)}{(\theta_{i(+)}+1)}$, $C_i^{(-)} = \frac{\varpi_i^{(-)}(\theta_{i(-)}-1)}{(\theta_{i(-)}+1)}$, and $\varpi_i^{(+)}$, $\varpi_i^{(-)}$ are rational numbers. Consequently, the search path in the right part of L_i, $i = 1, 2, ..., n$ can be defined as follows: for any $t\in R^+$, if $G_{(2k-3)i}^{(+)}\le t\le G_{(2k-1)i}^{(+)}$, we have

$$\phi_i(t) = \left(H_{(2k-1)i}^{(+)}-H_{(2k-3)i}^{(+)}\right) + \left(t-G_{(2k-1)i}^{(+)}\right) \qquad (2)$$

And, in the left part for any $t\in R^+$, if $G_{(2k-2)i}^{(-)}\le t\le G_{(2k)i}^{(-)}$, we have

$$\overline{\phi}_i(t) = -\left[\left(H_{(2k)i}^{(-)}-H_{(2k-2)i}^{(-)}\right) + \left(t-G_{(2k)i}^{(-)}\right)\right] \qquad (3)$$

Let the notations $\psi(G_{(2k-1)i}^{(+)}) = B(G_{(2k-1)i}^{(+)})-C_i^{(+)}(G_{(2k-1)i}^{(+)})$, $\tilde{\psi}(G_{(2k)i}^{(-)}) = B(G_{(2k)i}^{(-)}) + C_i^{(-)}(G_{(2k)i}^{(-)})$, $i = 1, 2, ..., n; k = 1, 2, ...$ are held. Then,

$$\tau = \inf\ \{t: \text{either one of } \phi_i(t) = Z_{0i} + B(t) \text{ or } \overline{\phi}_i(t) = Z_{0i} + B(t)\}, i = 1, 2, ..., n\}, \quad (4)$$

where Z_{0i} is a random variable which represents the target's initial position on L_i, $i = 1, 2, ..., n$ and independent with $\{B(t), t > 0\}$. Also, let the search plan be represented by $(\phi_1, \phi_2, ..., \phi_n, \overline{\phi}_1, \overline{\phi}_2, ..., \overline{\phi}_n)\in\Phi_0$, where $\Phi_0 = \{(\Phi, \overline{\Phi}): \phi_i \in \Phi, \overline{\phi}_i \in \overline{\Phi}\ \forall i = 1, 2, ..., n\}$.

For $t > 0$, we have an infinite number of possible outcomes, thus the probability density function which represents the target's position can be described as a calendar for the continuity of the relative frequencies of the data in a given interval. Consequently, the probability of any certain value of the target's position at time step k is equal to zero. Logically, the first interviewing time event and the target's position $z_i \in Z_i$ on any line L_i, $i = 1, 2, ..., n$ at time step k on $[0, \infty]$ or $[-\infty, 0]$ have a known probability value greater than zero. This leads us to study our problem in a new space (probability space). Let this probability space be (Ω, Σ, γ) where Ω is the sample space of all expected meeting points, Σ is the $\sigma-$ algebra which represents the collection of all mutually exclusive events that show the target's position at any time step k, and γ is the probability measure which used as a measure of an integrator factor into our

probability space. Now, we will study the existence of the finite search plan as in the following theorems.

Theorem 1: The combination of the search plans $(\phi_1, \phi_2, ..., \phi_n, \overline{\phi}_1, \overline{\phi}_2, ..., \overline{\phi}_n) \in \Phi_0$ is finite if the following $\int_0^\infty \sum_{k=2}^\infty \theta_{i(+)}^{2k-2} P(\psi(G_{(2k-3)i}^{(+)}) > -z_i)\gamma_i(dz_i)$ and $\int_{-\infty}^0 \sum_{k=2}^\infty \theta_{(-)}^{2i-2} P(\tilde{\psi}(G_{(2k-2)}^{(-)})$ $< -z_i)\gamma_i(dz_i)$ are finite for all $i = 1, 2, ..., n$, where $\sum_{i=1}^n \gamma_i = 1$.

Proof Since the target may be met at one of n lines, then $\tau_{\phi_i}, i = 1, 2, ..., n$ "the first interviewing time between one of the searchers $S_{2j-1}, j = 1, 2, ..., 2n$ in the right parts of all lines $L_i, i = 1, 2, ..., n$ and the Brownian target" and $\tau_{\overline{\phi}_i}, i = 1, 2, ..., n$ "the first interviewing time between one of the searchers $S_{2j}, j = 1, 2, ..., 2n$ in the left parts and the Brownian target" of all lines $L_i, i = 1, 2, ..., n$ are mutually exhaustive events. Then, for any $k \geq 0$, we have

$$P(\tau > t) = P\left(\tau_{\phi_1} > t \text{ or } \tau_{\phi_2} > t \text{ or } ... \text{ or } \tau_{\phi_n} > t \text{ or } \tau_{\overline{\phi}_1} > t \text{ or } \tau_{\overline{\phi}_2} > t \text{ or } ... \text{ or } \tau_{\overline{\phi}_n} > t\right)$$
$$= \bigcup_{i=1}^n \left(P(\tau_{\phi_i} > t) + P(\tau_{\overline{\phi}_i} > t)\right)$$
$$= \sum_{i=1}^n \left(P(\tau_{\phi_i} > t) + P(\tau_{\overline{\phi}_i} > t)\right).$$

