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Generalized ψ^*-closed sets in bitopological spaces

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$ij - {}^{\psi^*}T_{1/5}$ spaces

Abstract In this paper, we introduce and study a new class of sets in a bitopological space (X, τ_1, τ_2), namely, ij-ψ^*-closed sets, which settled properly in between the class of ji-α-closed sets and the class of ij-gα-closed sets. We also introduce and study new classes of spaces, namely, $ij - T_{1/5}$ spaces, ij-T_e spaces, ij-αT_e spaces, ij-T_l spaces and ij-αT_l spaces. As applications of ij-ψ^*-closed sets, we introduce and study four new classes of spaces, namely, $ij - T^{\psi^*}_{1/5}$ spaces, $ij - {}^{\psi^*}T_{1/5}$ spaces (both classes contain the class of $ij - T_{1/5}$ spaces), ij-αT_k spaces and ij-T_k spaces. The class of ij-T_k spaces is properly placed in between the class of ij-T_e spaces and the class of ij-T_l spaces. It is shown that dual of the class of $ij - T^{\psi^*}_{1/5}$ spaces to the class of ij-αT_e spaces is the class of ij-αT_k spaces and the dual of the class of $ij - {}^{\psi^*}T_{1/5}$ spaces to the class of $ij - T_{1/5}$ spaces is the class of $ij - T^{\psi^*}_{1/5}$ spaces and also that the dual of the class of ij-T_l spaces to the class of ij-T_k spaces is the class of ij-αT_k spaces. Further we introduce and study ij-ψ^*-continuous functions and ij-ψ^*-irresolute functions.

2010 MATHEMATICAL SUBJECT CLASSIFICATION: 54 C 55; 54 C 10; 54 C 10; 54 E 55

1. Introduction

Recently the topological structure τ on a set X has a lot of applications in many real life applications. The abstractness of a set X enlarges the range of its applications. For example, a special type of this structure is the basic structure for rough set theory [1]. Alexandroff topologies are widely applied in the field of digital topologies [2]. Moreover, τ and its generalizations are applied in biochemical studies [3].

The work presented in this paper will open the way for using two viewpoints in these applications. That is, to apply two topologies at the same time. The concepts of g-closed sets, gs-closed sets, sg-closed sets, gα-closed sets, αg-closed sets, gp-closed sets, gsp-closed sets and spg-closed sets have been introduced in topological spaces (cf. [4–10]). El-Tantawy and Abu-Donia [11] introduced the concepts of (ij-$GC(X)$, ij-$GSC(X)$, ij-$SGC(X)$, ij-$G\alpha C(X)$, ij-$\alpha GC(X)$, ij-$GPC(X)$, ij-$GSPC(X)$, and ij-$SPGC(X)$) subset of (X, τ_1, τ_2). Abd Allah and Nawar [12] introduced The concept of ψ^*-open sets and studied The properties of $T_{1/5}$, T_e, αT_e, T_l, αT_l. In this paper, we introduce a new class of sets in a bitopological space (X, τ_1, τ_2), namely, ij-ψ^*-closed sets, which settled properly in between the class of ji-α-closed sets and the class of ij-gα-closed sets. And we extend the properties to a bitopological space (X, τ_1, τ_2). Also we use

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the family of ij-ψ^*-closed sets to introduce some types of properties in (X, τ_1, τ_2), and we study the relation between these properties. The concepts of pre-continuous, semi-continuous, α-continuous, sp-continuous, g-continuous, αg-continuous, gα-continuous, gs-continuous, sg-continuous, gsp-continuous, spg-continuous, gp-continuous, gc-irresolute, gs-irresolute, αg-irresolute and gα-irresolute functions have been introduced in topological spaces (cf. [7,10,13–22]). El-Tantawy and Abu-Donia [11] introduced the concepts of (*ij-pre*-continuous, *ij*-semi-continuous, *ij*-α-continuous, *ij-sp*-continuous, *ij-g*-continuous, *ij-αg*-continuous, *ij-gα*-continuous, *ij-gs*-continuous, *ij-sg*-continuous, *ij-gsp*-continuous, *ij-spg*-continuous, *ij-gp*-continuous, *ij*-gc-irresolute, *ij-gs*-irresolute, *ij-αg*-irresolute and *ij-gα*-irresolute) functions in bitopological spaces. In this paper, we introduce a new functions in a bitopological space (X, τ_1, τ_2), namely, ij-ψ^*-continuous functions and ij-ψ^*-irresolute functions.

2. Preliminaries

Definition 2.1. [23] A subset A of a bitopological space (X, τ_1, τ_2) is called:

(1) *ij*-preopen if $A \subseteq \tau_i\text{-int}(\tau_j\text{-cl}(A))$ and *ij*-preclosed if $\tau_i\text{-cl}(\tau_j\text{-int}(A)) \subseteq A$.
(2) *ij*-semi-open if $A \subseteq \tau_j\text{-cl}(\tau_i\text{-int}(A))$ and *ij*-semi-closed if $\tau_j\text{-int}(\tau_i\text{-cl}(A)) \subseteq A$.
(3) *ij*-α-open if $A \subseteq \tau_i\text{-int}(\tau_j\text{-cl}(\tau_i\text{-int}(A)))$ and *ij*-α-closed if $\tau_i\text{-cl}(\tau_j\text{-int}(\tau_i\text{-cl}(A))) \subseteq A$.
(4) *ij*-semi-preopen if $A \subseteq \tau_j\text{-cl}(\tau_i\text{-int}(\tau_j\text{-cl}(A)))$ and *ij*-semi preclosed if $\tau_j\text{-int}(\tau_i\text{-cl}(\tau_j\text{-int}(A))) \subseteq A$.

The class of all *ij*-preopen (resp. *ij*-semi-open, *ij*-α-open and *ij*-semi-preopen) sets in a bitopological space (X, τ_1, τ_2) is denoted by $ij\text{-}PO(X)$ (resp. $ij\text{-}SO(X)$, $ij\text{-}\alpha O(X)$ and $ij\text{-}SPO(X)$). The class of all *ij*-preclosed (resp. *ij*-semi-closed, *ij*-α-closed and *ij*-semi-preclosed) sets in a bitopological space (X, τ_1, τ_2) is denoted by $ij\text{-}PC(X)$ (resp. $ij\text{-}SC(X)$, $ij\text{-}\alpha C(X)$ and $ij\text{-}SPC(X)$).

Definition 2.2. [23] For a subset A of a bitopological space (X, τ_1, τ_2), the *ij*-pre-closure (resp. *ij*-semi-closure, *ij*-α-closure and *ij*-semi-pre-closure) of A are denoted and defined as follow:

(1) $ij\text{-}pcl(A) = \cap\{F \subset X: F \in ij\text{-}PC(X), F \supseteq A\}$.
(2) $ij\text{-}scl(A) = \cap\{F \subset X: F \in ij\text{-}SC(X), F \supseteq A\}$.
(3) $ij\text{-}\alpha cl(A) = \cap\{F \subset X: F \in ij\text{-}\alpha C(X), F \supseteq A\}$.
(4) $ij\text{-}spcl(A) = \cap\{F \subset X: F \in ij\text{-}SPC(X), F \supseteq A\}$.

Dually, the *ij*-preinterior (resp. *ij*-semi-interior, *ij*-α-interior and *ij*-semi-preinterior) of A, denoted by $ij\text{-}pint(A)$ (resp. $ij\text{-}sint(A)$, $ij\text{-}\alpha int(A)$ and $ij\text{-}spint(A)$) is the union of all *ij*-preopen (resp. *ij*-semi-open, *ij*-α-open and *ij*-semi-preopen) subsets of X contained in A.

