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ORIGINAL ARTICLE

On the dynamics of the nonlinear rational difference equation $x_{n+1} = Ax_n + Bx_{n-k} + Cx_{n-l} + \frac{bx_{n-k}}{dx_{n-k} - ex_{n-l}}$

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Abstract In this article, we study the periodicity, the boundedness and the global stability of the positive solutions of the following nonlinear difference equation

$$x_{n+1} = Ax_n + Bx_{n-k} + Cx_{n-l} + \frac{bx_{n-k}}{dx_{n-k} - ex_{n-l}}, \quad n = 0, 1, 2, \ldots$$

where the coefficients $A, B, C, b, d, e \in (0, \infty)$, while k and l are positive integers. The initial conditions $x_{-l}, \ldots, x_{-k}, \ldots, x_{-1}, x_0$ are arbitrary positive real numbers such that $k < l$. Some numerical examples will be given to illustrate our results.

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1. Introduction

The qualitative study of difference equations is a fertile research area and increasingly attracts many mathematicians. This topic draws its importance from the fact that many real life phenomena are modeled using difference equations. Examples from economy, biology, etc. can be found in [1–6]. It is known that nonlinear difference equations are capable of producing a complicated behavior regardless its order. This can be easily seen from the family $x_{n+1} = g_\mu(x_n)$, $\mu > 0$, $n \geqslant 0$. This behavior is ranging according to the value of μ, from the existence of a bounded number of periodic solutions to chaos.

There has been a great interest in studying the global attractivity, the boundedness character and the periodicity nature of nonlinear difference equations. For example, in the articles [7–13] closely related global convergence results were obtained which can be applied to nonlinear difference equations in proving that every solution of these equations converges to a period two solution. For other closely related results (see [14–23]) and the references cited therein. The study of these equations is challenging and rewarding and is still in its infancy. We believe that the nonlinear rational difference equations are of paramount importance in their own right. Furthermore the results about such equations offer prototypes for the development of

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the basic theory of the global behavior of nonlinear difference equations.

The objective of this article is to investigate some qualitative behavior of the solutions of the nonlinear difference equation

$$x_{n+1} = Ax_n + Bx_{n-k} + Cx_{n-l} + \frac{bx_{n-k}}{dx_{n-k} - ex_{n-l}}, \quad n = 0, 1, 2, \ldots \tag{1}$$

where the coefficients $A, B, C, b, d, e \in (0, \infty)$, while k and l are positive integers. The initial conditions $x_{-l}, \ldots, x_{-k}, \ldots, x_{-1}, x_0$ are arbitrary positive real numbers such that $k < l$. Note that the special cases of Eq. (1) have been studied discussed in [24] when $B = C = 0$, and $k = 0, l = 1, b$ is replaced by $-b$ and in [25] when $B = C = 0$, and $k = 0, b$ is replaced by $-b$ and in [26] when $A = C = 0, l = 0, b$ is replaced by $-b$ and in [27] when $B = C = 0, l = 0$.

Our interest now is to study behavior of the solutions of Eq. (1) in its general form. For the related work (see [28–30]). Let us now recall some well known results [2] which will be useful in the sequel.

Definition 1. Consider a difference equation in the form

$$x_{n+1} = F(x_n, x_{n-k}, x_{n-l}), \quad n = 0, 1, 2, \ldots \tag{2}$$

where F is a continuous function, while k and l are positive integers such that $k < l$. An equilibrium point $\tilde{x}$ of this equation is a point that satisfies the condition $\tilde{x} = F(\tilde{x}, \tilde{x}, \tilde{x})$. That is, the constant sequence $\{x_n\}$ with $x_n = \tilde{x}$ for all $n \geqslant -k \geqslant -l$ is a solution of that equation.

Definition 2. Let $\tilde{x} \in (0, \infty)$ be an equilibrium point of Eq. (2). Then we have

(i) An equilibrium point $\tilde{x}$ of Eq. (2) is called locally stable if for every $\varepsilon > 0$ there exists $\delta > 0$ such that, if $x_{-l}, \ldots, x_{-k}, \ldots, x_{-1}, x_0 \in (0, \infty)$ with $|x_{-l} - \tilde{x}| + \cdots + |x_{-k} - \tilde{x}| + \cdots + |x_{-1} - \tilde{x}| + |x_0 - \tilde{x}| < \delta$, then $|x_n - \tilde{x}| < \varepsilon$ for all $n \geqslant -k \geqslant -l$.

