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ORIGINAL ARTICLE

Properties of superposition operators acting between $\mathcal{B}_\mu^*$ and Q_K^*

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Abstract In this paper we introduce natural metrics in the hyperbolic Bloch and Q_K-type spaces with respect to which these spaces are complete. Moreover, Lipschitz continuous, bounded and compact superposition operators S_ϕ from the hyperbolic Bloch type space to the hyperbolic Q_K-type space are characterized by conditions depending only on the analytic symbol ϕ.

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1. Introduction

In 1979, Yamashita [1] introduced originally the concept of systematically hyperbolic function classes. Subsequently, this concept has studied for hyperbolic Hardy, BMOA and Dirichlet-classes (see, e.g., [1,3–7]). In the last decades, Smith [8] studied inner functions in the hyperbolic little Bloch-class. The hyperbolic counter parts of the Q_p-spaces were studied by Li [9] and Li et al. [10].

On the other hand, Cámera and Giménez [11,12] studied the Bergman space A^p, the space of all L^p functions (with respect to Lebesgue area measure) which is analytic in the unit disk. They showed that $S_\phi(A^p) \subset A^q$ if and only if ϕ is a polynomial of degree at most p/q where $S_\phi : L^p(\mathbb{D}) \to L^q(\mathbb{D})$ is the superposition operator. Later, Buckley and Vukotic [13,14] introduced superposition operators from Besov spaces into Bergman spaces and univalent interpolation in Besov spaces. Also, in [15], Alvarez et al. characterized superposition operators between the Bloch space and Bergman spaces. Recently, Wen Xu [16] studied superposition operators on Bloch-type spaces.

Let X and Y be two metric spaces of analytic functions on the unit disk $\mathbb{D} = \{z \in \mathbb{C} :| z |< 1\}$. Assume that ϕ denotes a complex-valued function in the plane $\mathbb{C}$. The superposition operator S_ϕ on Xdefined by

$$S_\phi(f) := \phi \circ f, \quad f \in X.$$

If $\phi \circ f \in Y$ for $f \in X$, we say that ϕ acts by superposition from Xinto Y. As in Wen Xu [16] if X contains linear functions, ϕ must be an analytic function.

Let $H(\mathbb{D})$ be the class of analytic functions on $\mathbb{D}$. Also, $B(\mathbb{D})$ denotes the class of all analytic functions on $\mathbb{D}$ such that $|f(z)| < 1$ for all $z \in \mathbb{D}$. It is clear that $B(\mathbb{D}) \subset H(\mathbb{D})$.

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Hyperbolic derivative for analytic functions on the unit disk $\mathbb{D}$.

$f^*(z) = \frac{|f'(z)|}{1-|f(z)|^2}$ (cf. [17]).

The spaces of analytic functions, have been actively appearing in different areas of mathematical sciences such as dynamical systems, theory of semigroups, probability, mathematical physics and quantum mechanics (see [18–20] and others). Now, we list the following definitions.

Definition 1.1 [2]. Let f be an analytic function in $\mathbb{D}$ and $0 < \alpha < \infty$. The α-Bloch space $\mathcal{B}^\alpha$ is defined by

$$\mathcal{B}^\alpha = \left\{ f \in H(\mathbb{D}) : \|f\|_{\mathcal{B}^\alpha} = \sup_{z\in\mathbb{D}} (1-|z|^2)^\alpha |f'(z)| < \infty \right\},$$

the little α-Bloch space $\mathcal{B}_0^\alpha$ is given as follows

$$\mathcal{B}_0^\alpha = \left\{ f \in H(\mathbb{D}) : \|f\|_{\mathcal{B}_0^\alpha} = \lim_{|z|\to 1^-} (1-|z|^2)^\alpha |f'(z)| = 0 \right\}.$$

The spaces $\mathcal{B}^1$ and $\mathcal{B}_0^1$ are called as the Bloch space, and little Bloch space and denoted by $\mathcal{B}$ and $\mathcal{B}_0$ respectively (see [21]).