Definition 2.3. [11] A subset A of a bitopological space (X, τ_1, τ_2) is called:

(1) *ij-g*-closed (denoted by $ij\text{-}GC(X)$) if, $A \subseteq U$, $U \in \tau_i \Rightarrow j\text{-cl}(A) \subseteq U$.
(2) *ij-gs*-closed (denoted by $ij\text{-}GSC(X)$) if, $A \subseteq U$, $U \in \tau_i \Rightarrow ji\text{-scl}(A) \subseteq U$.
(3) *ij-sg*-closed (denoted by $ij\text{-}SGC(X)$) if, $A \subseteq U$, $U \in ij\text{-}SO(X) \Rightarrow ji\text{-scl}(A) \subseteq U$.
(4) *ij-gα*-closed (denoted by $ij\text{-}G\alpha C(X)$) if, $A \subseteq U$, $U \in ij\text{-}\alpha O(X) \Rightarrow ji\text{-}\alpha\text{cl}(A) \subseteq U$.
(5) *ij-αg*-closed (denoted by $ij\text{-}\alpha GC(X)$) if, $A \subseteq U$, $U \in \tau_i \Rightarrow ji\text{-}\alpha\text{cl}(A) \subseteq U$.
(6) *ij-gp*-closed (denoted by $ij\text{-}GPC(X)$) if, $A \subseteq U$, $U \in \tau_i \Rightarrow ji\text{-pcl}(A) \subseteq U$.
(7) *ij-gsp*-closed (denoted by $ij\text{-}GSPC(X)$) if, $A \subseteq U$, $U \in \tau_i \Rightarrow ji\text{-spcl}(A) \subseteq U$.
(8) *ij-spg*-closed (denoted by $ij\text{-}SPGC(X)$) if, $A \subseteq U$, $U \in ji\text{-}SPO(X)) \Rightarrow ji\text{-spcl}(A) \subseteq U$.

The complement of an $ij\text{-}GC(X)$ (resp. $ij\text{-}GSC(X)$, $ij\text{-}SGC(X)$, $ij\text{-}G\alpha C(X)$, $ij\text{-}\alpha GC(X)$, $ij\text{-}GPC(X)$, $ij\text{-}GSPC(X)$, and $ij\text{-}SPGC(X)$) subset of (X, τ_1, τ_2) is called an $ij\text{-}GO(X)$ (resp. $ij\text{-}GSO(X)$, $ij\text{-}SGO(X)$, $ij\text{-}G\alpha O(X)$, $ij\text{-}\alpha GO(X)$, $ij\text{-}GPO(X)$, $ij\text{-}GSPO(X)$, and $ij\text{-}SPGO(X)$) subset of (X, τ_1, τ_2).

Definition 2.4. [11] A function f: $(X, \tau_1, \tau_2) \to (Y, \sigma_1, \sigma_2)$ is called:

(1) *ij*-pre-continuous if $\forall\ V \in i\text{-}C(Y)$, $f^{-1}(V) \in ij\text{-}PC(X)$.
(2) *ij*-semi-continuous if $\forall\ V \in i\text{-}C(Y)$, $f^{-1}(V) \in ij\text{-}SC(X)$.
(3) *ij*-α-continuous if $\forall\ V \in i\text{-}C(Y)$, $f^{-1}(V) \in ij\text{-}\alpha C(X)$.
(4) *ij-sp*-continuous if $\forall\ V \in i\text{-}C(Y)$, $f^{-1}(V) \in ij\text{-}SPC(X)$.
(5) *ij-g*-continuous if $\forall\ V \in j\text{-}C(Y)$, $f^{-1}(V) \in ij\text{-}GC(X)$.
(6) *ij-αg*-continuous if $\forall\ V \in j\text{-}C(Y)$, $f^{-1}(V) \in ij\text{-}\alpha GC(X)$.
(7) *ij-gα-continuous* if $\forall\ V \in j\text{-}C(Y)$, $f^{-1}(V) \in ij\text{-}G\alpha C(X)$.
(8) *ij-gs*-continuous if $\forall\ V \in j\text{-}C(Y)$, $f^{-1}(V) \in ij\text{-}GSC(X)$.
(9) *ij-sg*-continuous if $\forall\ V \in j\text{-}C(Y)$, $f^{-1}(V) \in ij\text{-}SGC(X)$.
(10) *ij-gsp*-continuous if $\forall\ V \in j\text{-}C(Y)$, $f^{-1}(V) \in ij\text{-}GSPC(X)$.
(11) *ij-spg*-continuous if $\forall\ V \in j\text{-}C(Y)$, $f^{-1}(V) \in ij\text{-}SPGC(X)$.
(12) *ij-gp*-continuous if $\forall\ V \in j\text{-}C(Y)$, $f^{-1}(V) \in ij\text{-}GPC(X)$.
(13) *i*-continuous if $\forall\ V \in i\text{-}C(Y)$, $f^{-1}(V) \in i\text{-}C(X)$.
(14) *ij-gc*-irresolute if $\forall\ V \in ij\text{-}GC(Y)$, $f^{-1}(V) \in ij\text{-}GC(X)$.
(15) *ij-gs*-irresolute if $\forall\ V \in ij\text{-}GSC(Y)$, $f^{-1}(V) \in ij\text{-}GSC(X)$.
(16) *ij-αg*-irresolute if $\forall\ V \in ij\text{-}\alpha GC(Y)$, $f^{-1}(V) \in ij\text{-}\alpha GC(X)$.
(17) *ij-gα*-irresolute if $\forall\ V \in ij\text{-}G\alpha C(Y)$, $f^{-1}(V) \in ij\text{-}G\alpha C(X)$.

Definition 2.5. [12] A subset A of (X, τ) is called ψ^*-closed if $A \subseteq U$, $U \in G\alpha O(X) \Rightarrow \alpha\text{cl}(A) \subseteq U$. The complement of ψ^*-closed set is said to be ψ^*-open.

Definition 2.6. [12] A space (X, τ) is called:

(1) $T_{1/5}$ space if $G\alpha C(X) = \alpha C(X)$.
(2) $T^{\psi^*}_{1/5}$ space if $\psi^* C(X) = \alpha C(X)$.
(3) ${}^{\psi^*}T_{1/5}$ space if $G\alpha C(X) = \psi^* C(X)$.
(4) T_e space if $GSC(X) = \alpha C(X)$.
(5) αT_e space if $\alpha GC(X) = \alpha C(X)$.
(6) T_k space if $GSC(X) = \psi^* C(X)$.
(7) αT_k space if $\alpha GC(X) = \psi^* C(X)$.
(8) T_l space if $GSC(X) = G\alpha C(X)$.
(9) αT_l space if $\alpha GC(X) = G\alpha C(X)$.

Definition 2.7. [12] A function f: $(X, \tau) \to (Y, \sigma)$ is called:

(1) ψ^*-continuous if $\forall\ V \in C(Y)$, $f^{-1}(V) \in \psi^* C(X)$.
(2) ψ^*-irresolute if $\forall\ V \in \psi^* C(Y)$, $f^{-1}(V) \in \psi^* C(X)$.
(3) pre-ψ^*-closed if $A \in \psi^* C(X)$, $f(A) \in \psi^* C(Y)$.

3. Basic properties of *ij*-ψ^*-closed sets

We introduce the following definition.

Definition 3.1. A subset A of a bitopological space (X, τ_1, τ_2) is called *ij*-ψ^*-closed set if, $A \subseteq U$, $U \in$ *ji*-$G\alpha O(X) \Rightarrow$ *ji*-αcl$(A) \subseteq U$.

The class of *ij*-ψ^*-closed subsets of (X, τ_1, τ_2) is denoted by *ij*-$\psi^* C(X)$.

The following diagram shows the relationships of *ij*-ψ^*-closed sets with some other sets discussed in this section (see Diagram 1).

Definition 3.1 is a particular case of Definition 8 from Noiri [24].