(ii) An equilibrium point $\tilde{x}$ of Eq. (2) is called locally asymptotically stable if it is locally stable and there exists $\gamma > 0$ such that, if $x_{-l}, \ldots, x_{-k}, \ldots, x_{-1}, x_0 \in (0, \infty)$ with $|x_{-l} - \tilde{x}| + \ldots + |x_{-k} - \tilde{x}| + \ldots + |x_{-1} - \tilde{x}| + |x_0 - \tilde{x}| < \gamma$, then

$$\lim_{n\to\infty} x_n = \tilde{x}.$$

(iii) An equilibrium point $\tilde{x}$ of Eq. (2) is called a global attractor if for every $x_{-l}, \ldots, x_{-k}, \ldots, x_{-1}, x_0 \in (0, \infty)$ we have

$$\lim_{n\to\infty} x_n = \tilde{x}.$$

(iv) An equilibrium point $\tilde{x}$ of Eq. (2) is called globally asymptotically stable if it is locally stable and a global attractor.

(v) An equilibrium point $\tilde{x}$ of Eq. (2) is called unstable if it is not locally stable.

Definition 3. A sequence $\{x_n\}_{n=-l}^{\infty}$ is said to be periodic with period r if $x_{n+r} = x_n$ for all $n \geqslant -p$. A sequence $\{x_n\}_{n=-l}^{\infty}$ is said to be periodic with prime period r if r is the smallest positive integer having this property.

Definition 4. Eq. (2) is called permanent and bounded if there exists numbers m and M with $0 < m < M < \infty$ such that for any initial conditions $x_{-l}, \ldots, x_{-k}, \ldots, x_{-1}, x_0 \in (0, \infty)$ there exists a positive integer N which depends on these initial conditions such that

$$m \leqslant x_n \leqslant M \quad \text{for all } n \geqslant N.$$

Definition 5. The linearized equation of Eq. (2) about the equilibrium point $\tilde{x}$ is defined by the equation

$$z_{n+1} = \rho_0 z_n + \rho_1 z_{n-k} + \rho_2 z_{n-l} = 0, \tag{3}$$

where

$$\rho_0 = \frac{\partial F(\tilde{x}, \tilde{x}, \tilde{x})}{\partial x_n}, \quad \rho_1 = \frac{\partial F(\tilde{x}, \tilde{x}, \tilde{x})}{\partial x_{n-k}}, \quad \rho_2 = \frac{\partial F(\tilde{x}, \tilde{x}, \tilde{x})}{\partial x_{n-l}}.$$

The characteristic equation associated with Eq. (3) is

$$\rho(\lambda) = \lambda^{l+1} - \rho_0 \lambda^l - \rho_1 \lambda^{l-k} - \rho_2 = 0. \tag{4}$$

Theorem 1 [2]. *Assume that F is a C^1 – function and let $\tilde{x}$ be an equilibrium point of Eq. (2). Then the following statements are true.*

(i) *If all roots of Eq. (4) lie in the open unit disk $|\lambda| < 1$, then the equilibrium point $\tilde{x}$ is locally asymptotically stable.*

(ii) *If at least one root of Eq. (4) has absolute value greater than one, then the equilibrium point $\tilde{x}$ is unstable.*

(iii) *If all roots of Eq. (4) have absolute value greater than one, then the equilibrium point $\tilde{x}$ is a source.*

Theorem 2 [3]. *Assume that ρ_0, ρ_1 and $\rho_2 \in R$. Then*

$$|\rho_0| + |\rho_1| + |\rho_2| < 1, \tag{5}$$

is a sufficient condition for the asymptotic stability of Eq. (2).

Theorem 3 [2]. *Consider the difference Eq. (2). Let $\tilde{x} \in I$ be an equilibrium point of* Eq. (2). *Suppose also that*

(i) *F is a nondecreasing function in each of its arguments.*

(ii) *The function F satisfies the negative feedback property*

$$[F(x, x, x) - x](x - \tilde{x}) < 0 \quad \text{for all } x \in I - \{\tilde{x}\},$$

where I is an open interval of real numbers. Then $\tilde{x}$ is global attractor for all solutions of Eq. (2).