A positive continuous function μ on $[0,1)$ is called normal if there are three constants $0 \leqslant \delta < 1$ and $0 < a < b$ such that.

i. $\frac{\mu(r)}{(1-r)^a}$ is decreasing on $[\delta,1)$ and $\lim_{r\to 1}\frac{\mu(r)}{(1-r)^a} = 0$;
ii. $\frac{\mu(r)}{(1-r)^b}$ is increasing on $[\delta,1)$ and $\lim_{r\to 1}\frac{\mu(r)}{(1-r)^b} = \infty$.

Definition 1.2 [22]. A function $f \in H(\mathbb{D})$ such that

$$\|f\|_\mu := \sup_{z\in\mathbb{D}} \mu(|z|) f'(z) < \infty$$

is called a μ-Bloch function. The space of all μ-Bloch functions is denoted by $\mathcal{B}_\mu$.

It is readily seen that $\mathcal{B}_\mu$ is a Banach space with the norm $\|f\|_{\mathcal{B}_\mu} := |f(0)| + \|f\|_\mu$. Also, when $\mu(z) = 1-|z|^2$, the space $\mathcal{B}_\mu$ is just the Bloch space which is denoted by $\mathcal{B}$; while when $\mu(z) = (1-|z|^2)^\alpha$ with $\alpha > 0$, the space $\mathcal{B}_\mu$ becomes the α-Bloch space which is denoted by $\mathcal{B}_\alpha$.

The hyperbolic μ-Bloch space is defined as follows:

Definition 1.3 [23]. The sets of $f \in B(\mathbb{D})$ for which

$$\mathcal{B}_\mu^* = \left\{ f : f \text{ analytic in } \mathbb{D} \text{ and } \sup_{z\in\mathbb{D}} \mu(|z|) f^*(z) < \infty \right\}.$$

The little hyperbolic Bloch space $\mathcal{B}_{\mu,0}^*$ is a subspace of $\mathcal{B}_\mu^*$ consisting of all $f \in \mathcal{B}_\mu^*$ such that

$$\lim_{|z|\to 1^-} \mu(|z|) f^*(z) = 0.$$

Following [23], the authors defined a natural metric on the hyperbolic μ-Bloch space $\mathcal{B}_\mu^*$ in the following way:

$$d(f,g;\mathcal{B}_\mu^*) := d_{\mathcal{B}_\mu^*}(f,g) + \|f-g\|_{\mathcal{B}_\mu} + |f(0)-g(0)|,$$

where

$$d_{\mathcal{B}_\mu^*}(f,g) := \sup_{a\in\mathbb{D}} \left| \frac{f'(z)}{1-|f(z)|^2} - \frac{g'(z)}{1-|g(z)|^2} \right| \mu(|z|)$$

for $f,\ g \in \mathcal{B}_\mu^*$.

The following conditions have played crucial roles in the study of Q_K spaces:

$$\int_0^1 \phi_K(s) \frac{ds}{s} < \infty. \quad (1)$$

$$\int_1^\infty \phi_K(s) \frac{ds}{s^2} < \infty. \quad (2)$$

Lemma 1.1 [24]. *If K satisfy the condition (2), then the function*

$$K_1(t) = t\int_t^\infty \frac{K(s)}{s^2} ds \quad (\text{where, } 0 < t < \infty),$$

has the following properties:

(A) K_1 *is nondecreasing on* $(0,\infty)$.
(B) $K_1(t)/t$ *is nondecreasing on* $(0,\infty)$.
(C) $K_1(t) \geqslant K(t)$ *for all* $t \in (0,\infty)$.
(D) $K_1 \lesssim K$ *on* $(0,1]$.
If $K(t) = K(1)$ *for* $t \geqslant 1$, *then we also have*
(E) $K_1(t) = K_1(1) = K(1)$ *for* $t \geqslant 1$, *so* $K_1 \approx K$ *on* $(0,\infty)$.