Then, we obtain

$$E(\tau) = \int_0^\infty P(\tau > t)dt \leq \sum_{1=1}^n \sum_{k=1}^\infty \left[\int_{G_{(2k-3)i}^{(+)}}^{G_{(2k-1)i}^{(+)}} P(\tau_{\phi_i} > t)\,dt + \int_{G_{(2k-2)i}^{(-)}}^{G_{(2k-1)i}^{(-)}} P(\tau_{\overline{\phi}_i} > t)\,dt\right]$$
$$\leq \sum_{i=1}^n \sum_{k=1}^\infty \left[\left(G_{(2k-1)i}^{(+)} - G_{(2k-3)i}^{(+)}\right)P\left(\tau_{\phi_i} > G_{(2k-3)i}^{(+)}\right) + \left(G_{(2k)i}^{(-)} - G_{(2k-2)i}^{(-)}\right)P\left(\tau_{\overline{\phi}_i} > G_{(2k-2)i}^{(-)}\right)\right]$$
$$\leq \sum_{i=1}^n \sum_{k=1}^\infty \left[\lambda_{i(+)}\left(\theta_{i(+)}^{2k-1} - \theta_{i(+)}^{2k-3}\right)P\left(\tau_{\phi_i} > G_{(2k-3)i}^{(+)}\right) + \lambda_{i(-)}\left(\theta_{i(-)}^{2k} - \theta_{i(-)}^{2k-2}\right)P\left(\tau_{\overline{\phi}_i} > G_{(2k-2)i}^{(-)}\right)\right] \quad (5)$$
$$= \sum_{i=1}^n \left(\lambda_{i(+)}\theta_{i(+)}P\left(\tau_{\phi_i} > 0\right) + \lambda_{i(+)}\theta_{i(+)}\left(\theta_{i(+)}^2 - 1\right)\sum_{k=2}^\infty \theta_{i(+)}^{2k-2}P\left(\tau_{\phi_i} > G_{(2k-3)i}^{(+)}\right)\right.$$
$$\left. + \lambda_{i(-)}\theta_{i(-)}P\left(\tau_{\overline{\phi}_i} > 0\right) + \lambda_{i(-)}\theta_{i(-)}^2\left(\theta_{i(-)}^2 - 1\right)\sum_{k=2}^\infty \theta_{(-)}^{2k-2}P\left(\tau_{\overline{\phi}_i} > G_{(2k-2)i}^{(-)}\right)\right).$$

Also, at time step k on the line $L_i, i = 1, 2, ..., n$, we have

$$P\left(\tau_{\phi_i} > G_{(2k-3)i}^{(+)}\right) \leq \int_0^\infty P\left(Z_{0i} + B\left(G_{(2k-3)i}^{(+)}\right) > H_{(2k-3)i}^{(+)}|Z_{0i} = z_i\right)\gamma_i(dz_i), k \geq 1.$$

By using the above notation $\psi(G_{(2k-3)i}^{(+)}) = B(G_{(2k-3)i}^{(+)}) - C_i^{(+)}(G_{(2k-3)i}^{(+)})$, we can get $\psi(G_{(2k-3)i}^{(+)}) > -z_i$ (El-Hadidy et al. [1, 2]), then $P(\tau_{\phi_i} > G_{(2k-3)i}^{(+)}) \leq \int_0^\infty P(\psi(G_{(2k-3)i}^{(+)}) > -z_i)\gamma_i(dz_i)$, and by using the another notation $\tilde{\psi}(G_{(2k)i}^{(-)}) = B(G_{(2k)i}^{(-)}) + C_i^{(-)}(G_{(2k)i}^{(-)})$, we get $P(\tau_{\overline{\phi}_i} > G_{(2k-2)}^{(-)}) \leq \int_{-\infty}^0 P(\tilde{\psi}(G_{(2k-2)i}^{(-)}) < -z_i)\gamma_i(dz_i)$. Consequently,

$$E(\tau) = \sum_{i=1}^n \left[g_i + \lambda_{i(+)}\theta_{i(+)}\left(\theta_{i(+)}^2 - 1\right)\int_0^\infty M_i(z_i)\gamma_i(dz_i) + \lambda_{i(-)}\theta_{i(-)}^2\left(\theta_{i(-)}^2 - 1\right)\int_{-\infty}^0 L_i(z_i)\gamma_i(dz_i)\right],$$

where $g_i = (\lambda_{i(+)}\theta_{i(+)} + \lambda_{i(-)}\theta_{i(-)})P(\tau > 0)$, $P(\tau > 0)$ is the knowing initial probability of τ and $M_i(z_i) = \sum_{k=2}^\infty \theta_{i(+)}^{2k-2}P(\psi(G_{(2k-3)i}^{(+)}) > -z_i)$ and $L_i(z_i) = \sum_{k=2}^\infty \theta_{i(-)}^{2k-2}P(\tilde{\psi}(G_{(2k-2)i}^{(-)}) < -z_i)$.

Thus, $E(\tau)$ is finite if $\int_0^\infty M_i(z_i)\gamma_i(dz_i)$ and $\int_{-\infty}^0 L_i(z)\gamma_i(dz_i)$ are finite. The prove is completed.∎

This model is the first investigation of the cooperative search technique by using $2n$ searchers which reduces $E(\tau)$. But the above conditions in Theorem 1 are not sufficient to get the finite expected search time. Thus, we want to get more conditions to make this model more applicable and effective.

