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ORIGINAL ARTICLE

Spacelike submanifolds of de-Sitter space and application of index form ☆

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Abstract In this paper we develop the total umbilicity of the spacelike submanifold M of a de-Sitter space with the help of some integral formulas, index form of a nonnull geodesic and the Jacobi equation.

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1. Introduction

Let $L_p^{n+p}(c)$ be an $(n+p)$-dimensional connected semi-Riemannian manifold of constant curvature c whose index is p. It is called an indefinite space form of index p and simply a space form when $p = 0$. If $c > 0$, we call it as a de-Sitter space of index p, denote it by $S_p^{n+p}(c)$. Let M^n be an n-dimensional Riemannian manifold immersed in $S_p^{n+p}(c)$. The semi-Riemannian metric of $S_p^{n+p}(c)$ induces the Riemannian metric of M^n, M^n is called a spacelike submanifold.

The study of spacelike submanifolds in de-Sitter space has been recently of substantial interest for both physics and mathematical point of view. In [1], Ximin achieved the total umbilicity of spacelike submanifolds with certain conditions on curvatures under the assumption that the normal bundle is flat and the normalized mean curvature vector is parallel. Further, in [2], it is seen that the index form and Jacobi equation provide nice relations to obtain interesting results on spacelike submanifolds using techniques of integral formulas [3]. Motivated from this literature, in this article we apply index form together with integral formulas on the Laplacian of the squared norm of the second fundamental form and obtain our main result in the form of the following theorem.

Theorem. *Let M^n be a compact spacelike submanifold of de Sitter space $S_p^{n+p}(c)$ with parallel mean curvature vector field ξ in the normal bundle. Let $x : [a,b] \times (-\delta,\delta) \to M$ be the fixed end point geodesic variation such that $|V'|^2 \leqslant \langle R(V,\alpha')V,\alpha'\rangle$, where V and α' are the Jacobi vector field and the tangent vector field to any nonnull geodesic α respectively, then M^n is totally umbilical and the second fundamental form of M^n is parallel.*

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2. Preliminaries

Let M^n be an n-dimensional Riemannian manifold, $e_1, e_2, \ldots, e_n$, a local orthonormal frame field on M^n, and let $\omega_1, \omega_2, \ldots, \omega_n$ be its dual coframe field. Then the structure equations of M^n are given by

$$d\omega_i = \sum_j \omega_{ij} \Lambda \omega_j; \quad \omega_{ij} + \omega_{ji} = 0 \tag{2.1}$$

$$d\omega_{ij} = \sum_k \omega_{ik} \Lambda \omega_{kj} - \frac{1}{2} \sum_{k,l} R_{ijkl} \omega_k \Lambda \omega_l \tag{2.2}$$

where ω_{ij} is the Levi–Civita connection form and R_{ijkl} are the components of the curvature tensor of M^n.

For any C^2-function f defined on M^n, we define its gradient and Hessian by the following formulas:

$$df = \sum_i f_i \omega_i \tag{2.3}$$

$$\sum_j f_{ij} \omega_j = df_i + \sum_j f_j \omega_{ji} \tag{2.4}$$

Further let $\phi = \sum_{ij} \phi_{ij} \omega_i \otimes \omega_j$ be a symmetric tensor defined on M^n. The covariant derivative of ϕ_{ij} is defined by

$$\sum_k \phi_{ijk} \omega_k = d\phi_{ij} + \sum_k \phi_{kj} \omega_{ki} + \sum_k \phi_{ik} \omega_{kj} \tag{2.5}$$

We call the symmetric tensor $\phi = \sum_i \phi_{ij} \omega_i \otimes \omega_j$, a Codazzi tensor if $\phi_{ijk} = \phi_{ikj}$.

The second covariant derivative of ϕ_{ij} is defined by

$$\sum_l \phi_{ijkl} \omega_l = d\phi_{ijk} + \sum_m \phi_{mjk} \omega_{mi} + \sum_m \phi_{imk} \omega_{mj} + \sum_m \phi_{ijm} \omega_{mk} \tag{2.6}$$

By the exterior differentiation of Eq. (2.5) we derive

$$\sum_k \{d\phi_{ijk} \Lambda \omega_k\} = d^2 \phi_{ij} + \sum_k \{d\phi_{kj} \Lambda \omega_{ki}\} + \sum_k \{d\phi_{ik} \Lambda \omega_{kj}\}$$

or

$$\sum_{k,l} \{\phi_{ijkl} \omega_l \Lambda \omega_k\} = \sum_k \{d\phi_{kj} \Lambda \omega_{ki}\} + \sum_k \{d\phi_{ik} \Lambda \omega_{kj}\}$$

or

$$\sum_{k,l} \{\phi_{ijkl} \omega_l \Lambda \omega_k\} = \sum_m \phi_{mj} \Omega_{mi} + \sum_m \phi_{im} \Omega_{mj} \tag{2.7}$$