Lemma 1.2 [24]. *If K satisfy the condition (2), then we can find another non-negative weight function given by*

$$K_1(t) = t\int_t^\infty \frac{K(s)}{s^2} ds \quad (\text{where, } 0 < t < \infty),$$

such that $Q_K = Q_{K_1}$ and that the new function K_1 has the following properties:

(A) K_1 *is nondecreasing on* $(0,\infty)$.
(B) $K_1(t)/t$ *is nondecreasing on* $(0,\infty)$.
(c) $K_1(t)$ *satisfies condition (1).*
(d) $K_1(2t) \approx K_1(t)$ *on* $(0,\infty)$.
(e) $K_1(t) \approx K(t)$ *on* $(0,1]$.
(f) K_1 *is differentiable on* $(0,\infty)$.
(g) K_1 *is concave on* $(0,\infty)$.
(h) $K_1(t) = K_1(1)$ *for* $t \geqslant 1$.

Definition 1.4 (*see [25]*). Let a function $K : [0,\infty) \to [0,\infty)$. The space Q_K is defined by

$$Q_K = \left\{ f \in H(\mathbb{D}) : \sup_{a\in\mathbb{D}} \int_{\mathbb{D}} |f'(z)|^2 K(g(z,a))\, dA(z) < \infty \right\}.$$

If

$$\lim_{|a|\to 1^-} \sup_{a\in\mathbb{D}} \int_{\mathbb{D}} |f'(z)|^2 K(g(z,a))\, dA(z) = 0,$$

then $f \in Q_{K,0}$. Clearly, if $K(t) = t^p$, then $Q_K = Q_p$.

Li et al. [10] defined the hyperbolic Q_K type space Q_K^* as follows.

Definition 1.5. Let $K : [0,\infty) \to [0,\infty)$. The hyperbolic space Q_K^* consists of those functions $f \in B(\mathbb{D})$ for which

$$\|f\|_{Q_K^*}^2 = \sup_{a\in\mathbb{D}} \int_{\mathbb{D}} (f^*(z))^2 K(g(z,a))\, dA(z) < \infty.$$

Moreover, we say that $f \in Q_K^*$ belongs to the space $Q_{K,0}^*$ if

$$\lim_{|a|\to 1^-} \int_{\mathbb{D}} (f^*(z))^2 K(g(z,a))\, dA(z) = 0,$$

where dA is the normalized 2-dimensional Lebesgue measure on $\mathbb{D}$, $g(z,a) = \log \frac{1}{|\varphi_a(z)|}$ is the Green's function of $\mathbb{D}$ where $\varphi_a(z) = \frac{a-z}{1-\bar{a}z}$ is the Möbius transformation related to the point $a \in \mathbb{D}$. Note that hyperbolic classes are not linear spaces, since they consist of functions that are self-maps of $\mathbb{D}$.

For $f, g \in Q_K^*$, we define their distance by

$$d(f,g;Q_K^*) := d_{Q_K^*}(f,g) + \|f-g\|_{Q_K} + |f(0)-g(0)|,$$

where

$$d_{Q_K^*}(f,g) := \left(\sup_{z\in\mathbb{D}} \int_{\mathbb{D}} |f^*(z) - g^*(z)|^2 K(g(z,a)) dA(z)\right)^{\frac{1}{2}}.$$

Now, we introduce the following results of the complete metric spaces $\mathcal{B}_\mu^*$ and Q_K^*.

Proposition 1.1. *The class $\mathcal{B}_\mu^*$ is equipped with a complete metric. Moreover, $\mathcal{B}_{\mu,0}^*$ is a closed (and therefore complete) subspace of $\mathcal{B}_\mu^*$.*

Proof. The proof of Propositions 1.1 is very similar to that of Proposition 2.1 in [10]. □

Proposition 1.2. *The class Q_K^* equipped with a complete metric space. Moreover, $Q_{K,0}^*$ is a closed (and therefore complete) subspace of Q_K^*.*

Proof. Let $f,\ g,\ h \in Q_K^*$. Then clearly

(i) $d(f,f;Q_K^*) = 0$.
(ii) $d(f,g;Q_K^*) \geqslant 0$ and $d(f,g;Q_K^*) = 0$ implies $f = g$.
(iii) $d(f,g;Q_K^*) = d(g,f;Q_K^*)$
(iv) $d(f,g;Q_K^*) \leqslant d(f,h;Q_K^*) + d(h,g;Q_K^*)$.

Hence, d is a metric on Q_K^*, and (Q_K^*, d) is a metric space.