2. The local stability of the solutions

The equilibrium point $\tilde{x}$ of Eq. (1) is the positive solution of the equation

$$\tilde{x} = (A + B + C)\tilde{x} + \frac{b\tilde{x}}{(d - e)\tilde{x}}, \tag{6}$$

where $d \neq e$. If $[(A + B + C) - 1](e - d) > 0$, then the only positive equilibrium point $\tilde{x}$ of Eq. (1) is given by

$$\tilde{x} = \frac{b}{[(A + B + C) - 1](e - d)}. \tag{7}$$

Let us now introduce a continuous function $F:(0,\infty)^3 \longrightarrow (0,\infty)$ which is defined by

$$F(u_0,u_1,u_2) = Au_0 + Bu_1 + Cu_2 + \frac{bu_1}{(du_1 - eu_2)}, \quad (8)$$

provided $du_1 \neq eu_2$. Consequently, we get

$$\begin{cases} \frac{\partial F(\widetilde{x},\widetilde{x},\widetilde{x})}{\partial u_0} = A = \rho_0, \\ \frac{\partial F(\widetilde{x},\widetilde{x},\widetilde{x})}{\partial u_1} = B - \frac{e[(A+B+C)-1]}{(e-d)} = \rho_1, \\ \frac{\partial F(\widetilde{x},\widetilde{x},\widetilde{x})}{\partial u_2} = C + \frac{e[(A+B+C)-1]}{(e-d)} = \rho_2, \end{cases} \quad (9)$$

where $e \neq d$. Thus, the linearized equation of Eq. (1) about $\widetilde{x}$ takes the form

$$z_{n+1} - \rho_0 z_n - \rho_1 z_{n-k} - \rho_1 z_{n-l} = 0, \quad (10)$$

where ρ_0, ρ_1 and ρ_2 are given by (9).

Theorem 4. *Assume that $e \neq d$, $A+B+C \neq 1$ and*

$$|A(e-d)| + |B(e-d) - e[(A+B+C)-1]| + |C(e-d) + e[(A+B+C)-1]| < |e-d|, \quad (11)$$

then the equilibrium point (7) of Eq. (1) is locally asymptotically stable.

Proof. From (9) and (11) we deduce that $|\rho_0| + |\rho_1| + |\rho_2| < 1$, and hence the proof follows with the aid of Theorem 2. □

3. Periodic solutions

In this section, we study the existence of periodic solutions of Eq. (1). The following theorem states the necessary and sufficient conditions that the Eq. (1) has periodic solutions of prime period two.

Theorem 5. *If k and l are both even positive integers, then Eq. (1) has no prime period two solution.*

Proof. Assume that there exist distinct positive solutions

$$\ldots, P, Q, P, Q, \ldots \quad (12)$$

of prime period two of Eq. (1). If k and l are both even positive integers, then $x_n = x_{n-k} = x_{n-l}$. It follows from Eq. (1) that

$$P = (A+B+C)Q - \frac{b}{(e-d)}, \quad (13)$$

and

$$Q = (A+B+C)P - \frac{b}{(e-d)}. \quad (14)$$

By subtracting (14) from (13), we get

$$(P-Q)[(A+B+C)+1] = 0. \quad (15)$$

Since $A+B+C+1 \neq 0$, then $P = Q$. This is a contradiction. Thus, the proof is now completed. □

Theorem 6. *If k and l are both odd positive integers and $A+1 \neq B+C$, then Eq. (1) has no prime period two solution.*

Proof. Following the proof of Theorem 5, we deduce that if k and l are both odd positive integers, then $x_{n+1} = x_{n-k} = x_{n-l}$. It follows from Eq. (1) that

$$P = AQ + (B+C)P - \frac{b}{(e-d)}, \quad (16)$$

and

$$Q = AP + (B+C)Q - \frac{b}{(e-d)}. \quad (17)$$

By subtracting (17) from (16), we get

$$(P-Q)[A-(B+C)+1] = 0. \quad (18)$$

Since $A-(B+C)+1 \neq 0$, then $P = Q$. This is a contradiction. Thus, the proof is now completed. □

Theorem 7. *If k is even and l is odd positive integers, then Eq. (1) has prime period two solution if the condition*

$$(1-C)(3e-d) < (e+d)(A+B), \quad (19)$$

is valid, provided $C < 1$ and $e(1-C) - d(A+B) > 0$.