Theorem 2: *At any time step k, the chosen search plan $\Phi_0 = \{(\Phi, \overline{\Phi}) : \phi_i \in \Phi, \overline{\phi}_i \in \overline{\Phi} \, \forall i = 1, 2, ..., n\}$ should satisfy $[M_1(z_{k1}), M_2(z_{k2}), ..., M_n(z_{kn})] \leq [\tilde{M}_1(|z_{k1}|), \tilde{M}_2(|z_{k2}|), ..., \tilde{M}_n(|z_{kn}|)]$ and $[L_1(z_{k1}), L_2(z_{k2}), ..., L_n(z_{kn})] \leq [\tilde{L}_1(|z_{k1}|), \tilde{L}_2(|z_{k2}|), ..., \tilde{L}_n(|z_{kn}|)]$ where $[\tilde{M}_1(|z_{k1}|), \tilde{M}_2(|z_{k2}|), ..., \tilde{M}_n(|z_{kn}|)]$ and $[\tilde{L}_1(|z_{k1}|), \tilde{L}_2(|z_{k2}|), ..., \tilde{L}_n(|z_{kn}|)]$ are vectors of linear functions.*

Proof At time step k, if $z_{ki} \leq 0$ on the line L_i, $i = 1, 2, ..., n$, then $[M_1(z_{k1}), M_2(z_{k2}), ..., M_n(z_{kn})] \leq [M_1(0), M_2(0), ..., M_n(0)]$ but for $z_{ki} > 0$, we get $M_i(0) = \sum_{k=2}^{\infty} \theta_{i(+)}^{2k-2} P(\psi(G_{(2k-1)i}^{(+)}) > 0)$. Consequently,

$$[M_1(z_{k1}), M_2(z_{k2}), ..., M_n(z_{kn})] = [M_1(0) + \sum_{k=2}^{\infty} \theta_{1(+)}^{2k-2} P\left(\psi\left(G_{(2k-1)1}^{(+)}\right) \leq 0\right), M_2(0) + \sum_{k=2}^{\infty} \theta_{2(+)}^{2k-2} P\left(\psi\left(G_{(2k-1)2}^{(+)}\right) \leq 0\right), ..., M_n(0)$$
$$+ \sum_{k=2}^{\infty} \theta_{n(+)}^{2k-2} P\left(\psi\left(G_{(2k-1)n}^{(+)}\right) \leq 0\right)].$$

$$(6)$$

If the target starts its motion at time step k on the real line L_i, $i = 1, 2, ..., n$ from the random point z_{ki} with drift μ and variance σ^2, then on the right part of L_i and for $t \geq G_{(2k-1)i}^{(+)} > 0$, we have

$$P(B(t) \geq \alpha_i t) \leq P\left(B\left(G_{(2k-1)i}^{(+)}\right) \geq \alpha_i G_{(2k-1)i}^{(+)}\right) = P\left(\sigma \sqrt{G_{(2k-1)i}^{(+)}} Z + \mu G_{(2k-1)i}^{(+)} \geq \alpha_i G_{(2k-1)i}^{(+)}\right),$$

where α_i, $i = 1, 2, ..., n$ are constants. This leads to

$$p\left(Z \geq \frac{(\alpha_i - \mu) G_{(2k-1)i}^{(+)}}{\sigma \sqrt{G_{(2k-1)i}^{(+)}}}\right) = \int_k^{\infty} \frac{1}{\sqrt{2\pi}} e^{-\frac{z^2}{2}} dz = \int_0^{\infty} \frac{1}{\sqrt{2\pi}} e^{-\frac{(x+\kappa_i)^2}{2}} dx \leq \frac{1}{2} \varepsilon^{G_{(2k-1)i}^{(+)}}, \qquad (7)$$

where $\kappa_i = \frac{(\alpha_i - \mu) G_{(2k-1)i}^{(+)}}{\sigma \sqrt{G_{(2k-1)i}^{(+)}}}$, $i = 1, 2, ..., n$. Also, in the left part of L_i, we can get

$$P\left(Z < \frac{(\alpha_i - \mu) G_{(2k-2)i}^{(-)}}{\sigma \sqrt{G_{(2k-2)i}^{(-)}}}\right) \leq 1 - \frac{1}{2} \varepsilon^{G_{(2k-2)i}^{(-)}}. \qquad (8)$$

From (7) and (8) in (6), we obtain

$$[M_1(0), M_2(0), ..., M_n(0)] < [\sum_{k=2}^{\infty} \theta_{1(+)}^{2k-2} \varepsilon_1^{G_{(2k-1)1}^{(+)}}, \sum_{k=2}^{\infty} \theta_{2(+)}^{2k-2} \varepsilon_2^{G_{(2k-1)2}^{(+)}}, ..., \sum_{k=2}^{\infty} \theta_{n(+)}^{2k-2} \varepsilon_n^{G_{(2k-1)n}^{(+)}}], \quad 0 < \varepsilon_i < 1, i = 1, 2, ..., n.$$