Through an standard calculation by using Eqs. (2.2) and (2.7), we have

$$\sum_{k,l} \{\phi_{ijkl} - \phi_{ijlk}\} \omega_l \Lambda \omega_k = 2 \sum_m \phi_{mj} \Omega_{mi} + 2 \sum_m \phi_{im} \Omega_{mj}$$

from which it follows that

$$\phi_{ijkl} - \phi_{ijlk} = \sum_m \phi_{mj} R_{mikl} + \sum_m \phi_{im} R_{mjkl} \tag{2.8}$$

Above Eq. (2.8) is called the Ricci identity.

Now we derive the expression of the Laplacian $\Delta\phi$ of the tensor ϕ_{ij} which is defined to be $\sum_k \phi_{ijkk}$. From this definition of the Laplacian we write

$$\Delta\phi = \sum_k \phi_{ijkk} = \sum_k (\phi_{ijkk} - \phi_{ikjk}) + \sum_k (\phi_{ikjk} - \phi_{ikkj}) + \sum_k (\phi_{ikkj} - \phi_{kkij}) + \sum_k \phi_{kkij}$$

The above equation yields

$$\Delta\phi = \sum_{m,k} \phi_{mk} R_{mijk} + \sum_{m,k} \phi_{im} R_{mkjk} + \sum_k (\phi_{ijkk} - \phi_{ikjk}) + \sum_k (\phi_{ikkj} - \phi_{kkij}) + \left(\sum_k \phi_{kk} \right)_{ij} \tag{2.9}$$

Since the tensor ϕ_{ij} is Codazzi, we have $\phi_{ijk} = \phi_{ikj}$ from which we conclude

$$\phi_{ijkk} = \phi_{ikjk} \tag{2.10}$$

Also we know that ϕ_{ij} is symmetric i.e. $\phi_{ij} = \phi_{ji}$ from which we get $\phi_{ijk} = \phi_{jik}$. Taking this into account, we find that

$$\phi_{ikk} = \phi_{kik}$$

or

$$\phi_{ikk} = \phi_{kki}$$

that is

$$\phi_{ikkj} = \phi_{kkij} \tag{2.11}$$

Using Eqs. (2.9)–(2.11) we calculate

$$\Delta\phi_{ij} = \left(\sum_k \phi_{kk} \right)_{ij} + \sum_{m,k} \phi_{mk} R_{mijk} + \sum_{m,k} \phi_{im} R_{mkjk}$$

Let $\|\phi\|^2 = \sum_{i,j} \phi_{ij}^2$; $\|\nabla\phi\|^2 = \sum_{i,j,k} \phi_{ijk}^2$ and $tr\phi = \sum_k \phi_{kk}$. Then from the above we deduce

$$\frac{1}{2} \Delta \|\phi\|^2 = \|\nabla\phi\|^2 + \sum_{i,j} \phi_{ij} (tr\phi)_{ij} + \sum_{i,j,m,k} \phi_{ij} \phi_{mk} R_{mijk} + \sum_{i,j,m,k} \phi_{ij} \phi_{im} R_{mkjk}$$

Near a given point $p \in M^n$, we choose a local orthonormal frame field $\{e_1, e_2, \ldots, e_n\}$ and its dual frame field $\{\omega_1, \omega_2, \ldots, \omega_n\}$ such that $\phi = \sum_{i,j} \phi_{ij} \omega_i \otimes \omega_j$, $\phi_{ij} = \lambda_i \delta_{ij}$ at p. Then the above equation is simplified to

$$\frac{1}{2} \Delta \|\phi\|^2 = \|\nabla\phi\|^2 + \sum_i \lambda_i (tr\phi)_{ii} + \sum_{i,j} \lambda_i \lambda_j R_{jiij} + \sum_{i,j} \lambda_i^2 R_{ijij}$$

or

$$\frac{1}{2} \Delta \|\phi\|^2 = \|\nabla\phi\|^2 + \sum_i \lambda_i (tr\phi)_{ii} + \frac{1}{2} \{ \sum_{i,j} \lambda_i^2 R_{ijij} + \sum_{i,j} \lambda_j^2 R_{jiji} - 2 \sum_{i,j} \lambda_i \lambda_j R_{ijij} \}$$

from which we finally get

$$\frac{1}{2} \Delta \|\phi\|^2 = \|\nabla\phi\|^2 + \sum_i \lambda_i (tr\phi)_{ii} + \frac{1}{2} R_{ijij} (\lambda_i - \lambda_j)^2 \tag{2.12}$$