Proof. If k is even and l is odd positive integers, then $x_n = x_{n-k}$ and $x_{n+1} = x_{n-l}$. It follows from Eq. (1) that

$$P = (A+B)Q + CP - \frac{bQ}{(eP - dQ)}, \quad (20)$$

and

$$Q = (A+B)P + CQ - \frac{bP}{(eQ - dP)}. \quad (21)$$

Consequently, we get

$$eP^2 - dPQ = e(A+B)PQ - d(A+B)Q^2 + eCP^2 - CdPQ - bQ, \quad (22)$$

and

$$eQ^2 - dPQ = e(A+B)PQ - d(A+B)P^2 + eCQ^2 - CdPQ - bP. \quad (23)$$

By subtracting (23) from (22), we get

$$P + Q = \frac{b}{[e(1-C) - d(A+B)]}, \quad (24)$$

where $e(1-C) - d(A+B) > 0$. By adding (22) and (23), we obtain

$$PQ = \frac{eb^2(1-C)}{(e+d)[(1-C)+(A+B)][e(1-C)-d(A+B)]^2}, \quad (25)$$

where $C < 1$. Assume that P and Q are two positive distinct real roots of the quadratic equation

$$t^2 - (P+Q)t + PQ = 0. \quad (26)$$

Thus, we deduce that

$$(P+Q)^2 > 4PQ. \quad (27)$$

Substituting (24) and (25) into (17), we get the condition (19). Thus, the proof is now completed. □

Theorem 8. *If k is odd and l is even positive integers, then Eq. (1) has prime period two solution if the condition*

$$(A+C)(3e-d) < (e+d)(1-B), \tag{28}$$

is valid, provided $B < 1$ and $d(1-B) - e(A+C) > 0$.

Proof. If k is odd and l is even positive integers, then $x_{n+1} = x_{n-k}$ and $x_n = x_{n-l}$. It follows from Eq. (1) that

$$P = (A+C)Q + BP - \frac{bP}{(eQ-dP)}, \tag{29}$$

and

$$Q = (A+C)P + BQ - \frac{bQ}{(eP-dQ)}. \tag{30}$$

Consequently, we get

$$P+Q = \frac{b}{[d(1-B)-e(A+C)]}, \tag{31}$$

where $d(1-B) - e(A+C) > 0$,

$$PQ = \frac{eb^2(A+C)}{(e+d)[(1-B)+(A+C)][d(1-B)-e(A+C)]^2}, \tag{32}$$

where $B < 1$. Substituting (31) and (32) into (27), we get the condition (28). Thus, the proof is now completed. □

4. Boundedness of the solutions

In this section, we investigate the boundedness of the positive solutions of Eq. (1).

Theorem 9. *Let $\{x_n\}$ be a solution of Eq. (1). Then the following statements are true:*

(i) *Suppose $b < d$ and for some $N \geqslant 0$, the initial conditions*

$$x_{N-l+1}, \ldots, x_{N-k+1}, \ldots, x_{N-1}, x_N \in \left[\frac{b}{d}, 1\right],$$

are valid, then for $b \neq e$ and $d^2 \neq be$, we have the inequality

$$\frac{b}{d}(A+B+C) + \frac{b^2}{(d^2-be)} \leqslant x_n \leqslant (A+B+C) + \frac{b}{(b-e)}, \tag{33}$$

for all $n \geqslant N$.

(ii) *Suppose $b > d$ and for some $N \geqslant 0$, the initial conditions*

$$x_{N-l+1}, \ldots, x_{N-k+1}, \ldots, x_{N-1}, x_N \in \left[1, \frac{b}{d}\right],$$

are valid, then for $b \neq e$ and $d^2 \neq be$, we have the inequality

$$(A+B+C) + \frac{b}{(b-e)} \leqslant x_n \leqslant \frac{b}{d}(A+B+C) + \frac{b^2}{(d^2-be)}, \tag{34}$$

for all $n \geqslant N$.