For any two random positions $z_{2i} \leq z_{1i}$ of the target on the right part of L_i, we get $P(z_{2i} \leq B(t) \leq z_{1i}) \leq P(z_{2i} \leq B(G_{(2k-1)i}^{(+)}) \leq z_{1i})$ is non-increasing with time $G_{(2k-1)i}^{(+)}$, where $t \geq G_{(2k-1)i}^{(+)} \geq \max(\frac{z_{1i}}{\mu}, \frac{z_{2i}}{\mu})$,

because

$$P\left(z_{2i} \le B\left(G^{(+)}_{(2k-1)i}\right) \le z_{1i}\right) = P\left(z_{2i} \le \sigma \sqrt{G^{(+)}_{(2k-1)i}}\, Z + \mu G^{(+)}_{(2k-1)i} \le z_{1i}\right)$$

$$= P\left(\frac{z_{2i} - \mu G^{(+)}_{(2k-1)i}}{\sigma \sqrt{G^{(+)}_{(2k-1)i}}} \le Z \le \frac{z_{1i} - \mu G^{(+)}_{(2k-1)i}}{\sigma \sqrt{G^{(+)}_{(2k-1)i}}}\right).$$

If $\mu < 0$, then $\dfrac{z_{2i} - \mu G^{(+)}_{(2k-1)i}}{\sigma \sqrt{G^{(+)}_{(2k-1)i}}} \ge 0$, $\dfrac{d}{dG^{(+)}_{(2k-1)i}}\left(\dfrac{z_{2i} - \mu G^{(+)}_{(2k-1)i}}{\sigma \sqrt{G^{(+)}_{(2k-1)i}}}\right) \ge 0$, $\dfrac{z_{1i} - \mu G^{(+)}_{(2k-1)i}}{\sigma \sqrt{G^{(+)}_{(2k-1)i}}} \le 0$ and $\dfrac{d}{dG^{(+)}_{(2k-1)i}}\left(\dfrac{z_{1i} - \mu G^{(+)}_{(2k-1)i}}{\sigma \sqrt{G^{(+)}_{(2k-1)i}}}\right) \le 0.$

Thus, $P(z_{2i} \le B(G^{(+)}_{(2k-1)i}) \le z_{1i})$ be non-increasing.

Assuming that $[\psi_1(k), \psi_2(k), ..., \psi_n(k)] = [\sum_{\ell=1}^{k} y_{\ell 1}, \sum_{\ell=1}^{k} y_{\ell 2}, ..., \sum_{\ell=1}^{k} y_{\ell n}]$, where $\{y_{\ell i}\}_{\ell \ge 1,\ i=1,}$ $_{2, ..., n}$ is a sequence of i.i.d.r.vs which represents the target position at time step k on the line L_i, $i = 1, 2, ..., n$ and $y_{ik} \sim N(\mu - \alpha_i, \sigma^2)$. Consider the following: $[d_{k1}, d_{k2}, ..., d_{kn}]$ $= [G^{(+)}_{(2k+1)1}, G^{(+)}_{(2k+1)2}, ..., G^{(+)}_{(2k+1)n}] = [\lambda_{1(+)}(\theta^{2k+1}_{1(+)} - 1), \lambda_{2(+)}(\theta^{2k+1}_{2(+)} - 1), ..., \lambda_{n(+)}(\theta^{2k+1}_{n(+)} - 1)]$ and $[U_1(j, j+1), U_2(j, j+1), ..., U_n(j, j+1)] = [\sum_{k=1}^{\infty} P[-(j+1) < \psi_1(k) \le -j], \sum_{k=1}^{\infty} P[-(j+1) < \psi_2(k) \le -j], ..., \sum_{k=1}^{\infty} P[-(j+1) < \psi_n(k) \le -j]]$. By choosing $[d_{m1}, d_{m2}, ..., d_{mn}] = [\max(0, \frac{z_{m1}}{\mu}), \max(0, \frac{z_{m2}}{\mu}), ..., \max(0, \frac{z_{mn}}{\mu})]$ and also for any line L_i, $i = 1, 2, ..., n$, we choose $a_i(k) = P[-z_{ki} < \psi_i(k) \le 0] = \sum_{j=1}^{|z_{ki}|} P[-(j+1) < \psi_i(n) \le -j]$. Thus, if $k > d_{mi}$ for all $k \ge 1$, $i = 1, 2, ..., n$ and since $P(z_{2i} \le B(G^{(+)}_{(2k-1)i}) \le z_{1i})$ is non-increasing with time $G^{(+)}_{(2k-1)i}$ at step k, then we have $a_i(k)$ is a non-increasing also. Consequently, $[M_1(z_{k1}), M_2(z_{k2}), ...,$ $M_n(z_{kn})] - [M_1(0), M_2(0), ..., M_n(0)] = [\sum_{k=1}^{m} \theta^{2k+1}_{1(+)} a_1(d_{k1}) + \alpha_1 \sum_{k=m+1}^{\infty} (d_{k1} - d_{(k-1)1}) a_1(d_{k1})$, $\sum_{k=1}^{m} \theta^{2k+1}_{2(+)} a_2(d_{k2}) + \alpha_2 \sum_{k=m+1}^{\infty} (d_{k2} - d_{(k-1)2}) a_2(d_{k2}), ..., \sum_{k=1}^{m} \theta^{2k+1}_{n(+)} a_n(d_{kn}) + \alpha_n \sum_{k=m+1}^{\infty} (d_{kn} - d_{(k-1)n}) a_n(d_{kn})]$.