Theorem 3.1. *Every ji-α-closed set is an ij-ψ^*-closed set.*

The following example supports that an *ij*-ψ^*-closed set need not be a *ji*-α-closed set in general.

Example 3.1. Let $X = \{a, b, c, d\}$, $\tau_1 = \{X, \phi, \{a\}, \{a, d\}\}$ and $\tau_2 = \{X, \phi, \{a, b\}, \{c, d\}\}$. Then we have $A = \{b, c\} \in$ *ij*-$\psi^* C(X)$ but $A \notin$ *ji*-$\alpha C(X)$.

Therefore the class of *ij*-ψ^*-closed sets is properly contains the class of *ji*-α-closed sets. Next we show that the class of *ij*-ψ^*-closed sets is properly contained in the class of *ij*-gα-closed set.

Theorem 3.2. *Every ij-ψ^*-closed set is an ij-gα-closed set.*

The following example supports that the converse of the above theorem is not true in general.

Example 3.2. Let X, τ_1, and τ_2 are as in the Example 3.1. Then the subset $B = \{b\} \in$ *ij*-$G\alpha C(X)$ but $B \notin$ *ij*-$\psi^* C(X)$.

Remark 3.1. The intersection of two sets in *ij*-ψ^*-closed set is not in general a set in *ij*-ψ^*-closed set, as shown by the following example.

Example 3.3. Let X, τ_1, and τ_2 be as in the Example 3.1. Then we have $\{a, b\}$ and $\{b, c\} \in$ *ij*-$\psi^* C(X)$ but $\{a, b\} \cap \{b, c\} = \{b\} \notin$ *ij*-$\psi^* C(X)$.

Theorem 3.3. *For any bitopological space (X, τ_1, τ_2).*

(1) *ij*-$\psi^* C(X) \cap$ *ji*-$G\alpha O(X) \subseteq$ *ji*-$\alpha C(X)$.
(2) *If $A \in$ ij-$\psi^* C(X)$ and $A \subseteq B \subseteq$ ji-αcl(A), then $B \in$ ij-$\psi^* C(X)$.*

Proof.

(1) Let $A \in$ *ij*-$\psi^* C(X) \cap$ *ji*-$G\alpha O(X)$. Then we have *ji*-αcl$(A) \subseteq A$. Consequently, $A \in$ *ji*-$\alpha C(X)$.
(2) Let $U \in$ *ji*-$G\alpha O(X)$ such that $B \subseteq U$. Since $A \subseteq B$ and $A \in$ *ij*-$\psi^* C(X)$, then *ji*-αcl$(A) \subseteq U$. Since $B \subseteq$ *ji*-αcl(A), then we have *ji*-αcl$(B) \subseteq$ *ji*-αcl$(A) \subseteq U$. Therefore, $B \in$ *ij*-$\psi^* C(X)$. □

Theorem 3.4. *Let (X, τ_1, τ_2) be a bitopological space, $A \in$ ij-$G\alpha C(X)$. Then $A \in$ ij-$\psi^* C(X)$ if ij-$\alpha O(X)$ = ji-$G\alpha O(X)$.*

Proof. Let $A \in$ *ij*-$G\alpha C(X)$ i.e. $A \subseteq U$ and $U \in$ *ij*-$\alpha O(X)$, then *ji*-αcl$(A) \subseteq U$. Since *ij*-$\alpha O(X)$ = *ji*-$G\alpha O(X)$. Consequently, $A \subseteq U$ and $U \in$ *ji*-$G\alpha O(X)$, then *ji*-αcl$(A) \subseteq U$ i.e. $A \in$ *ij*-$\psi^* C(X)$. □

Theorem 3.5. Let (X_1, τ_1, τ_2) and $(X_2, \tau_1^*, \tau_2^*)$ be two bitopological spaces. Then the following statement is true. If $A \in$ *ij*-$\psi^* O(X_1)$ and $B \in$ *ij*-$\psi^* O(X_2)$, then $A \times B \in$ *ij*-$\psi^* O(X_1 \times X_2)$.

Proof. Let $A \in$ *ij*-$\psi^* O(X_1)$ and B $\in$ *ij*-$\psi^* O(X_2)$ and $W = A \times B \subseteq X_1 \times X_2$. Let $F = F_1 \times F_2 \subseteq W$, $F \in$ *ji*-$G\alpha C(X_1 \times X_2)$. Then there are $F_1 \in$ *ji*-$G\alpha C(X_1)$, $F_2 \in$ *ji*-$G\alpha C(X_2)$, $F_1 \subseteq A$, $F_2 \subseteq B$ and so, $F_1 \subseteq \tau_{ji} - \alpha\text{int}(A)$ and $F_2 \subseteq \tau^*_{ji} - \alpha\text{int}(B)$. Hence $F_1 \times F_2 \subseteq A \times B$ and $F_1 \times F_2 \subseteq \tau_{ji} - \alpha\text{int}(A) \times \tau^*_{ji} - \alpha\text{int}(B) = \tau_{ji} \times \tau^*_{ji} - \alpha\text{int}(A \times B)$.

Therefore $A \times B \in$ *ij*-$\psi^* O(X_1 \times X_2)$. □

Theorem 3.6. *A subset A of X is ij-$\psi^* O(X)$ if and only if F is a subset of ij-αint(A) whenever $F \subseteq A$ and $F \in$ ji-$G\alpha C(X)$.*

Theorem 3.7. *For each $x \in X$, either $\{x\}$ is ji-$G\alpha C(X)$ or $\{x\}$ is ij-$\psi^* O(X)$.*

Theorem 3.8. *A subset A of X is ij-$\psi^* C(X)$ if and only if ji-$\alpha C(A) \cap F = \emptyset$, whenever $A \cap F = \emptyset$, where F is ji-$G\alpha C(X)$.*

4. Applications of *ij*-ψ^*-closed sets

As applications of *ij*-ψ^*-closed sets, four new classes of spaces, namely, $ij - T^{\psi^*}_{1/5}$ spaces, $ij - {}^{\psi^*}T_{1/5}$ spaces, *ij*-T_k spaces and *ij*-αT_k spaces are introduced.

We introduce the following definitions.

Definition 4.1. A bitopological space (X, τ_1, τ_2) is called an $ij - T_{1/5}$ space if *ij*-$G\alpha C(X)$ = *ji*-$\alpha C(X)$.

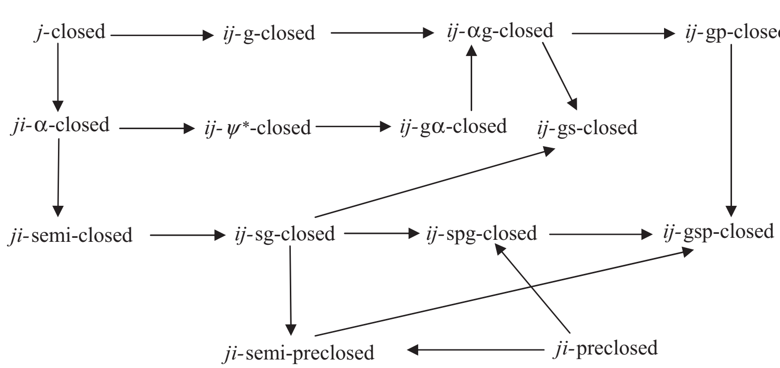

Diagram 1

Definition 4.2. A bitopological space (X, τ_1, τ_2) is called an $ij-T^{\psi^*}_{1/5}$ space if $ij\text{-}\psi^*C(X) = ji\text{-}\alpha C(X)$.

We prove that the class of $ij-T^{\psi^*}_{1/5}$ spaces properly contains the class of $ij-T_{1/5}$ spaces.

Theorem 4.1. *Every $ij-T_{1/5}$ space is an $ij-T^{\psi^*}_{1/5}$ space.*

Proof. Follows from the fact that every $ij\text{-}\psi^*$-closed set is an ij-gα-closed set. □

The converse of the above theorem is not true as it can be seen from the following example.