3. Spacelike submanifolds in de Sitter space and index form

Let M^n be an n-dimensional space-like submanifold in $S_p^{n+p}(c)$. We choose a local field of semi-Riemannian orthonormal frames $e_1, e_2, \ldots, e_{n+p}$ in $S_p^{n+p}(c)$ such that at each point of

M^n, $e_1, e_2, \ldots, e_n$ span the tangent space of M^n and form an orthonormal frame there. We use the following convention on the range of indices:

$$1 \leqslant A, B, C, \ldots \leqslant n+p, \quad 1 \leqslant i, j, k, \ldots \leqslant n, \quad n+1 \leqslant \alpha, \beta, \gamma, \ldots \leqslant n+p$$

In terms of dual frame field the semi-Riemannian metric of $S_p^{n+p}(c)$ is given by $d\bar{s}^2 = \sum_i \omega_i^2 - \sum_\alpha \omega_\alpha^2 = \sum_A \varepsilon_A \omega_A^2$ where $\varepsilon_i = 1$ and $\varepsilon_\alpha = -1$. Then the structural equations of $S_p^{n+p}(c)$ are given by [1]

$$d\omega_A = \sum_B \varepsilon_B \omega_{AB} \Lambda \omega_B; \quad \omega_{AB} + \omega_{BA} = 0 \tag{3.1}$$

$$d\omega_{AB} = \sum_C \varepsilon_C \omega_{AC} \Lambda \omega_{CB} - \frac{1}{2} \sum_{C,D} K_{ABCD} \omega_C \Lambda \omega_D \tag{3.2}$$

where $K_{ABCD} = c\varepsilon_A \varepsilon_B (\delta_{AC}\delta_{BD} - \delta_{AD}\delta_{BC})$.

Now restricting these forms on M^n, we have

$$\omega_\alpha = 0, \quad n+1 \leqslant \alpha \leqslant n+p \tag{3.3}$$

From Cartan's lemma, we write

$$\omega_{\alpha i} = \sum_j h_{ij}^\alpha \omega_j, \quad h_{ij}^\alpha = h_{ji}^\alpha \tag{3.4}$$

From these formulas, we obtain the structure equations of M^n as follows:

$$d\omega_i = \sum_j \omega_{ij} \Lambda \omega_j; \quad \omega_{ij} + \omega_{ji} = 0 \tag{3.5}$$

$$d\omega_{ij} = \sum_k \omega_{ik} \Lambda \omega_{kj} - \frac{1}{2} \sum_{k,l} R_{ijkl} \omega_k \Lambda \omega_l \tag{3.6}$$

$$R_{ijkl} = c(\delta_{ik}\delta_{jl} - \delta_{il}\delta_{jk}) - \sum_\alpha (h_{ik}^\alpha h_{jl}^\alpha - h_{il}^\alpha h_{jk}^\alpha) \tag{3.7}$$

Here we have

$$h = \sum_\alpha h_\alpha e_\alpha = \sum_{i,j,\alpha} h_{ij}^\alpha \omega_i \otimes \omega_j \otimes e_\alpha$$

The mean curvature vector field ξ, the mean curvature H and the square of the length of the second fundamental form S are expressed as

$$\xi = \sum_\alpha H_\alpha e_\alpha, \quad H = \|\xi\|, \quad S = \sum_{i,j,\alpha} (h_{ij}^\alpha)^2 \tag{3.8}$$

respectively, where the matrix of h_α is given by $L_\alpha = (h_{ij}^\alpha)_{n \times n}$ and $H_\alpha = \frac{1}{n}\sum_i h_{ii}^\alpha$ for $\alpha = n+1, n+2, \ldots, n+p$. Moreover the normal curvature tensor $\{R_{\alpha\beta kl}\}$ and the normalized scalar curvature R are expressed as

$$R_{\alpha\beta kl} = \sum_m (h_{km}^\alpha h_{ml}^\beta - h_{lm}^\alpha h_{mk}^\beta)$$

and

$$R = c + \frac{1}{n(n-1)}(S - n^2 H^2)$$

If $R_{\alpha\beta kl} = 0$ at any point x of M^n, we say that the normal connection is flat at x. It is well known that $R_{\alpha\beta kl} = 0$ at x if and only if $h_\alpha' s$ are simultaneously diagonalizable at x [4]. Now suppose that the mean curvature vector ξ is parallel in the normal bundle i.e. the length of ξ is constant which gives $H =$ constant. Further assume that H is a positive constant on M^n and choose $e_{n+1} = \frac{\xi}{H}$. Then it follows that $H_{n+1} = H$ and $H_\alpha = 0$, for $\alpha > n+1$. The following definitions are essential for proving the main result of this article:

Definition [2]. A variation of a curve segment $\alpha : [a, b] \to M$ is a two parameter mapping

$$x : [a, b] \times (-\delta, \delta) \to M$$

such that $\alpha(u) = x(u, 0)$ for all $a \leqslant u \leqslant b$. The vector field V on α given by $V(u) = x_v(u, 0)$ is called the variation vector field of x. Similarly the vector field $A(u) = x_{vv}(u, 0)$ gives the acceleration and we call it the transverse acceleration vector field of x.

As a particular case of variational vector field we have Jacobi vector field defined as follows:

Definition [2]. If γ is a geodesic, a vector field Y on γ that satisfies the Jacobi differential equation $Y'' = R_{Y\gamma'}(\gamma')$ is called a Jacobi vector field.

Also we know that if L is the arc length function of x then the first variation of arc length function is given by [2].

$$L_x'(0) = \varepsilon \int_a^b g\left(\frac{\alpha'}{|\alpha'|}, V'\right) du$$

where ε is the sign of α.

The second variation of arc length of $L_x(v)$ is possible in case α is a geodesic and is given by [2]

$$L_x''(0) = \frac{\varepsilon}{c} \int_a^b \{\langle V', V' \rangle - \langle R(V, \alpha')V, \alpha \rangle\} du + \frac{\varepsilon}{c} [\langle \alpha', A \rangle]_a^b$$

where $\|\alpha'\| = c > 0$ and A is the transverse acceleration vector field of the variation$x$.

We recall that the Riemannian curvature tensor is defined as:

Definition [2]. Let M be a semi-Riemannian manifold with Levi–Civita connection ∇. The function $R : TM \otimes TM \otimes TM \to TM$ given by

$$R(X, Y)Z = \nabla_{[X,Y]} Z - [\nabla_X, \nabla_Y] Z$$

is a $(1, 3)$-tensor field on M called Riemannian curvature tensor.

It is clear that for a fixed endpoint variation the last term of the above expression is zero and hence we have

$$L_x''(0) = \frac{\varepsilon}{c} \int_a^b \{\langle V', V' \rangle - \langle R(V, \alpha')V, \alpha \rangle\} du$$

Definition [2]. The index form I_α of a nonnull geodesic $\alpha \in \Omega(p, q)$, is the unique symmetric bilinear form

$$I_\alpha : T_\alpha(\Omega) \times T_\alpha(\Omega) \to R$$

such that if $V \in T_\alpha(\Omega)$, then

$$I_\alpha(V, V) = L_x''(0)$$

where $\Omega(p, q)$ is the collection of all piecewise smooth curve segments $\alpha : [a, b] \to M$ from p to q.

Moreover in the proof of our main result we make use of Green's theorem which states that

Theorem [4]. *For any function f on an orientable closed Riemannian manifold M, we have*

$$\int_M \Delta f dV = 0$$

We are now in a position to prove the main result of this article as follows:

4. Proof of the main theorem

Theorem 1. *Let M^n be a compact spacelike submanifold of de Sitter space $S_p^{n+p}(c)$ with parallel mean curvature vector field ξin the flat normal bundle. Let $x : [a,b] \times (-\delta,\delta) \to M$ be the fixed end point geodesic variation such that $|V'|^2 \leqslant \langle R(V,\alpha')V,\alpha'\rangle$, where V and α' are the variation vector field and the tangent vector field to any nonnull geodesic αrespectively, then M^n is totally umbilical and the second fundamental form of M^n is parallel.*

Proof. Taking $\phi = h$, where h is the second fundamental form as defined previously, we obtain from Eq. (2.12)

$$\frac{1}{2}\Delta\|h\|^2 = \|\nabla h\|^2 + \sum_i \lambda_i(trh)_{ii} + \frac{1}{2}R_{ijij}(\lambda_i - \lambda_j)^2$$

Since h is simultaneously diagonalizable, we have $h_{ij} = \lambda_i\delta_{ij}$ which from the above equation yields

$$\frac{1}{2}\Delta\|h\|^2 = \|\nabla h\|^2 + \sum_i h_{ii}(nH)_{ii} + \frac{1}{2}R_{ijij}(h_{ii} - h_{jj})^2$$

But from the assumption ξ is parallel in the normal bundle i.e. $|\xi| = H =$ constant. Using this result we get $(nH)_{ii} = 0$. Putting this value in the above equation we find

$$\frac{1}{2}\Delta\|h\|^2 = \|\nabla h\|^2 + \frac{1}{2}\langle R(X,Y)X,Y\rangle(h(X,X) - h(Y,Y))^2$$

for any tangent vector fields X and Y of M^n.