Proof. First of all, if for some $N \geqslant 0$, $\frac{b}{d} \leqslant x_N \leqslant 1$ and $b \neq e$, we have

$$x_{N+1} = Ax_N + Bx_{N-k} + Cx_{N-l} + \frac{bx_{N-k}}{dx_{N-k} - ex_{N-l}} \leqslant A+B+C + \frac{bx_{N-k}}{dx_{N-k} - ex_{N-l}}. \tag{35}$$

But, it is easy to see that $dx_{N-k} - ex_{N-l} \geqslant b - e$, then for $b \neq e$, we get

$$x_{N+1} \leqslant A+B+C + \frac{b}{b-e}. \tag{36}$$

Similarly, we can show that

$$x_{N+1} \geqslant \frac{b}{d}(A+B+C) + \frac{bx_{N-k}}{dx_{N-k} - ex_{N-l}}. \tag{37}$$

But, one can see that $dx_{N-k} - ex_{N-l} \leqslant \frac{d^2-be}{d}$, then for $d^2 \neq be$, we get

$$x_{N+1} \geqslant \frac{b}{d}(A+B+C) + \frac{b^2}{d^2-be}. \tag{38}$$

From (36) and (38) we deduce for all $n \geqslant N$ that the inequality (33) is valid. Hence, the proof of part (i) is completed.

Similarly, if $1 \leqslant x_N \leqslant \frac{b}{d}$, then we can prove part (ii) which is omitted here for convenience. Thus, the proof is now completed. □

5. Global stability

In this section we study the global asymptotic stability of the positive solutions of Eq. (1).

Theorem 10. *If $0 < A+B+C < 1$ and $e \neq d$, then the equilibrium point $\tilde{x}$ given by (7) of Eq. (1) is global attractor.*

Proof. We consider the following function

$$F(x,y,z) = Ax + By + Cz + \frac{by}{(dy-ez)}, \tag{39}$$

where $dy \neq ez$, provided that $B(dy-ez)^2 > ez$ and $C(dy-ez)^2 + bey > 0$. It is easy to verify the condition (i) of Theorem 3. Let us now verify the condition (ii) of Theorem 3 as follows:

$$\begin{aligned}[F(x,x,x)-x](x-\tilde{x}) &= \left\{(A+B+C)x - \frac{b}{e-d} - x\right\} \times \left\{x - \frac{b}{[(A+B+C)-1](e-d)}\right\} \\ &= \left\{\frac{x(e-d)[(A+B+C)-1]-b}{e-d}\right\}^2 \times \frac{1}{[(A+B+C)-1]}.\end{aligned} \tag{40}$$

Since $0 < A+B+C < 1$ and $e \neq d$, then we deduce from (40) that

$$[F(x,x,x)-x](x-\tilde{x}) < 0. \tag{41}$$

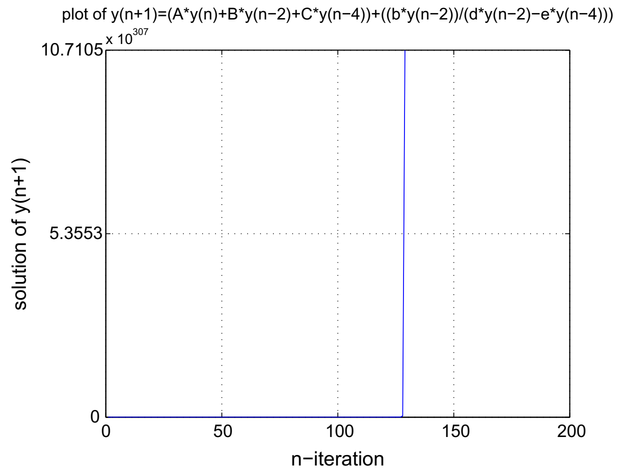

Fig. 1 $x_{n+1} = 300x_n + 200x_n + 200x_{n-2} + 100x_{n-4} + \frac{50x_{n-2}}{30x_{n-2}+20x_{n-4}}$.