Example 4.1. Let $X = \{a, b, c\}$, $\tau_1 = \{X, \phi, \{a\}\}$ and $\tau_2 = \{X, \phi, \{b\}\}$. Then (X, τ_1, τ_2) is an $ij-T^{\psi^*}_{1/5}$ space but not an $ij-T_{1/5}$ space since $\{b, c\} \in ij\text{-}G\alpha C(X)$ but $\{b, c\} \notin ji\text{-}\alpha C(X)$.

We introduce the following definition.

Definition 4.3. A bitopological space (X, τ_1, τ_2) is called an $ij-{}^{\psi^*}T_{1/5}$ space if $ij\text{-}G\alpha C(X) = ij\text{-}\psi^*C(X)$.

Theorem 4.2. *Every $ij-T_{1/5}$ space is an $ij-{}^{\psi^*}T_{1/5}$ space.*

Proof. Let (X, τ_1, τ_2) be an $ij-T_{1/5}$ space. Let $A \in ij\text{-}G\alpha C(X)$. Since (X, τ_1, τ_2) is an $ij-T_{1/5}$ space, then $A \in ji\text{-}\alpha C(X)$. Hence, by using Theorem 3.1, we have $A \in ij\text{-}\psi^*C(X)$. Therefore (X, τ_1, τ_2) is an $ij-{}^{\psi^*}T_{1/5}$ space. □

The converse of the above theorem is not true as we see in the following example.

Example 4.2. Let $X = \{a, b, c\}$, $\tau_1 = \{X, \phi, \{a\}\}$ and $\tau_2 = \{X, \phi, \{a\}, \{b, c\}\}$. Then (X, τ_1, τ_2) is an $ij-{}^{\psi^*}T_{1/5}$ space but not an $ij-T_{1/5}$ space since $\{a, b\} \in ij\text{-}G\alpha C(X)$ but $\{a, b\} \notin ji\text{-}\alpha C(X)$.

We show that $ij-T^{\psi^*}_{1/5}$ ness is independent from $ij-{}^{\psi^*}T_{1/5}$ ness.

Remark 4.1. $ij-T^{\psi^*}_{1/5}$ ness and $ij-{}^{\psi^*}T_{1/5}$ ness are independent as it can be seen from the next two examples.

Example 4.3. Let X, τ_1, and τ_2 be as in the Example 4.1. Then (X, τ_1, τ_2) is an $ij-T^{\psi^*}_{1/5}$ space but not an $ij-{}^{\psi^*}T_{1/5}$ space since $\{b, c\} \in ij\text{-}G\alpha C(X)$ but $\{b, c\} \notin ij\text{-}\psi^*C(X)$.

Example 4.4. Let X, τ_1, and τ_2 be as in the Example 4.2. Then (X, τ_1, τ_2) is an $ij-{}^{\psi^*}T_{1/5}$ space but not an $ij-T^{\psi^*}_{1/5}$ space since $\{a, c\} \in ij\text{-}\psi^*C(X)$ but $\{a, c\} \notin ji\text{-}\alpha C(X)$.

Theorem 4.3. *If (X, τ_1, τ_2) is an $ij-{}^{\psi^*}T_{1/5}$ space, then for each $x \in X$, $\{x\}$ is either ji-α-closed or ij-ψ^*-open.*

Proof. Suppose that (X, τ_1, τ_2) is an $ij-{}^{\psi^*}T_{1/5}$ space. Let $x \in X$ and assume that $\{x\} \notin ji\text{-}\alpha C(X)$. Then $\{x\} \notin ij\text{-}G\alpha C(X)$ since every ji-α-closed set is an ij-gα-closed set. So $X\text{-}\{x\} \notin ji\text{-}\alpha O(X)$. Therefore $X\text{-}\{x\} \in ij\text{-}G\alpha C(X)$ since X is the only ji-α-open set which contains $X\text{-}\{x\}$. Since (X, τ_1, τ_2) is an $ij-{}^{\psi^*}T_{1/5}$ space, then $X\text{-}\{x\} \in ij\text{-}\psi^*C(X)$ or equivalently $\{x\} \in ij\text{-}\psi^*O(X)$. □

Theorem 4.4. *A space (X, τ_1, τ_2) is an $ij-T_{1/5}$ space if and only if it is $ij-{}^{\psi^*}T_{1/5}$ and $ij-T^{\psi^*}_{1/5}$ space.*

Proof. The necessity follows from the Theorems 4.1 and 4.2. For the sufficiency, suppose that (X, τ_1, τ_2) is both $ij-{}^{\psi^*}T_{1/5}$ and $ij-T^{\psi^*}_{1/5}$ space. Let $A \in ij\text{-}G\alpha C(X)$. Since (X, τ_1, τ_2) is an $ij-{}^{\psi^*}T_{1/5}$ space, then $A \in ij\text{-}\psi^*C(X)$. Since (X, τ_1, τ_2) is an $ij-T^{\psi^*}_{1/5}$ space, then $A \in ji\text{-}\alpha C(X)$. Thus (X, τ_1, τ_2) is an $ij-T_{1/5}$ space. □

We introduce the following definitions ij-T_e spaces and ij-αT_e spaces respectively and show that every ij-T_e (ij-αT_e) space is an $ij-T_{1/5}$ space.

Definition 4.4. A space (X, τ_1, τ_2) is called an ij-T_e space if $ij\text{-}GSC(X) = ji\text{-}\alpha C(X)$.

Definition 4.5. A space (X, τ_1, τ_2) is called an ij-αT_e space if $ij\text{-}\alpha GC(X) = ji\text{-}\alpha C(X)$.

Theorem 4.5. *Every ij-T_e space is an $ij-T_{1/5}$ space.*

Proof. Follows from the fact that every ij-gα-closed set is an ij-gs-closed set. □

An $ij-T_{1/5}$ space need not be an ij-T_e space as we see the next example.

Example 4.5. Let $X = \{a, b, c\}$, $\tau_1 = \{X, \phi, \{a\}, \{b\}, \{a, b\}, \{a, c\}\}$ and $\tau_2 = \{X, \phi, \{a\}, \{a, b\}\}$. Then (X, τ_1, τ_2) is an $ij-T_{1/5}$ space but not an ij-T_e space since $\{b\} \in ij\text{-}GSC(X)$ but $\{b\} \notin ji\text{-}\alpha C(X)$.

Theorem 4.6. *Every ij-αT_e space is an $ij-T_{1/5}$ space.*

Proof. Follows from the fact that every ij-gα-closed set is an ij-αg-closed set. □

An $ij-T_{1/5}$ space need not be an ij-αT_e space as we see the next example.

Example 4.6. Let $X = \{a, b, c\}$, $\tau_1 = \{X, \phi, \{a\}, \{b\}, \{a, b\}\}$ and $\tau_2 = \{X, \phi, \{a\}, a, c\}\}$. Then (X, τ_1, τ_2) is an $ij-T_{1/5}$ space but not an ij-αT_e space since $\{a, c\} \in ij\text{-}\alpha GC(X)$ but $\{a, c\} \notin ji\text{-}\alpha C(X)$.

Theorem 4.7. *Every ij-T_e space is an ij-αT_e space.*

Proof. Follows from the fact that every ij-αg-closed set is an ij-gs-closed set. □

The converse of the above theorem is not true in general as the following example supports.

Example 4.7. Let X, τ_1, and τ_2 be as in the Example 4.5. Then (X, τ_1, τ_2) is an ij-αT_e space but not an ij-T_e space since $\{b\} \in ij\text{-}GSC(X)$ but $\{b\} \notin ji\text{-}\alpha C(X)$.