To proof the completeness, let $(f_n)_{n=1}^\infty$ be a Cauchy sequence in the metric space (Q_K^*, d), that is, for any $\varepsilon > 0$ there is an $N = N(\varepsilon) \in \mathbb{N}$ such that $d(f_n, f_m; Q_K^*) < \varepsilon$, for all $n, m > N$. Since $(f_n) \subset B(\mathbb{D})$, such that f_{n_j} converges to f uniformly on compact subsets of $\mathbb{D}$. It follows that also f_n converges to f uniformly on compact subsets, now let $m > N$, and $0 < r < 1$. Then Fatou's lemma yields

$$\begin{aligned}
&\int_{D(0,r)} |f^*(z) - f_m^*(z)|^2 K(1-|\varphi_a(z)|^2) dA(z) \\
&= \int_{D(0,r)} \lim_{n\to\infty} |f_n^*(z) - f_m^*(z)|^2 K(1-|\varphi_a(z)|^2) dA(z) \\
&\leqslant \lim_{n\to\infty} \int_{D(0,r)} |f^*(z) - f_m^*(z)|^2 K(1-|\varphi_a(z)|^2) dA(z) \leqslant \varepsilon^2,
\end{aligned}$$

and by letting $r \to 1^-$, it follows that,

$$\begin{aligned}
&\int_{\mathbb{D}} (f^*(z))^2 K(1-|\varphi_a(z)|^2) dA(z) \\
&\leqslant 2\varepsilon^2 + 2\int_{\mathbb{D}} (f_m^*(z))^2 K(1-|\varphi_a(z)|^2) dA(z)
\end{aligned}$$

this yields,

$$|\,|f|\,|_{Q_K^*}^2 \leqslant 2\,|\,|f_m|\,|_{Q_K^*}^2 + 2\varepsilon^2.$$

Thus $f \in Q_K^*$. We also find that $f_n \to f$ with respect to the metric of (Q_K^*, d) and (Q_K^*, d) is complete metric space. The second part of the assertion follows.

Our objective in this paper is to study Lipschitz continuity, boundedness and compactness of the superposition operator S_ϕ between the hyperbolic spaces $\mathcal{B}_\mu^*$ and Q_K^*. □

2. Main results

First, we study Lipschitz continuity of the superposition operator S_ϕ between the hyperbolic spaces $\mathcal{B}_\mu^*$ and Q_K^* equipped with a complete metric space. Throughout this section we assume that

$$(\phi^*(f(z)) + \phi^*(g(z))) \geqslant \frac{\epsilon}{\mu(|f(z)|)} > 0, \quad \forall\, z \in \mathbb{D}. \tag{3}$$

Now, we give the following result.

Theorem 2.1. *Assume ϕ is non-constant analytic mapping from $\mathbb{D}$ into itself and let $K : [0,\infty) \to [0,\infty)$. Suppose that (3) is satisfied. Then the following statements are equivalent:*

(i) $S_\phi : \mathcal{B}_\mu^* \to Q_K^*$ *is bounded;*
(ii) $S_\phi : \mathcal{B}_\mu^* \to Q_K^*$ *is Lipschitz continuous;*
(iii) $$\sup_{a\in\mathbb{D}} \int_{\mathbb{D}} \frac{|f'(z)|^2}{\mu(|f(z)|)^2} K(g(z,a)) dA(z) < \infty.$$

Proof. To prove (i) $\Longleftrightarrow$ (iii), first assume that (iii) holds, for any $f \in \mathcal{B}_\mu^*$, and $|f(z)|$ is bounded. Then, we obtain

$$\begin{aligned}
\|S_\phi f\|_{Q_K^*} &= \sup_{a\in\mathbb{D}} \int_{\mathbb{D}} ((\phi\circ f)^*(z))^2\; K(g(z,a)) dA(z) \\
&= \sup_{a\in\mathbb{D}} \int_{\mathbb{D}} (\phi^*(f(z)))^2 |f'(z)|^2 K(g(z,a)) dA(z) \\
&\leqslant \|\phi(f(z))\|_{\mathcal{B}_\mu^*}^2 \sup_{a\in\mathbb{D}} \int_{\mathbb{D}} \frac{|f'(z)|^2}{\mu(|f(z)|)^2} K(g(z,a)) dA(z) < \infty.
\end{aligned}$$

Hence, it follows that (i) holds.