From Lemma 1 in El-Hadidy et al [1], we have $[M_1(z_{k1}), M_2(z_{k2}), ..., M_n(z_{kn})] - [M_1(0)$, $M_2(0), ..., M_n(0)] \le [\sum_{k=1}^{m} \theta^{2k+1}_{1(+)} + \alpha_1 \sum_{k=d_m}^{\infty} a_1(k), \sum_{k=1}^{m} \theta^{2k+1}_{2(+)} + \alpha_2 \sum_{k=d_m}^{\infty} a_2(k), ... \sum_{k=1}^{m} \theta^{2k+1}_{n(+)} + \alpha_2 \sum_{k=d_m}^{\infty} a_n$ $(k)] \le [d_{m1} + \lambda_{1(+)}(1 - \theta_{n(+)}) \sum_{j=0}^{|z_{k1}|} U_1(j, j+1), d_{m2} + \lambda_{2(+)}(1 - \theta_{2(+)}) + \sum_{j=0}^{|z_{k2}|} U_2(j, j+1), ... d_{mn}$ $+ \lambda_{n(+)}(1 - \theta_{n(+)} + \sum_{j=0}^{|z_{kn}|} U_n(j, j+1))]$.

Since $[\psi_1(k), \psi_2(k), ..., \psi_n(k)] = [\sum_{\ell=1}^{k} y_{\ell 1}, \sum_{\ell=1}^{k} y_{\ell 2}, ..., \sum_{\ell=1}^{k} y_{\ell n}]$, and $\{y_{\ell i}\}_{\ell \ge 1, i=1,2,...,n}$, then

$[U_1(j, j+1), U_2(j, j+1), ..., U_n(j, j+1)]$ satisfies the renewal theorem as in Feller [29]. Hence, $[U_1(j, j+1), U_2(j, j+1), ..., U_n(j, j+1)]$ is bounded $\forall j$ by a constant. Hence, $[M_1(z_{k1}), M_2(z_{k2}), ..., M_n(z_{kn})] \le [M_1(0), M_2(0), ..., M_n(0)] + [M_1, M_2, ..., M_n] + [M_1(|z_{k1}|), M_2(|z_{k2}|), ..., M_n(|z_{kn}|)][\tilde{M}_1(|z_{k1}|), \tilde{M}_2(|z_{k2}|), ..., \tilde{M}_n(|z_{kn}|)]$. By the same method, we can show that $[L_1(z_{k1}), L_2(z_{k2}), ..., L_n(z_{kn})] \le [\tilde{L}_1(|z_{k1}|), \tilde{L}_2(|z_{k2}|), ..., \tilde{L}_n(|z_{kn}|)]$ in the left parts of L_i, $i = 1, 2, ..., n$.∎

In addition, to the conditions in the previous theorems, we need to prove another important condition that is $E \mid Z_0 \mid\ < \infty$. This condition confirms the existence of the finiteness.

Theorem 3: *If* $(\phi_1, \phi_2, ..., \phi_n, \overline{\phi}_1, \overline{\phi}_2, ..., \overline{\phi}_n) \in \Phi_0$ *are a combination of finite search, then* $E \mid Z_0 \mid$ *is finite.*

Proof It is clear that $E(\tau) < \infty$ if $(P(\tau$ is finite$) = 1$. Thus, we have

$$P(\tau \text{ is } finite) = P(\tau_{\phi_1} \text{ is } finite \text{ or } \tau_{\phi_2} \text{ is } finite \text{ or} ... \text{or } \tau_{\phi_n} \text{ is } finite \text{ or } \tau_{\overline{\phi}_1} \text{ is } finite \text{ or } \tau_{\overline{\phi}_2} \text{ is } finite$$

$$\text{or} ... \text{ or } \tau_{\overline{\phi}_n} \text{ is } finite)$$

$$= \overset{n}{\underset{i=1}{\cup}} \left(P\left(\tau_{\phi_i} \text{ is } finite\right) + P\left(\tau_{\overline{\phi}_i} \text{ is } finite\right) \right)$$

$$= \sum_{i=1}^{n} \left(P\left(\tau_{\phi_i} \text{ is } finite\right) + P\left(\tau_{\overline{\phi}_i} \text{ is } finite\right) \right).$$