Theorem 4.8. *Every ij-T_e space is an $ij-T^{\psi^*}_{1/5}$ space.*

Proof. Follows from the fact that every $ij\text{-}\psi^*$-closed set is an ij-gs-closed set. □

The converse of the above theorem is not true in general as the following example supports.

Example 4.8. Let $X = \{a, b, c, d, e\}$, $\tau_1 = \{X, \phi, \{a\}, \{c, d\}, \{a, c, d\}, \{b, c, d, e\}\}$ and $\tau_2 = \{X, \phi, \{a\}, \{a, b\}, \{a, b, e\}, \{a, c, d\}, \{a, b, c, d\}\}$. Then (X, τ_1, τ_2) is an $ij - T^{\psi^*}_{1/5}$ space but not an ij-T_e space since $\{d\} \in ij$-$GSC(X)$ but $\{d\} \notin ji$-$\alpha C(X)$.

Theorem 4.9. *Every ij-αT_e space is an $ij - T^{\psi^*}_{1/5}$ space.*

Proof. Follows from the fact that every ij-ψ^*-closed set is an ij-αg-closed set. □

An $ij - T^{\psi^*}_{1/5}$ space need not be an ij-αT_e space as we see the next example.

Example 4.9. Let X, τ_1, and τ_2 be as in the Example 4.8. Then (X, τ_1, τ_2) is an $ij - T^{\psi^*}_{1/5}$ space but not an ij-αT_e space $\{c\} \in ij$-$\alpha GC(X)$ but $\{c\} \notin ji$-$\alpha C(X)$.

We introduce the following definitions.

Definition 4.6. A space (X, τ_1, τ_2) is called an ij-T_k space if ij-$GSC(X) = ij$-$\psi^*C(X)$.

Definition 4.7. A space (X, τ_1, τ_2) is called an ij-αT_k space if ij-$\alpha GC(X) = ij$-$\psi^*C(X)$

Definition 4.8. A space (X, τ_1, τ_2) is called an ij-T_l space if ij-$GSC(X) = ij$-$G\alpha C(X)$.

Definition 4.9. A space (X, τ_1, τ_2) is called an ij-αT_l space if ij-$\alpha GC(X) = ij$-$G\alpha C(X)$.

We show that the class of ij-αT_k spaces properly contains the class of ij-αT_e spaces and is properly contained in the class of ij-αT_l spaces. We also show that the class of ij-αT_k spaces is the dual of the class of $ij - T^{\psi^*}_{1/5}$ spaces to the class of ij-αT_e spaces. Moreover we prove that ij-αT_k ness and $ij - T^{\psi^*}_{1/5}$ ness are independent from each other.

Theorem 4.10. *Every ij-αT_e space is an ij-αT_k space.*

Proof. Let (X, τ_1, τ_2) be an ij-αT_e space. Let $A \in ij$-$\alpha GC(X)$. Since (X, τ_1, τ_2) is an ij-αT_e space, then $A \in ji$-$\alpha C(X)$. Hence, by using Theorem 3.1, we have $A \in ij$-$\psi^*C(X)$. Therefore (X, τ_1, τ_2) is an ij-αT_k space. □

The following example supports that the converse of the above theorem is not true in general.

Example 4.10. Let X, τ_1, and τ_2 be as in the Example 4.2. Then (X, τ_1, τ_2) is an ij-αT_k space but not an ij-αT_e space since $\{a, c\} \in ij$-$\alpha GC(X)$ but $\{a, c\} \notin ji$-$\alpha C(X)$.

Theorem 4.11. *Every ij-αT_k space is an ij-αT_l space.*

Proof. Let (X, τ_1, τ_2) be an ij-αT_k space. Let $A \in ij$-$\alpha GC(X)$. Since (X, τ_1, τ_2) is an ij-αT_k space, then $A \in ij$-$\psi^*C(X)$. Hence, by using Theorem 3.2, we have $A \in ij$-$G\alpha C(X)$. Therefore (X, τ_1, τ_2) is an ij-αT_l space. □

The following example supports that the converse of the above theorem is not true in general.

Example 4.11. Let X, τ_1, and τ_2 be as in the Example 4.1. Then (X, τ_1, τ_2) is an ij-αT_l space but not an ij-αT_k space since $\{b\} \in ij$-$\alpha GC(X)$ but $\{b\} \notin ij$-$\psi^*C(X)$.

Theorem 4.12. *A space (X, τ_1, τ_2) is an ij-αT_e space if and only if it is ij-αT_k and $ij - T^{\psi^*}_{1/5}$ space.*

Proof. The necessity follows from the Theorems 4.9 and 4.10. For the sufficiency, suppose that (X, τ_1, τ_2) is both ij-αT_k and $ij - T^{\psi^*}_{1/5}$ space. Let $A \in ij$-$\alpha GC(X)$. Since (X, τ_1, τ_2) is an ij-αT_k space, then $A \in ij$-$\psi^*C(X)$. Since (X, τ_1, τ_2) is an $ij - T^{\psi^*}_{1/5}$ space, then $A \in ji$-$\alpha C(X)$. Thus (X, τ_1, τ_2) is an ij-αT_e space. □

Remark 4.2. ij-αT_k ness and $ij - T^{\psi^*}_{1/5}$ ness are independent as it can be seen from the next two examples.

Example 4.12. Let X, τ_1, and τ_2 be as in the Example 4.2. Then (X, τ_1, τ_2) is an ij-αT_k space but not an $ij - T^{\psi^*}_{1/5}$ space since $\{a, b\} \in ij$-$\psi^*C(X)$ but $\{a, b\} \notin ji$-$\alpha C(X)$.

Example 4.13. Let X, τ_1, and τ_2 be as in the Example 4.1. Then (X, τ_1, τ_2) is an $ij - T^{\psi^*}_{1/5}$ space but not an ij-αT_k space since $\{b, c\} \in ij$-$\alpha GC(X)$ but $\{b, c\} \notin ij$-$\psi^*C(X)$.

Definition 4.10. A subset A of a bitopological space (X, τ_1, τ_2) is called an ij-ψ^*-open if its complement is an ij-ψ^*-closed of (X, τ_1, τ_2).

Theorem 4.13. *If (X, τ_1, τ_2) is an ij-αT_k space, then for each $x \in X$, $\{x\}$ is either ij-αg-closed or ij-ψ^*-open.*

Proof. Suppose that (X, τ_1, τ_2) is an ij-αT_k space. Let $x \in X$ and assume that $\{x\} \notin ij$-$\alpha GC(X)$. Then $\{x\} \notin ji$-$\alpha C(X)$ since every ji-α-closed set is an ij-αg-closed set. So X-$\{x\} \notin ji$-$\alpha O(X)$. Therefore X-$\{x\} \in ij$-$\alpha GC(X)$ since X is the only ji-α-open set which contains X-$\{x\}$. Since (X, τ_1, τ_2) is an ij-αT_k space, then X-$\{x\} \in ij$-$\psi^*C(X)$ or equivalently $\{x\} \in ij$-$\psi^*O(X)$. □

Theorem 4.14. *Every ij-αT_k space is an $ij - {}^{\psi^*}T_{1/5}$ space.*

Proof. Let (X, τ_1, τ_2) be an ij-αT_k space. Let $A \in ij$-$G\alpha C(X)$, then $A \in ij$-$\alpha GC(X)$. Since (X, τ_1, τ_2) is an ij-αT_k space, then $A \in ij$-$\psi^*C(X)$. Therefore (X, τ_1, τ_2) is an $ij - {}^{\psi^*}T_{1/5}$ space. □

The following example supports that the converse of the above theorem is not true in general.

Example 4.14. Let X, τ_1, and τ_2 be as in the Example 4.8. Then (X, τ_1, τ_2) is an $ij - {}^{\psi^*}T_{1/5}$ space but not an ij-αT_k space since $\{c\} \in ij$-$\alpha GC(X)$ but $\{c\} \notin ij$-$\psi^*C(X)$.