Conversely, by assuming that (i) holds and (3), there exists a constant $\epsilon > 0$ such that $(\phi^*(f(z)) + \phi^*(g(z))) \geqslant \frac{\epsilon}{\mu(|f(z)|)} > 0$, where $f,\ g \in \mathcal{B}_\mu^*$, and $\|S_\phi f\|_{Q_K^*} \leqslant C\|\phi(f(z))\|_{\mathcal{B}_\mu^*}$.

We can assume $|f'(z)| < |g'(z)|$. Then, we have

$$\begin{aligned}
&\|S_\phi f\|_{Q_K^*} + \|S_\phi g\|_{Q_K^*} \\
&\geqslant \sup_{a\in\mathbb{D}} \int_{\mathbb{D}} \left[((\phi\circ f)^*(z))^2 + ((\phi\circ g)^*(z))^2\right] K(g(z,a)) dA(z) \\
&\geqslant \sup_{a\in\mathbb{D}} \int_{\mathbb{D}} \left[(\phi^*(f(z)))^2 |f'(z)|^2 + (\phi^*(g(z)))^2 |g'(z)|^2\right] K(g(z,a)) dA(z) \\
&\geqslant \sup_{a\in\mathbb{D}} \int_{\mathbb{D}} \left[(\phi^*(f(z)))^2 + (\phi^*(g(z)))^2\right] |f'(z)|^2\, K(g(z,a)) dA(z)
\end{aligned}$$

$$\geqslant \frac{1}{2}\sup_{a\in\mathbb{D}}\int_{\mathbb{D}}[\phi^*(f(z))+\phi^*(g(z))]^2|f'(z)|^2 K(g(z,a))dA(z)$$
$$\geqslant \frac{\epsilon^2}{2}\sup_{a\in\mathbb{D}}\int_{\mathbb{D}}\frac{|f'(z)|^2}{\mu(|f(z)|)^2}K(g(z,a))dA(z).$$

Then, we have

$$\sup_{a\in\mathbb{D}}\int_{\mathbb{D}}\frac{|f'(z)|^2}{\mu(|f(z)|)^2}K(g(z,a))dA(z)\leqslant \|S_\phi f\|^2_{Q_K^*}+\|S_\phi g\|^2_{Q_K^*}<\infty.$$

So (iii) is satisfied.

To prove (ii) $\iff$ (iii), assume first that $S_\phi:\mathcal{B}_\mu^*\to Q_K^*$ is Lipschitz continuous, that is, there exists a positive constant C such that

$$d(\phi\circ f,\phi\circ g;Q_K^*)\leqslant Cd(\phi(f(z)),\phi(g(z));\mathcal{B}_\mu^*),\quad \text{for all} f,\ g\in\mathcal{B}_\mu^*.$$

Taking $\phi(g)=0$, this implies

$$\|\phi\circ f\|_{Q_K^*}\leqslant C\Big(\|\phi(f(z))\|_{\mathcal{B}_\mu^*}+\|\phi(f(z))\|_{\mathcal{B}_\mu}+|\phi(f(0))|\Big),\text{for all} f\in\mathcal{B}_\mu^*. \tag{4}$$

The assertion (iii) follows by choosing $f(z)=z$ in (4). Moreover, from (3), for $f,\ g\in\mathcal{B}_\mu^*$, we deduce that

$$(\phi^*(f(z))+\phi^*(g(z)))\mu(|f(z)|)\geqslant\epsilon>0,\quad \text{for all } z\in\mathbb{D}. \tag{5}$$

Therefore, combining (4) and (5), we have

$$\|\phi(f(z))\|_{\mathcal{B}_\mu^*}+\|\phi(g(z))\|_{\mathcal{B}_\mu^*}+\|\phi(f(z))\|_{\mathcal{B}_\mu}$$
$$+\|\phi(g(z))\|_{\mathcal{B}_\mu}+|\phi(f(0))|+|\phi(g(0))|$$
$$\geqslant\|\phi\circ f\|_{Q_K^*}+\|\phi\circ g\|_{Q_K^*}\geqslant\frac{\epsilon^2}{2}\int_{\mathbb{D}}\frac{|f'(z)|^2}{\mu(|f(z)|)^2}K(g(z,a))dA(z).$$

For which the assertion (iii) follows.