But, we have only one of $(P(\tau_{\phi_i} \text{ is } finite) = 1$ or $P(\tau_{\overline{\phi}_i}$ is finite$) = 1$ for all $i = 1, 2, ..., n$. If we suppose that $P(\tau_{\phi_i} \text{ is } finite) + P(\tau_{\overline{\phi}_i} \text{ is } finite) = 1$, then we have $Z_0 = \phi_i(\tau_{\phi_i}) - B(\tau_{\phi_i}) + \overline{\phi}_i(\tau_{\overline{\phi}_i}) - B(\tau_{\overline{\phi}_i}) = \phi_i(\tau_{\phi_i}) + \overline{\phi}_i(\tau_{\overline{\phi}_i}) - (B(\tau_{\phi_i}) + B(\tau_{\overline{\phi}_i}))$ with probability one and hence, $\mid Z_0 \mid\ \leq\ \mid \phi_i(\tau_{\phi_i}) + \overline{\phi}_i(\tau_{\overline{\phi}_i}) \mid\ +\ \mid B(\tau_{\phi_i}) + B(\tau_{\overline{\phi}_i}) \mid\ \leq \tau_{\phi_i} + \tau_{\overline{\phi}_i} +\ \mid B(\tau_{\phi_i}) + B(\tau_{\overline{\phi}_i}) \mid, E \mid Z_0 \mid\ \leq E(\tau_{\phi_i}) + E(\tau_{\overline{\phi}_i}) + E \mid B(\tau_{\phi_i}) \mid +E \mid B(\tau_{\overline{\phi}_i}) \mid$. But, $\mid B(\tau_{\phi_i}) \mid\ \leq \tau_{\phi_i}$ and $\mid B(\tau_{\overline{\phi}_i}) \mid\ \leq \tau_{\overline{\phi}_i}$, then $E \mid B(\tau_{\phi_i}) + B(\tau_{\overline{\phi}_i}) \mid\ \leq E(\tau_{\phi_i} + \tau_{\overline{\phi}_i})$ leads to $E \mid B(\tau_{\phi_i}) \mid +E \mid B(\tau_{\overline{\phi}_i}) \mid\ \leq E(\tau_{\phi_i}) + E(\tau_{\overline{\phi}_i})$. If $E(\tau_{\phi_i}) < \infty$ and $E(\tau_{\overline{\phi}_i}) < \infty$, then $E \mid B(\tau_{\phi_i}) \mid< \infty, E \mid B(\tau_{\overline{\phi}_i}) \mid< \infty$, and $E \mid Z_0 \mid$ is finite. Also, if $P(\tau_{\phi_{\hbar}} \text{ is } finite) + p(\tau_{\overline{\phi}_{\hbar}} \text{ is } finite) = 1$ for all $\hbar \neq i, i = 1, 2, ..., n$ then $Z_0 = \phi_i(\tau_{\phi_i}) - B(\tau_{\phi_i}) + \overline{\phi}_i(\tau_{\overline{\phi}_i}) - B(\tau_{\overline{\phi}_i}) = \phi_i(\tau_{\phi_i}) + \overline{\phi}_i(\tau_{\overline{\phi}_i}) - (B(\tau_{\phi_i}) + B(\tau_{\overline{\phi}_i}))$ with probability one; similarly, we get $E \mid Z_0 \mid$ is finite. ∎

The direct result to the realization of the previous theorems confirming a finite search plan if $E \mid Z_0 \mid$ is finite.

Computational expected value for τ

From the Markovian property of the Brownian motion and its dependence on normal distribution, we can use this information to compute $E(\tau)$ which depends on the probability of meeting the target on the line L_i, $i = 1, 2, ..., n$ at the time step k. This probability is given by $G_{(2k-1)i}^{(+)}$ which is depending on the probability of the meeting at the time $G_{(2k-3)i}^{(+)}$.

Theorem 4: The approximated value of $E(\tau)$ is given by

$$E(\tau) = \sum_{i=1}^{n} \left(\lambda_{i(+)}\theta_{i(+)} + \lambda_{i(+)}\theta_{i(+)}\left(\theta_{i(+)}^2 - 1\right) \sum_{k=2}^{\infty} \theta_{i(+)}^{2k-2} \int_{0}^{\infty} \int_{-\infty}^{H_{(2k-3)i}^{(+)}-z_i} \left[1 - \Phi\left(G_{(2k-5)i}^{(+)} - z_i\right)\right] \right.$$

$$d\Phi\left(G_{(2k-5)i}^{(+)} - z_i\right)\gamma_i(dz_i) + \lambda_{i(-)}\theta_{i(-)} + \lambda_{i(-)}\theta_{i(-)}^2\left(\theta_{i(-)}^2 - 1\right)$$

$$\left. \sum_{k=2}^{\infty} \theta_{(-)}^{2k-2} \int_{-\infty}^{0} \int_{-\infty}^{H_{(2k-2)i}^{(-)}+z_i} \Phi\left(G_{(2k-4)i}^{(-)} + z_i\right) d\Phi\left(G_{(2k-4)i}^{(-)} + z_i\right)\gamma_i(dz_i) \right).$$

$$(9)$$

where $(G^{(+)}_{(2k-5)i}-z_i) \sim N(0, (G^{(+)}_{(2k-3)i}-G^{(+)}_{(2k-5)i}))$ and $(G^{(-)}_{(2k-4)i}+z_i) \sim N(0, (G^{(-)}_{(2k-2)i}-G^{(-)}_{(2k-4)i}))$ with the distribution functions $\Phi(G^{(+)}_{(2k-5)i}-z_i)$ and $\Phi(G^{(-)}_{(2k-4)i}+z_i)$, respectively, at time step k on the line L_i, $i = 1, 2, ..., n$.

Proof From (5), we get

$$E(\tau) = \sum_{i=1}^{n} \left(\lambda_{i(+)}\theta_{i(+)}P\left(\tau_{\phi_i} > 0\right) + \lambda_{i(+)}\theta_{i(+)}\left(\theta_{i(+)}^2-1\right) \right.$$

$$\sum_{k=2}^{\infty} \theta_{i(+)}^{2k-2}P\left(\tau_{\phi_i} > G^{(+)}_{(2k-3)i}\right) + \lambda_{i(-)}\theta_{i(-)}P\left(\tau_{\overline{\phi}_i} > 0\right)$$

$$\left. +\lambda_{i(-)}\theta_{i(-)}^2\left(\theta_{i(-)}^2-1\right) \sum_{k=2}^{\infty} \theta_{(-)}^{2k-2}P\left(\tau_{\overline{\phi}_i} > G^{(-)}_{(2k-2)i}\right)\right). \tag{10}$$

Also, at the time step k on the right part of the line L_i, $i = 1, 2, ..., n$, we have

$$P(\tau_{\phi_i} > \left(G^{(+)}_{(2k-3)i}\right) \le \int_0^\infty P\left(Z_{0i} + B\left(G^{(+)}_{(2k-3)i}\right) > \left(H^{(+)}_{(2k-3)i}\right)|Z_{0i} = z_i\right)\gamma_i(dz_i).$$