We show that the class of ij-T_k spaces properly contains the class of ij-T_e spaces, and is properly contained in the class of ij-αT_k spaces, the class of ij-T_l spaces, and the class of ij-αT_l spaces.

Theorem 4.15. *Every ij-T_e space is an ij-T_k space.*

Proof. Let (X, τ_1, τ_2) be an ij-T_e space. Let $A \in ij\text{-}GSC(X)$. Since (X, τ_1, τ_2) is an ij-T_e space, then $A \in ji\text{-}\alpha C(X)$. Hence, by using Theorem 3.1, we have $A \in ij\text{-}\psi^* C(X)$. Therefore (X, τ_1, τ_2) is an ij-T_k space. □

The following example supports that the converse of the above theorem is not true in general.

Example 4.15. Let X, τ_1, and τ_2 be as in the Example 4.2. Then (X, τ_1, τ_2) is an ij-T_k space but not an ij-T_e space since {a, c} $\in ij\text{-}GSC(X)$ but {a, c} $\notin ji\text{-}\alpha C(X)$.

Theorem 4.16. *Every ij-T_k space is an ij-αT_k space.*

Proof. Let (X, τ_1, τ_2) be an ij-T_k space. Let $A \in ij\text{-}\alpha GC(X)$, then $A \in ij\text{-}GSC(X)$. Since (X, τ_1, τ_2) is an ij-T_k space, then $A \in ij\text{-}\psi^* C(X)$. Therefore (X, τ_1, τ_2) is an ij-αT_k space. □

The converse of the above theorem is not true as it can be seen from the following example.

Example 4.16. Let X, τ_1, and τ_2 be as in the Example 4.5. Then (X, τ_1, τ_2) is an ij-αT_k space but not an ij-T_k space since {b} $\in ij\text{-}GSC(X)$ but {b} $\notin ij\text{-}\psi^* C(X)$.

Theorem 4.17. *Every ij-T_k space is an ij-T_l space.*

Proof. Let (X, τ_1, τ_2) be an ij-T_k space. Let $A \in ij\text{-}GSC(X)$. Since (X, τ_1, τ_2) is an ij-T_k space, then $A \in ij\text{-}\psi^* C(X)$. Hence, by using Theorem 3.2, we have $A \in ij\text{-}G\alpha C(X)$. Therefore (X, τ_1, τ_2) is an ij-T_l space. □

The converse of the above theorem is not true as it can be seen from the following example.

Example 4.17. Let X = {a, b, c}, $\tau_1 = \{X, \phi, \{a, b\}\}$ and $\tau_2 = \{X, \phi, \{a, c\}\}$. Then (X, τ_1, τ_2) is an ij-T_l space but not an ij-T_k space since {c} $\in ij\text{-}GSC(X)$ but {c} $\notin ij\text{-}\psi^* C(X)$.

Next we prove that the dual of the class of ij-T_l spaces to the class of ij-T_k spaces is the class of ij-αT_k spaces.

Theorem 4.18. *A space (X, τ_1, τ_2) is an ij-T_k space if and only if it is ij-αT_k and ij-T_l space.*

Proof. The necessity follows from the Theorems 4.16 and 4.17. For the sufficiency, suppose that (X, τ_1, τ_2) is both ij-αT_k and ij-T_l space. Let $A \in ij\text{-}GSC(X)$. Since (X, τ_1, τ_2) is an ij-T_l space, then $A \in ij\text{-}G\alpha C(X)$. Then $A \in ij\text{-}\alpha GC(X)$. Since (X, τ_1, τ_2) is an ij-αT_k space, then $A \in ij\text{-}\psi^* C(X)$. Therefore (X, τ_1, τ_2) is an ij-T_k space. □

Theorem 4.19. *A space (X, τ_1, τ_2) is an ij-T_e space if and only if it is ij-T_k and ij $- T^{\psi^*}_{1/5}$ space.*

Proof. The necessity follows from the Theorems 4.8 and 4.15. For the sufficiency, suppose that (X, τ_1, τ_2) is both ij-T_k and $ij - T^{\psi^*}_{1/5}$ space. Let $A \in ij\text{-}GSC(X)$. Since (X, τ_1, τ_2) is an ij-T_k space, then $A \in ij\text{-}\psi^* C(X)$. Since (X, τ_1, τ_2) is an $ij - T^{\psi^*}_{1/5}$ space, then $A \in ji\text{-}\alpha C(X)$. Therefore (X, τ_1, τ_2) is an ij-T_e space. □

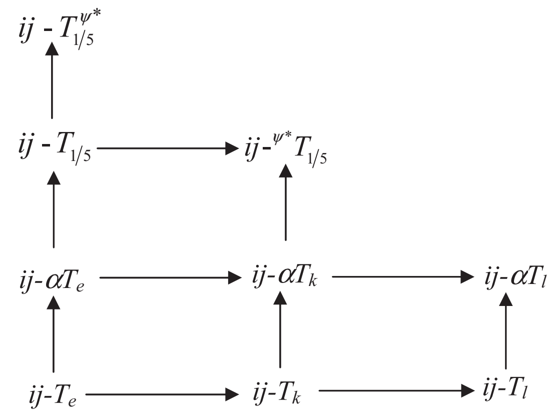

Diagram 2

The following diagram shows the relationships between the separation axioms discussed in this section (see Diagram 2).

5. ij-ψ^*-continuous and ij-ψ^*-irresolute functions

We introduce the following definition.

Definition 5.1. A function f: $(X, \tau_1, \tau_2) \to (Y, \sigma_1, \sigma_2)$ is called ij-ψ^*-continuous if $\forall\ V \in j\text{-}C(Y)$, $f^{-1}(V) \in ij\text{-}\psi^* C(X)$.

The following diagram shows the relationships of ij-ψ^*-continuous functions with some other functions discussed in this section (see Diagram 3).

Theorem 5.1. *Every ji-α-continuous function is ij-ψ^*-continuous.*

The following example supports that the converse of the above theorem is not true in general.

Example 5.1. Let X = {a, b, c, d}, Y = {u, v, w}, $\tau_1 = \{X, \phi, \{a\},\{a, d\}\}$, $\tau_2 = \{X, \phi, \{a, b\},\{c, d\}\}$, $\sigma_1 = \{Y, \phi, \{u\}, \{v\}, \{u, v\}, \{u, w\}\}$ and $\sigma_2 = \{Y, \phi, \{u\}, \{u, v\}\}$. Define f: $(X, \tau_1, \tau_2) \to (Y, \sigma_1, \sigma_2)$ by $f(a)$ = u, $f(b)$ = v and $f(c) = f(d)$ = w. f is not ji-α-continuous function since {v, w} $\in j\text{-}C(Y)$ but $f^{-1}(\{v, w\}) = \{b, c, d\} \notin ji\text{-}\alpha C(X)$. However f is ij-ψ^*-continuous function.

Theorem 5.2. *Every ij-ψ^*-continuous function is ij-gα-continuous.*

The following example supports that the converse of the above theorem is not true in general.

Example 5.2. Let X, Y, τ_1, τ_2, σ_1 and σ_2 be as in the example 5.1. Define f: $(X, \tau_1, \tau_2) \to (Y, \sigma_1, \sigma_2)$ by f (a) = u, f (b) = w and f (c) = f (d) = v. f is not ij-ψ^*-continuous function since {w} $\in j\text{-}C(Y)$ but $f^{-1}(\{w\}) = \{b\} \notin ij\text{-}\psi^* C(X)$. However f is ij-gα-continuous function.