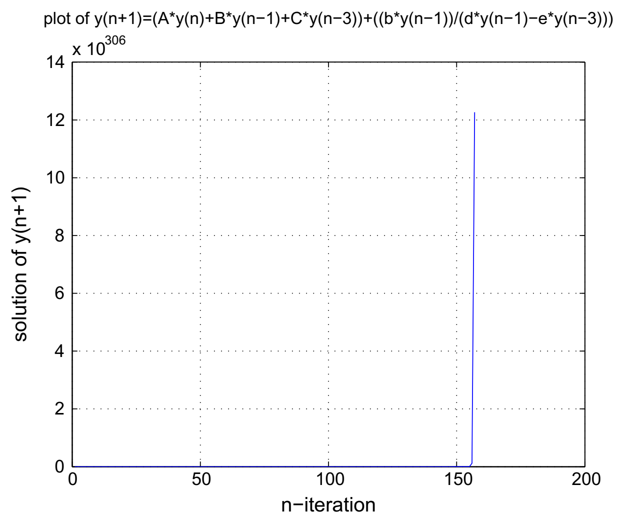

Fig. 2 $x_{n+1} = 100x_n + 50x_{n-1} + 25x_{n-3} + \frac{5x_{n-1}}{3x_{n-1}+2x_{n-3}}$.

According to Theorem 3, $\tilde{x}$ is global attractor. Thus, the proof is now completed. □

On combining the two Theorems 4 and 10, we have the result.

Theorem 11. *The equilibrium point $\tilde{x}$ given by (7) of Eq. (1) is globally asymptotically stable.*

6. Numerical examples

In order to illustrate the results of the previous section and to support our theoretical discussions, we consider some numerical examples in this section. These examples represent different types of qualitative behavior of solutions of Eq. (1).

Example 1. Fig. 1, shows that Eq. (1) has no prime period two solutions if $k = 2$, $l = 4$, $x_{-4} = 1$, $x_{-3} = 2$, $x_{-2} = 3, x_{-1} = 4$, $x_0 = 5$, $A = 300$, $B = 200$, $C = 100$, $b = 50$, $d = 30$, $e = 20$.

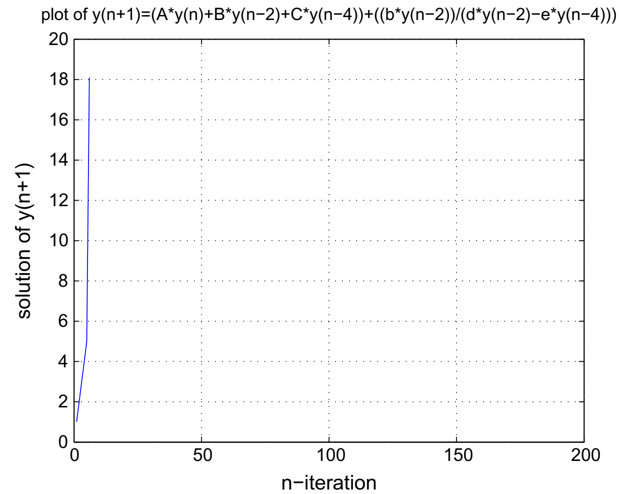

Fig. 3 $x_{n+1} = 0.4x_n + 0.3x_{n-2} + 0.2x_{n-4} + \frac{5x_{n-2}}{x_{n-2}+2x_{n-4}}$.

Example 2. Fig. 2, shows that Eq. (1) has no prime period two solutions if $k = 1$, $l = 3$, $x_{-3} = 1$, $x_{-2} = 2$, $x_{-1} = 3$, $x_0 = 4$, $A = 100$, $B = 50$, $C = 25$, $b = 5$, $d = 3$, $e = 2$.

Example 3. Fig. 3, shows that Eq. (1) is globally asymptotically stable if $x_{-4} = 1$, $x_{-3} = 2$, $x_{-2} = 3$, $x_{-1} = 4$, $x_0 = 5$, $A = 0.4$, $B = 0.3$, $C = 0.2$, $b = 5$, $d = 1$, $e = 2$.

7. Conclusions

We have discussed some properties of the nonlinear rational difference Eq. (1), namely the periodicity, the boundedness and the global stability of the positive solutions of this equation. We gave some figures to illustrate the behavior of these solutions. Our results in this article can be considered as a more generalization than the results obtained in Refs. [24–27]. Note that Example 1 verifies Theorem 5 which shows that if k and l are both even positive integers, then Eq. (1) has no prime period two solution. But Example 2 verifies Theorem 6 which shows that if k and l are both odd positive integers, then Eq. (1) has no prime period two solution, while Example 3 verifies Theorem 11 which shows that Eq. (1) is globally asymptotically stable.

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