Consequently, we can find that

$$P\left(Z_{0i} + B\left(G^{(+)}_{(2k-3)i}\right) > H^{(+)}_{(2k-3)i}|Z_{0i} = z_i\right) = P((B\left(G^{(+)}_{(2k-3)i}\right) > \left(H^{(+)}_{(2k-3)i}-z_i\right)$$
$$= 1-P(B\left(G^{(+)}_{(2k-3)i}\right) \le \left(H^{(+)}_{(2k-3)i}-z_i\right)). \tag{11}$$

Generally, for any times t, s, we have $B(t) – B(s)$ is a normal random variable with mean 0 and variance $t – s$ where $s < t$. Therefore, we can find that

$$P\left(B\left(G^{(+)}_{(2k-3)i}\right) \le \left(H^{(+)}_{(2k-3)i}-z_i\right)\right) \le \int_{-\infty}^{\infty} P\left(B\left(G^{(+)}_{(2k-5)i}\right) \le \left(H^{(+)}_{(2k-5)i}-z_i\right), [B\left(G^{(+)}_{(2k-3)i}\right)\right.$$

$$+B\left(G^{(+)}_{(2k-5)i}\right)-B\left(G^{(+)}_{(2k-5)i}\right)] \le \left(H^{(+)}_{(2k-5)i}-z_i\right) \mid B\left(G^{(+)}_{(2k-5)i}\right) = \left(G^{(+)}_{(2k-5)i}-z_i\right)\right)$$

$$\times \frac{1}{\left(G^{(+)}_{(2k-3)i}-G^{(+)}_{(2k-5)i}\right)\sqrt{2\pi}} \times e^{-\frac{1}{2}\left(\frac{G^{(+)}_{(2k-5)i}-z_i}{(G^{(+)}_{(2k-3)i}-G^{(+)}_{(2k-5)i})}\right)^2} d\left(G^{(+)}_{(2k-5)i}-z_i\right)$$

$$= \int_{-\infty}^{\infty} P\left([B\left(G^{(+)}_{(2k-3)i}\right) + B\left(G^{(+)}_{(2k-5)i}\right)]-B\left(G^{(+)}_{(2k-5)i}\right) \le \left(H^{(+)}_{(2k-5)i}-z_i\right)|B\left(G^{(+)}_{(2k-5)i}\right) = \left(G^{(+)}_{(2k-5)i}-z_i\right)\right)$$

$$\times \frac{1}{\left(G^{(+)}_{(2k-3)i}-G^{(+)}_{(2k-5)i}\right)\sqrt{2\pi}} e^{-\frac{1}{2}\left(\frac{G^{(+)}_{(2k-5)i}-z_i}{\left(G^{(+)}_{(2k-3)i}-G^{(+)}_{(2k-5)i}\right)}\right)^2} d\left(G^{(+)}_{(2k-5)i}-z_i\right)$$

(see Klebaner [30]). By considering that the first meeting is not done at the time $t = G^{(+)}_{(2k-5)i})$, then

$$P(B\left(G^{(+)}_{(2k-3)i}\right) \le \left(H^{(+)}_{(2k-3)i}-z_i\right)) = \int_{-\infty}^{H^{(+)}_{(2k-3)i}-z_i} P\left(\left[\left(B\left(G^{(+)}_{(2k-3)i}\right)\right) \le \left(H^{(+)}_{(2k-3)i}-z_i\right)\right] - \left[\left(B\left(G^{(+)}_{(2k-5)i}\right) \le \left(H^{(+)}_{(2k-5)i}\right)-z_i\right)\right]\right.$$

$$\le -\left(G^{(+)}_{(2k-5)i}-z_i\right) \times \frac{1}{\left(G^{(+)}_{(2k-3)i}-G^{(+)}_{(2k-5)i}\right)\sqrt{2\pi}}$$

$$\times e^{-\frac{1}{2}\left(\frac{G^{(+)}_{(2k-5)i}-z_i}{\left(G^{(+)}_{(2k-3)i}-G^{(+)}_{(2k-5)i}\right)}\right)} d\left(G^{(+)}_{(2k-5)i}-z_i\right)$$

$$= \int_{-\infty}^{H^{(+)}_{(2k-3)i}-z_i} \Phi\left(-\left(G^{(+)}_{(2k-5)i}-z_i\right)d\Phi\left(G^{(+)}_{(2k-5)i}-z_i\right) = \int_{-\infty}^{H^{(+)}_{2i-3}-z} \left[1-\Phi\left(G^{(+)}_{(2k-5)i}-z_i\right)\right]d\Phi\left(G^{(+)}_{(2k-5)i}-z_i\right),$$

where $(G^{(+)}_{(2k-5)i}-z_i) \sim N(0,(G^{(+)}_{(2k-3)i}-G^{(+)}_{(2k-5)i}))$ and $\Phi(G^{(+)}_{(2k-5)i}-z_i)$ is the distribution function of this variable. By a similar manner, on the left part of line L_i, $i = 1, 2, ..., n$, one can compute the probability at time step k by

$$P\left(Z_{0i} + B\left(G^{(-)}_{(2k-2)i}\right) < H^{(-)}_{(2k-2)i}|Z_{0i}=z_i\right) = P\left(B\left(G^{(-)}_{(2k-2)i}\right)-\left(H^{(-)}_{(2k-2)i}+z_i\right) < 0\right)$$

$$= \int_{-\infty}^{H^{(-)}_{(2k-2)i}+z_i} \Phi\left(G^{(-)}_{(2k-2)i}+z_i\right)d\Phi\left(G^{(-)}_{(2k-2)i}+z_i\right).$$