Theorem 5.3. *If f_1: $(X_1, \tau_1, \tau_2) \to (Y_1, \sigma_1, \sigma_2)$ and f_2: $(X_2, \tau_1^*, \tau_2^*) \to (Y_2, \sigma_1^*, \sigma_2^*)$ be two ij-ψ^*-continuous functions. Then the function f: $(X_1 \times X_2,\ \tau_1 \times \tau_1^*,\ \tau_2 \times \tau_2^*) \to (Y_1 \times Y_2,\ \sigma_1 \times \sigma_1^*,\ \sigma_2 \times \sigma_2^*)$ defined by $f(x_1, x_2) = (f(x_1), f(x_2))$ is ij-ψ^*-continuous.*

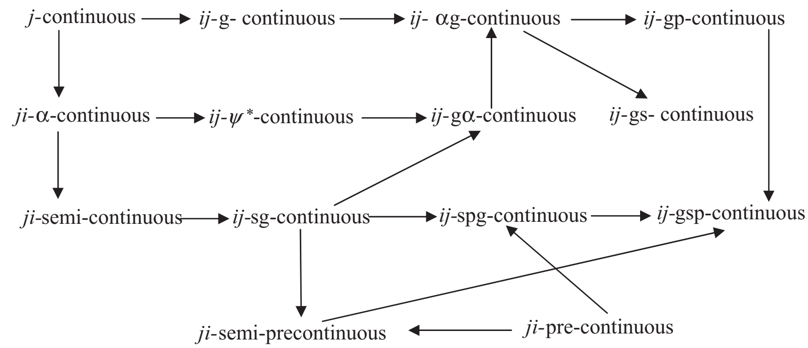

Diagram 3

Proof. Let $V_1 \in j\text{-}O(Y_1)$ and $V_2 \in j\text{-}O(Y_2)$. Since f_1 and f_2 are two *ij*-ψ^*-continuous, then $f^{-1}(V_1) \in ij - \psi^* O(X_1)$ and $f^{-1}(V_2) \in ij - \psi^* O(X_2)$. Hence, by using Theorem 3.5, we have $f^{-1}(V_1) \times f^{-1}(V_2) \in ij - \psi^* O(X_1 \times X_2)$. □

We introduce the following definition.

Definition 5.2. A function f: $(X, \tau_1, \tau_2) \rightarrow (Y, \sigma_1, \sigma_2)$ is called *ij*-ψ^*-irresolute if $\forall\ V \in ij\text{-}\psi^* C(Y)$, $f^{-1}(V) \in ij\text{-}\psi^* C(X)$.

Theorem 5.4. *Every ij-ψ^*-irresolute function is ij-ψ^*-continuous.*

The following example supports that the converse of the above theorem is not true in general.

Example 5.3. Let $X = \{a, b, c, d\}$, $Y = \{u, v, w\}$, $\tau_1 = \{X, \phi, \{a\}, \{a, d\}\}$, $\tau_2 = \{X, \phi, \{a, b\}, \{c, d\}\}$, $\sigma_1 = \{Y, \phi, \{u\}\}$ and $\sigma_2 = \{Y, \phi, \{u\}, \{v, w\}\}$. Define f: $(X, \tau_1, \tau_2) \rightarrow (Y, \sigma_1, \sigma_2)$ by $f(a) = v$, $f(b) = w$ and $f(c) = f(d) = u$. f is not *ij*-ψ^*-irresolute function since $\{u, v\} \in ij\text{-}\psi^* C(Y)$ but $f^{-1}(\{u, v\}) = \{a, c, d\} \notin ij - \psi^* C(X)$. However f is *ij*-ψ^*-continuous function.

Theorem 5.5. *Let f: $(X, \tau_1, \tau_2) \rightarrow (Y, \sigma_1, \sigma_2)$ and g: $(Y, \sigma_1, \sigma_2) \rightarrow (Z, \eta_1, \eta_2)$ be any two functions. Then*

(1) *g o f is ij-ψ^*-continuous if g is j-continuous and f is ij-ψ^*-continuous.*
(2) *g o f is ij-ψ^*-irresolute if both f and g are ij-ψ^*-irresolute.*
(3) *g o f is ij-ψ^*-continuous if g is ij-ψ^*-continuous and f is ij-ψ^*-irresolute.*

Proof. Let $V \in j\text{-}C(Z)$, since g is *j*-continuous, then $g^{-1}(V) \in j\text{-}C(Y)$. Since f is *ij*-ψ^*-continuous, then we have $f^{-1}(g^{-1}(V)) \in ij\text{-}\psi^* C(X)$. Consequently, g o f is *ij*-ψ^*-continuous.

(2)–(3) Similarly. □

Theorem 5.6. *Let f: $(X, \tau_1, \tau_2) \rightarrow (Y, \sigma_1, \sigma_2)$ be an ij-ψ^*-continuous function. If (X, τ_1, τ_2) is $ij - T^{\psi^*}_{1/5}$ space, then f is ji-α-continuous function.*

Proof. Let $V \in j\text{-}C(Y)$. Since f is *ij*-ψ^*-continuous, then $f^{-1}(V) \in ij\text{-}\psi^* C(X)$. Since (X, τ_1, τ_2) is an $ij - T^{\psi^*}_{1/5}$ space, then $f^{-1}(V) \in ji\text{-}\alpha C(X)$. Consequently, f is *ji*-α-continuous. □

Theorem 5.7. *Let f: $(X, \tau_1, \tau_2) \rightarrow (Y, \sigma_1, \sigma_2)$ be an ij-αg-continuous function. If (X, τ_1, τ_2) is an ij-αT_k space, then f is ij-ψ^*-continuous.*

Proof. Let $V \in j\text{-}C(Y)$. Since f is an *ij*-αg-continuous function, thus $f^{-1}(V) \in ij\text{-}\alpha GC(X)$. Since (X, τ_1, τ_2) is an *ij*-αT_k space, then $f^{-1}(V) \in ij\text{-}\psi^* C(X)$. Consequently, f is *ij*-ψ^*-continuous. □

Theorem 5.8. *Let f: $(X, \tau_1, \tau_2) \rightarrow (Y, \sigma_1, \sigma_2)$ be an ij-gα-continuous function. If (X, τ_1, τ_2) is $ij - {}^{\psi^*}T_{1/5}$ space, then f is ij-ψ^*-continuous.*

Proof. Let $V \in j\text{-}C(Y)$. Since f is an *ij*-gα-continuous function, thus $f^{-1}(V) \in ij\text{-}G\alpha C(X)$. Since (X, τ_1, τ_2) is an $ij - {}^{\psi^*}T_{1/5}$ space, then $f^{-1}(V) \in ij\text{-}\psi^* C(X)$. Consequently, f is *ij*-ψ^*-continuous. □

Theorem 5.9. *Let f: $(X, \tau_1, \tau_2) \rightarrow (Y, \sigma_1, \sigma_2)$ be an ij-gs-continuous function. If (X, τ_1, τ_2) is ij-T_k space, then f is ij-ψ^*-continuous.*

Proof. Let $V \in j\text{-}C(Y)$. Since f is an *ij*-gs-continuous function, thus $f^{-1}(V) \in ij\text{-}GSC(X)$. Since (X, τ_1, τ_2) is an *ij*-T_k space, then $f^{-1}(V) \in ij\text{-}\psi^* C(X)$. Consequently, f is *ij*-ψ^*-continuous. □

Theorem 5.10. *Let f: $(X, \tau_1, \tau_2) \rightarrow (Y, \sigma_1, \sigma_2)$ be onto, ij-ψ^*-irresolute and ji-α-closed. If (X, τ_1, τ_2) is $ij - T^{\psi^*}_{1/5}$ space, then (Y, σ_1, σ_2) is also an $ij - T^{\psi^*}_{1/5}$ space.*

Proof. Let $V \in ij\text{-}\psi^* C(Y)$. Since f is *ij*-ψ^*-irresolute, then $f^{-1}(V) \in ij\text{-}\psi^* C(X)$. Since (X, τ_1, τ_2) is $ij - T^{\psi^*}_{1/5}$ space, then $f^{-1}(V) \in ji\text{-}\alpha C(X)$. Since f is *ji*-α-closed and onto. Then we have $V \in ji\text{-}\alpha C(Y)$. Therefore (Y, σ_1, σ_2) is also an $ij - T^{\psi^*}_{1/5}$ space. □

We introduce the following definition.