Assume now that (iii) is satisfied, we have

$$d(\phi\circ f,\phi\circ g;Q_K^*)$$
$$=d_{Q_K^*}(\phi\circ f,\phi\circ g)+\|\phi\circ f-\phi\circ g\|_{Q_K}+|\phi(f(0))-\phi(g(0))|$$
$$\leqslant d_{\mathcal{B}_\mu^*}(\phi(f(z)),\phi(g(z)))\left(\sup_{a\in\mathbb{D}}\int_{\mathbb{D}}\frac{|f'(z)|^2}{\mu}(|f(z)|)^2K(g(z,a))dA(z)\right)^{\frac{1}{2}}$$
$$+\|\phi(f(z))-\phi(g(z))\|_{\mathcal{B}_\mu}\left(\sup_{a\in\mathbb{D}}\int_{\mathbb{D}}\frac{|f'(z)|^2}{\mu(|f(z)|)^2}K(g(z,a))dA(z)\right)^{\frac{1}{2}}$$
$$+|\phi(f(0))-\phi(g(0))|\leqslant Cd(\phi(f(z)),\phi(g(z));\mathcal{B}_\mu^*).$$

Thus $S_\phi:\mathcal{B}_\mu^*\to Q_K^*$ is Lipschitz continuous and this completes the proof.

Secondly, we state and prove compactness of the superposition operators S_ϕ between the hyperbolic spaces. Recall that a superposition operator $S_\phi:\mathcal{B}_\mu^*\to Q_K^*$ is said to be compact, if it maps any ball in $\mathcal{B}_\mu^*$ onto a pre-compact set in Q_K^*.

We state and prove the following proposition. □

Proposition 2.1. *Let ϕ be an analytic mapping from $\mathbb{D}$ into itself and let $K:[0,\infty)\to[0,\infty)$. If $S_\phi:\mathcal{B}_\mu^*\to Q_K^*$ is compact, then it maps closed balls onto compact sets.*

Proof. If $B\subset\mathcal{B}_\mu^*$ is a closed ball and $g\in Q_K^*$ belongs to the closure of $S_\phi(B)$, we can find a sequence $(f_n)_{n=1}^\infty\subset B$ such that $\phi\circ f_n$ converges to $g\in Q_K^*$ as $n\to\infty$. But $(f_n)_{n=1}^\infty$ is a normal family, hence it has a subsequence $(f_{n_j})_{j=1}^\infty$ converging uniformly on compact subsets of $\mathbb{D}$ to an analytic function f. It follows that also f_n converges to f uniformly on compact subsets, and by the Cauchy formula, the same also holds for the derivatives. Let $m>N$. Then the uniform convergence yields

$$\left|\frac{f'(z)}{1-|f(z)|^2}-\frac{f_m'(z)}{1-|f_m(z)|^2}\right|\mu(|z|)$$
$$=\lim_{n\to\infty}\left|\frac{f_n'(z)}{1-|f_n(z)|^2}-\frac{f_m'(z)}{1-|f_m(z)|^2}\right|\mu(|z|)$$
$$\leqslant\lim_{n\to\infty}d(f_n,f_m;\mathcal{B}_\mu^*)\leqslant\epsilon. \tag{6}$$

for all $z\in\mathbb{D}$, and it follows that $\|f\|_{\mathcal{B}_\mu^*}\leqslant\|f_m\|_{\mathcal{B}_\mu^*}+\epsilon$. Thus $f\in\mathcal{B}_\mu^*$. From (6) f belongs to the closed ball B. On the other hand, also the sequence $\phi\circ(f_{n_j})_{j=1}^\infty$ converges uniformly on compact subsets to an analytic function, which is $g\in Q_K^*$. We get $g=\phi\circ f$, i.e. g belongs to $S_\phi(B)$. Thus, this set is closed and also compact. □

Now, we give the main theorem for compactness of superposition operators acting between $\mathcal{B}_\mu^*$ and Q_K^* classes.