Since the target starts its motion on the line L_i, $i = 1, 2, ..., n$ from a random point z_i, then we can assume that $P(\tau_{\phi_i} > 0) = P(\tau_{\overline{\phi}_i} > 0) = 1$. In addition, the first interviewing time should be done on one real line by one searcher. Consequently, in (10), we can obtain the approximate value of $E(\tau)$ by

$$E(\tau) = \sum_{i=1}^{n}(\lambda_{i(+)}\theta_{i(+)} + \lambda_{i(+)}\theta_{i(+)}\left(\theta^2_{i(+)}-1\right) \sum_{k=2}^{\infty} \theta^{2k-2}_{i(+)} \int_{0}^{\infty} \int_{-\infty}^{H^{(+)}_{(2k-3)i}-z_i}$$

$$\left[1-\Phi\left(G^{(+)}_{(2k-5)i}-z_i\right)\right]d\Phi\left(G^{(+)}_{(2k-5)i}-z_i\right)\gamma_i(dz_i) + \lambda_{i(-)}\theta_{i(-)}$$

$$+\lambda_{i(-)}\theta^2_{i(-)}\left(\theta^2_{i(-)}-1\right) \sum_{k=2}^{\infty} \theta^{2k-2}_{(-)} \int_{-\infty}^{0} \int_{-\infty}^{H^{(-)}_{(2k-2)i}+z_i} \Phi\left(G^{(-)}_{(2k-4)i}+z_i\right)d\Phi\left(G^{(-)}_{(2k-4)i}+z_i\right)\gamma_i(dz_i)).\blacksquare$$

If we have $X\sim N(0,\sigma^2)$, then $\int_{-\infty}^{0}d\Phi(x)=\frac{1}{2}$. This leads to $\frac{1}{2} < \int_{-\infty}^{H^{(+)}_{2i-3}-z} d\Phi(G^{(+)}_{2i-5}-z) < 1$ where

$$\int_{-\infty}^{H^{(+)}_{(2k-3)i}-z_i} [1-\Phi(G^{(+)}_{(2k-5)i}-z_i)]d\Phi(G^{(+)}_{2i-5}-z_i) = \int_{-\infty}^{H^{(+)}_{(2k-3)i}-z_i} d\Phi(G^{(+)}_{(2k-5)i}-z_i) - \int_{-\infty}^{H^{(+)}_{(2k-3)i}-z_i} \Phi(G^{(+)}_{(2k-5)i}-z_i)d\Phi(G^{(+)}_{(2k-5)i}-$$

$z_i)$ and $H^{(+)}_{(2k-3)i}-z_i > 0$. From the fact $\int_{-\infty}^{0}\Phi(x)d\Phi(x) = \int_{0}^{\frac{1}{2}}xdx = \frac{1}{8} < \frac{1}{2}$, one can easily con-

clude that $0 < \int_{-\infty}^{H^{(+)}_{(2k-3)i}-z_i} \Phi(G^{(+)}_{(2k-5)i}-z_i)d\Phi(G^{(+)}_{(2k-5)i}-z_i) < \frac{1}{2}$. This also contributes to the assertiveness of the existence and finiteness of our search plan where the Brownian target is not evading the searchers.

Concluding remarks

1) The existence of a cooperative search plan with multiple searchers that used the coordinated linear search technique to find a Brownian target on one of n real line has been presented.

2) The random variable Z_0 is used to give the target initial position.

3) We showed the finiteness of our continuous search plan by giving the conditions that make $E(\tau)$ is finite as in Theorem 1.

4) At any time step k, we showed that the chosen search plan $\Phi_0 = \{(\Phi, \overline{\Phi}) : \phi_i \in \Phi, \overline{\phi}_i \in \overline{\Phi} \ \forall i = 1, 2, ..., n\}$ should satisfy $[M_1(z_{k1}), M_2(z_{k2}), ..., M_n(z_{kn})] \leq [\tilde{M}_1(|z_{k1}|), \tilde{M}_2(|z_{k2}|), ..., \tilde{M}_n(|z_{kn}|)]$, and $[L_1(z_{k1}), L_2(z_{k2}), ..., L_n(z_{kn})] \leq [\tilde{L}_1(|z_{k1}|), \tilde{L}_2(|z_{k2}|), ..., \tilde{L}_n(|z_{kn}|)]$ where $[\tilde{M}_1(|z_{k1}|), \tilde{M}_2(|z_{k2}|), ..., \tilde{M}_n(|z_{kn}|)]$ and $[\tilde{L}_1(|z_{k1}|), \tilde{L}_2(|z_{k2}|), ..., \tilde{L}_n(|z_{kn}|)]$ are vectors of linear functions, see Theorem 2.

5) Besides proving that $E \mid Z_0 \mid \ < \infty$ in Theorem 3, we compute the approximated value of $E(\tau)$ in Theorem 4.

6) In future research, one can study the finiteness of the expected value of the first meeting time between one of n-cooperative searchers and n-dimensional Brownian target and calculate this expected value.

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