Definition 5.3. A function f: $(X, \tau_1, \tau_2) \rightarrow (Y, \sigma_1, \sigma_2)$ is called an *ij-pre*-ψ^*-closed if $A \in ij\text{-}\psi^* C(X)$, $f(A) \in ij\text{-}\psi^* C(Y)$.

Theorem 5.11. *Let f: $(X, \tau_1, \tau_2) \rightarrow (Y, \sigma_1, \sigma_2)$ be onto, ij-gα-irresolute and ij-pre-ψ^*-closed. If (X, τ_1, τ_2) is $ij - {}^{\psi^*}T_{1/5}$ space, then (Y, σ_1, σ_2) is also an $ij - {}^{\psi^*}T_{1/5}$ space.*

Proof. Let $V \in ij\text{-}G\alpha C(Y)$. Since f is *ij*-gα-irresolute, then $f^{-1}(V) \in ij\text{-}G\alpha C(Y)$. Since (X, τ_1, τ_2) is an $ij - {}^{\psi^*}T_{1/5}$ space. Since f is *ij*-pre-ψ^*-closed and onto. Then we have $f(f^{-1}(V)) = V \in ij\text{-}\psi^* C(Y)$. Therefore (Y, σ_1, σ_2) is also an $ij - {}^{\psi^*}T_{1/5}$ space. □

Theorem 5.12. *Let f: $(X, \tau_1, \tau_2) \rightarrow (Y, \sigma_1, \sigma_2)$ be onto, ij-αg-irresolute and ij-pre-ψ^*-closed. If (X, τ_1, τ_2) is an ij-αT_k space, then (Y, σ_1, σ_2) is also an ij-αT_k space.*

Proof. Let $V \in ij\text{-}\alpha GC(Y)$. Since f is ij-αg-irresolute, then $f^{-1}(V) \in ij\text{-}\alpha GC(X)$. Since (X, τ_1, τ_2) is an $ij\text{-}\alpha T_k$ space, then $f^{-1}(V) \in ij\text{-}\psi^* C(X)$. Since f is ij-pre-ψ^*-closed and onto. Then we have $f(f^{-1}(V)) = V \in ij\text{-}\psi^* C(Y)$. Therefore (Y, σ_1, σ_2) is also an $ij\text{-}\alpha T_k$ space. □

Theorem 5.13. *Let* $f: (X, \tau_1, \tau_2) \to (Y, \sigma_1, \sigma_2)$ *be onto, ij-gs-irresolute and ij-pre-*ψ^**-closed. If* (X, τ_1, τ_2) *is an ij-*T_k *space, then* (Y, σ_1, σ_2) *is also an ij-*T_k *space.*

Proof. Let $V \in ij\text{-}GSC(Y)$. Since f is ij-gs-irresolute, then $f^{-1}(V) \in ij\text{-}GSC(X)$. Since (X, τ_1, τ_2) is an $ij\text{-}T_k$ space, then $f^{-1}(V) \in ij\text{-}\psi^* C(X)$. Since f is ij-pre-ψ^*-closed and onto. Then we have $f(f^{-1}(V)) = V \in ij\text{-}\psi^* C(Y)$. Therefore (Y, σ_1, σ_2) is also an $ij\text{-}T_k$ space. □

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References

[1] Z. Pawlak, Rough sets, Int. J. Comput. Inf. Sci. 11 (5) (1982) 341–356.
[2] T.Y. Kong, R. Kopperman, P.R. Meyer, A topological approach to digital topology, Am. Math. Monthly 98 (1991) 901–917.
[3] B.M.R. Stadler, P.F. Stadler, Generalized topological spaces in evolutionary theory and combinatorial chemistry, J. Chen. Inf. Comput. Sci. 42 (2002) 577–585.
[4] N. Levine, Generalized closed sets in topological spaces, Rend. Circ. Math. Palermo 19 (2) (1970) 89–96.
[5] S.P. Arya, T.M. Nour, Characterizations of s-normal spaces, Indian J. Pure. Appl. Math. 21 (8) (1990) 717–719.
[6] P. Bhattacharya, B.K. Lahiri, Semi-generalized closed sets in topology, Indian J. Math. 29 (1987) 375–382.
[7] H. Maki, R. Devi, K. Balachandran, Generalized α-closed sets in topology, Bull. Fukuoka Univ. Ed. Part III (42) (1993) 13–21.
[8] H. Maki, R. Devi, K. Balachandran, Associated topologies of generalized α-closed sets and α-generalized closed sets, Mem. Fac. Sci. Kochi Univ. Ser. A. Math. 15 (1994) 51–63.
[9] H. Maki, J. Umehara, T. Noiri, Every topological space is pre-T1/2, Mem. Fac. Sci. Koch Univ. Ser. A. Math. 17 (1996) 33–42.
[10] J. Dontchev, On generalizing semi-preopen sets, Mem. Fac. Sci. Kochi Univ. Ser. A. Math. 16 (1995) 35–48.
[11] O.A. El-Tantawi, H.M. Abu-Donia, Generalized separation axioms in bitopological spaces, Arab. J. Sci. Eng. 1 (2005) 117–129.
[12] M.A. Abd Allah, A.S. Nawar, ψ^*-closed sets in topological spaces, 'Wulfenia' J. 21 (9) (2014) 391–401.
[13] A.S. Mashhour, M.E. Abd El-Monsef, S.N. El-Deeb, On pre-continuous and weak pre-continuous mapings, Proc. Math. and Phys. Soc. Egypt 53 (1982) 47–53.
[14] N. Levine, Semi-open sets and semi-continuity in topological spaces, Amer. Math. Monthly 70 (1963) 36–41.
[15] A.S. Mashhour, I.A. Hasanein, S.N. El-Deeb, α-continuous and α-open mappings, Acta Math. Hung. 41 (1983) 213–218.
[16] K. Balachandran, P. Sundaram, H. Maki, On generalized continuous maps in topological spaces, Mem. Fac. Kochi Univ. Ser. A. Math. 12 (1991) 5–13.
[17] Y. Gnanambal, On generalized preregular closed sets in topological spaces, Indian J. Pure. Appl. Math. 28 (3) (1997) 351–360.
[18] R. Devi, H. Maki, K. Balachandran, Semi-generalized homeomorphisms and generalized semi-homeomorphism in topological spaces, Indian J. Pure. Appl. Math. 26 (3) (1995) 271–284.
[19] P. Sundaram, H. Maki, K. Balachandran, Semi-generalized continuous maps and semi-$T_{1/2}$ spaces, Bull. Fukuoka Univ. Ed. Part III (40) (1991) 33–40.
[20] I. Arokiarani, K. Balachandran, J. Dontchev, Some characterizations of gp-irresolute and gp-continuous maps between topological spaces, Mem. Fac. Sci. Kochi Univ. Ser. A. Math. 20 (1999) 93–104.
[21] R. Devi, K. Balachandran, H. Maki, Generalized α-closed maps and α-generalized closed maps, Indian J. Pure. Appl. Math. 29 (1) (1998) 37–49.
[22] R. Devi, K. Balachandran, H. Maki, On generalized α-continuous maps and α-generalized continuous maps, Far East J. Math. Sci. (1997) 1–15 (Special volume)
[23] S. Sampath Kumar, On a decomposition of pairwise continuity, Bull. Cal. Math. Soc. 89 (1997) 441–446.
[24] T. Noiri, The further unified theory for modifications of g-closed sets, Rend. Circ. Math. Palermo 57 (2008) 411–422.