Theorem 2.2. *Let ϕ be an analytic mapping from $\mathbb{D}$ into itself and let $K:[0,\infty)\to[0,\infty)$. Then $S_\phi:\mathcal{B}_\mu^*\to Q_K^*$ is compact if*

$$\lim_{r\to1^-}\sup_{a\in\mathbb{D}}\int_{|f(z)|>r}\frac{|f'(z)|^2}{\mu(|f(z)|)^2}K(g(z,a))dA(z)=0. \tag{7}$$

Proof. We first assume that (7) holds. Let $B:=\overline{B}(g,\delta)\subset\mathcal{B}_\mu^*$, $g\in\mathcal{B}_\mu^*$ and $\delta>0$, be a closed ball, and let $(f_n)_{n=1}^\infty\subset B$ be some sequence. We show that its image has a convergent subsequence in Q_K^*, which proves the compactness of S_ϕ by definition.

Again, $(f_n)_{n=1}^\infty\subset B(\mathbb{D})$ is normal, hence, there is a subsequence $(f_{n_j})_{j=1}^\infty$ which converges uniformly on the compact subsets of $\mathbb{D}$ to an analytic function f. By Cauchy formula for the derivative of an analytic function, also the sequence $(f'_{n_j})_{j=1}^\infty$ converges uniformly on the compact subsets of $\mathbb{D}$ to f_{n_j}. It follows that also the sequences $(\phi\circ f_{n_j})_{j=1}^\infty$ and $(\phi\circ f'_{n_j})_{j=1}^\infty$ converge uniformly on the compact subsets of $\mathbb{D}$ to $\phi\circ f$ and $\phi\circ f'$, respectively. Moreover, $f\in B\subset\mathcal{B}_\mu^*$ since for any fixed R, $0<R<1$, the uniform convergence yields

$$\sup_{|z|\leqslant R}\left|\frac{f'(z)}{1-|f(z)|^2}-\frac{g'(z)}{1-|g(z)|^2}\right|\mu(|z|)+\sup_{|z|\leqslant R}|f'(z)-g'(z)|\mu(|z|)$$
$$+|f(0)-g(0)|=\lim_{j\to\infty}\sup_{|z|\leqslant R}\left|\frac{f'_{n_j}(z)}{1-|f_{n_j}(z)|^2}-\frac{g'(z)}{1-|g(z)|^2}\right|\mu(|z|)$$
$$+\lim_{j\to\infty}\left(\sup_{|z|\leqslant R}|f'_{n_j}(z)-g'(z)|\mu(|z|)+|f_{n_j}(0)-g(0)|\right)<\delta.$$

Hence, $d(f,g;\mathcal{B}_\mu^*)\leqslant\delta$.

Let $\varepsilon > 0$. Since (7) is satisfied, we may fix $r, 0 < r < 1$, such that

$$\sup_{a\in\mathbb{D}} \int_{|f(z)|>r} \frac{|f'(z)|^2}{\mu(|f(z)|)^2} K(g(z,a))dA(z) \leqslant \varepsilon.$$

By the uniform convergence, we may fix $N_1 \in \mathbb{N}$ such that

$$|\phi(0)\circ f_{n_j} - \phi(0)\circ f| \leqslant \varepsilon, \quad \text{forall } j \geqslant N_1. \tag{8}$$

The condition (7) is known to imply the compactness of $S_\phi : \mathcal{B}_\mu \to Q_K$, hence possibly to passing once more to a subsequence and adjusting the notations, we may assume that

$$\|\phi\circ f_{n_j} - \phi\circ f\|_{Q_K} \leqslant \varepsilon, \text{ for all } j \geqslant N_2; \quad N_2 \in \mathbb{N}. \tag{9}$$