Taking in particular $X = V$ and $Y = \alpha'$, in the above equation where V and α' are as the supposition, we get

$$\frac{1}{2}\Delta\|h\|^2 = \|\nabla h\|^2 + \frac{1}{2}\langle R(V,\alpha')V,\alpha'\rangle(h(V,V) - h(\alpha',\alpha'))^2 \tag{4.1}$$

Now since M^n is compact, integrating the above equation and using Green's theorem, we derive

$$\int_M \|\nabla h\|^2 + \frac{1}{2}\int_M \{\langle R(V,\alpha')V,\alpha'\rangle(h(V,V) - h(\alpha',\alpha'))^2\} = 0 \tag{4.2}$$

As M^n is a spacelike submanifold, its index is 0; therefore, from [lemma-13, page-273, 2], it follows that the Index form I_α satisfies the inequality

$$I_\alpha \geqslant 0$$

or

$$\frac{1}{c}\int_a^b \{\langle V',V'\rangle - \langle R(V,\alpha')V,\alpha'\rangle\} du \geqslant 0$$

Further $|\alpha'| = c > 0$, implies that

$$\{\langle V',V'\rangle - \langle R(V,\alpha')V,\alpha'\rangle\} \geqslant 0$$

This gives

$$|V'|^2 \geqslant \langle R(V,\alpha')V,\alpha'\rangle\} \tag{4.3}$$

The above inequality along with Eq. (4.2), shows that

$$\int_M \|\nabla h\|^2 + \frac{1}{2}\int_M \{|V'|^2(h(V,V) - h(\alpha',\alpha'))^2\} \geqslant 0 \tag{4.4}$$

By assumption we have $|V'|^2 \leqslant \langle R(V,\alpha')V,\alpha'\rangle$which gives

$$\int_M \|\nabla h\|^2 + \frac{1}{2}\int_M \{|V'|^2(h(V,V) - h(\alpha',\alpha'))^2\} \leqslant 0 \tag{4.5}$$

Combining Eqs. (4.4) and (4.5), we conclude that

$$\int_M \{\|\nabla h\|^2 + \frac{1}{2}|V'|^2(h(V,V) - h(\alpha',\alpha'))^2\} = 0$$

or

$$\|\nabla h\| = 0 \text{ and } \{|V'|^2(h(V,V) - h(\alpha',\alpha'))^2\} = 0$$

This shows that $\nabla h = 0$ and either $V' = 0$ or $(h(V,V) - h(\alpha',\alpha'))^2 = 0$. Now we discuss two possibilities

(i) if $\nabla h = 0$ and $V' = 0$
(ii) if $\nabla h = 0$ and $(h(V,V) - h(\alpha',\alpha'))^2 = 0$

First possibility implies that the variational vector field is constant from starting point to the end point of the fixed endpoint geodesic variation but this is impossible for the fixed endpoint geodesic variation as $V(a) = V(b) = 0$ for this kind of motion and the variational vector field must be non-zero in between the endpoints. Hence we have $\nabla h = 0$ and $(h(V,V) - h(\alpha',\alpha'))^2 = 0$. These equations show that

$$\nabla h = 0 \quad \text{and } h(V,V) = h(\alpha',\alpha') \tag{4.6}$$

We know that there exists geodesic curves α through each point p and along all the directions at p of a smooth manifold. By our choice we can choose α' to be some e_i of the basis $\{e_i : i = 1,2,\ldots,n\}$ of M^n and V to be the Jacobi vector field so that $\langle V,\alpha'\rangle = 0$ i.e. V may be taken as one of the basis vectors e_j. Then from Eq. (4.6), we find that

$$h(e_i,e_i) = h(e_j,e_j) \quad \text{for all } i,j = 1,2,\ldots,n$$

Hence M^n is totally umbilical. This completes the proof of the theorem. □

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References

[1] L. Ximin, Spacelike submanifolds in de-Sitter space, J. Phys. A: Math. Gen. 34 (2001) 5463–5468.

[2] B. O'Neill, Semi-Riemannian Geometry with Applications to Relativity, Academic Press, 1983.

[3] K. Yano, Integral Formulas in Riemannian Geometry, Marcel Deccer, Inc., New York, 1970.

[4] B.Y. Chen, Geometry of Submanifolds, Marcel Dekker, Inc., New York, 1973.