Since $(f_{n_j})_{j=1}^{\infty} \subset B$, $f \in B$ and $|f'_{n_j}(z)| \leqslant |f'(z)|$ it follows that

$$\sup_{a\in\mathbb{D}} \int_{|f(z)|\geqslant r} \left[(\phi\circ f_{n_j})^*(z) - (\phi\circ f)^*(z)\right]^2 K(g(z,a))dA(z)$$
$$\leqslant \sup_{a\in\mathbb{D}} \int_{|f(z)|\geqslant r} \left[\phi^*(f_{n_j}(z))|f'_{n_j}(z)| - \phi^*(f(z))|f'(z)|\right]^2 K(g(z,a))dA(z)$$
$$\leqslant \sup_{a\in\mathbb{D}} \int_{|f(z)|\geqslant r} \left[\phi^*(f_{n_j}(z)) - \phi^*(f(z))\right]^2 |f'(z)|^2 K(g(z,a))dA(z)$$
$$\leqslant d_{\mathcal{B}^*_\alpha}(\phi(f_{n_j}(z)), \phi(f(z))) \sup_{a\in\mathbb{D}} \int_{|f(z)|>r} \frac{|f'(z)|^2}{\mu(|f(z)|)^2} K(g(z,a))dA(z),$$

hence,

$$\sup_{a\in\mathbb{D}} \int_{|f(z)|\geqslant r} \left[(\phi\circ f_{n_j})^*(z) - (\phi\circ f)^*(z)\right]^2 K(g(z,a))dA(z) \leqslant C\varepsilon. \tag{10}$$

On the other hand, by the uniform convergence on the compact disc $\mathbb{D}$, we can find an $N_3 \in \mathbb{N}$ such that for all $j \geqslant N_3$,

$$\left|\frac{\phi'(f_{n_j})(z)}{1-|\phi(f_{n_j}(z))|^2} - \frac{\phi'(f(z))}{1-|\phi(f(z))|^2}\right| \leqslant \varepsilon.$$

For all z with $|f(z)| \leqslant r$. Hence, for such j,

$$\sup_{a\in\mathbb{D}} \int_{|f(z)|\leqslant r} \left[(\phi\circ f_{n_j})^*(z) - (\phi\circ f)^*(z)\right]^2 K(g(z,a))dA(z)$$
$$\leqslant \sup_{a\in\mathbb{D}} \int_{|f(z)|\leqslant r} \left[\phi^*(f_{n_j}(z)) - \phi^*(f(z))\right]^2 |f'(z)|^2 K(g(z,a))dA(z)$$
$$\leqslant \varepsilon\left(\sup_{a\in\mathbb{D}} \int_{|f(z)|\leqslant r} |f'(z)|^2 \frac{K(g(z,a))}{\mu(|f(z)|)^2} dA(z)\right)^{\frac{1}{2}} \leqslant C\varepsilon,$$

hence,

$$\sup_{a\in\mathbb{D}} \int_{|f(z)|\leqslant r} \left[(\phi\circ f_{n_j})^*(z) - (\phi\circ f)^*(z)\right]^2 K(g(z,a))dA(z) \leqslant C\,\varepsilon. \tag{11}$$

where C is bounded which is obtained from (iii) of Theorem 2.1 combining (8)–(11) we deduce that $f_{n_j} \to f$ in Q_K^*. The proof is therefore completed. □

3. Conclusion

We know that a superposition operator $S_\phi : \mathcal{B}^*_\mu \to Q^*_K$ is said to be bounded if there is a positive constant C such that $\|S_\phi f\|_{Q^*_K} \leqslant C\|\phi(f(z))\|_{\mathcal{B}^*_\mu}$; for all $f \in \mathcal{B}^*_\mu$. Theorem 2.1 shows that $S_\phi : \mathcal{B}^*_\mu \to Q^*_K$ is bounded if and only if it is Lipschitz continuous, that is, if there exists a positive constant C such that $d(\phi\circ f, \phi\circ g; Q^*_K) \leqslant Cd(\phi(f(z)), \phi(g(z)); \mathcal{B}^*_\mu)$, for all $f, g \in \mathcal{B}^*_\mu$.

4. Future work

It is still an open problem to extend the obtained results in this paper by using the superposition operators in different hyperbolic classes of functions which introduced in [1,6,26,27] and others.

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