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Generalization of Herstein theorem and its applications to range inclusion problems ☆

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Abstract Let R be an associative ring. An additive mapping $d: R \to R$ is called a Jordan derivation if $d(x^2) = d(x)x + xd(x)$ holds for all $x \in R$. The objective of the present paper is to characterize a prime ring R which admits Jordan derivations d and g such that $[d(x^m), g(y^n)] = 0$ for all $x, y \in R$ or $d(x^m) \circ g(y^n) = 0$ for all $x, y \in R$, where $m \geqslant 1$ and $n \geqslant 1$ are some fixed integers. This partially extended Herstein's result in [6, Theorem 2], to the case of (semi)prime ring involving pair of Jordan derivations. Finally, we apply these purely algebraic results to obtain a range inclusion result of continuous linear Jordan derivations on Banach algebras.

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1. Introduction

Throughout this paper R will denote an associative ring with center $Z(R)$. Recall that a ring R is said to be prime if for any $a, b \in R, aRb = \{0\}$ implies $a = 0$ or $b = 0$, and R is semiprime if for any $a \in R, aRa = \{0\}$ implies $a = 0$. A ring R is said to be n-torsion free, where $n > 1$ is an integer, in case $nx = 0$ implies $x = 0$ for all $x \in R$. For any $x, y \in R$, the symbol $[x, y]$ will denote the commutator $xy - yx$ and the symbol $x \circ y$ will stand for the anti-commutator $xy + yx$. Following [1], an additive mapping $d: R \to R$ is said to be a derivation (resp. Jordan derivation) on R if $d(xy) = d(x)y + xd(y)$ (resp. $d(x^2) = d(x)x + xd(x)$) holds for all $x, y \in R$. Let S be a nonempty subset of R. A mapping $f: R \to R$ is called centralizing on S if $[f(x), x] \in Z(R)$ for all $x \in S$ and is called commuting on S if $[f(x), x] = 0$ for all $x \in S$. The study of such mappings were initiated by Posner. In [2, Lemma 3], Posner proved that if a prime ring R has a nonzero commuting derivation on R, then R is commutative. This result was subsequently refined and extended by a number of algebraists; we refer the reader to [3–5] for a state-of-art account and a comprehensive bibliography.

In [6], Herstein proved the following result: If R is a prime ring of characteristic not two admitting a nonzero derivation d such that $[d(x), d(y)] = 0$ for all $x, y \in R$, then R is commutative. Further, Daif [7] showed that a 2-torsion free semiprime

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ring R admits a derivation d such that $[d(x), d(y)] = 0$ for all $x, y \in I$, where I is a nonzero ideal of R and d is nonzero on I, then R contains a nonzero central ideal. Motivated by the above result, Ashraf and Rehman [8] proved that if R is a 2-torsion free prime ring admitting a nonzero derivation d such that $d(x) \circ d(y) = 0$ for all $x, y \in I$, where I is a nonzero ideal of R, then R is commutative. This result was further extended by first author together with Shuliang [9, Theorem 3.2] for semiprime rings. In Section 3, our aim is to generalize these results for pair of Jordan derivations d and g. More precisely, it was shown that if R is a $\max\{m, n, 2\}!$-torsion free prime ring, where $m \geqslant 1$ and $n \geqslant 1$ are some fixed integers, and d, g are nonzero Jordan derivations of R such that $[d(x^m), g(y^n)] = 0$ for all $x, y \in R$, then R is commutative. Further, some more related results have also been discussed. In Section 4, we apply purely algebraic results from Section 3 to discuss the range inclusion problems in the setting of continuous linear Jordan derivations on Banach algebras. Throughout this paper, we assume that $m \geqslant 1$ and $n \geqslant 1$ are some fixed integers.

2. Some preliminaries

We shall do a great deal of calculations with commutators and anti-commutators, routinely using the following basic identities: For all $x, y, z \in R$;

$$[xy, z] = x[y, z] + [x, z]y \text{ and } [x, yz] = [x, y]z + y[x, z]$$
$$(x + y) \circ z = x \circ z + y \circ z \text{ and } x \circ (y + z) = x \circ y + x \circ z$$
$$x \circ (yz) = (x \circ y)z - y[x, z] = y(x \circ z) + [x, y]z$$
$$(xy) \circ z = x(y \circ z) - [x, z]y = (x \circ z)y + x[y, z].$$

We begin with the following lemmas which are essential for developing the proof of our results.

Lemma 2.1 [4, Theorem 4]. *Let R be a prime ring and I a nonzero left ideal of R. If R admits a nonzero derivation d which is centralizing on I, then R is commutative.*

Lemma 2.2 [10, Lemma 4]. *Let R be a 2-torsion free semiprime ring and $a, b \in R$. If for all $x \in R$ the relation $axb + bxa = 0$ holds, then $axb = bxa = 0$ is fulfilled for all $x \in R$.*

Lemma 2.3 [11, Lemma 1]. *Let R be an $m!$-torsion free ring. Suppose $y_1, y_2, \ldots, y_m \in R$ satisfying $\alpha y_1 + \alpha^2 y_2 + \ldots + \alpha^m y_m = 0$ for $\alpha = 1, 2, \ldots, m$. Then $y_i = 0$ for all i.*

Lemma 2.4 [12, Lemma 3.2]. *A continuous Jordan derivation on a Banach algebra leaves invariant the primitive ideals in the algebra.*

3. Generalizations of the condition $d(x)d(y) = d(y)d(x)$

To state our results precisely, we fix some notations. From now, Q always denotes the maximal right ring of quotients of R. If R is a (semi)prime ring, then Q is also a (semi)prime ring. The center of Q is called the extended centroid of R and is denoted by C. For the explanation of maximal right ring of quotients we refer the reader to [13]. We shall use the fact that any semiprime ring R and its maximal right ring of quotients Q satisfy the same differential identities which is very useful since Q contains the identity element (see Theorem 3 in [14]). For the explanation of differential identities we refer the reader to [15,16]. Throughout this section, we will use the fact that image of the identity of a ring R is zero under any derivation. We begin our investigations with the following theorem which generalizes Theorem 2 in [6].

Theorem 3.1. *Let R be a $\max\{m, n, 2\}!$-torsion free prime ring, and d, g be nonzero Jordan derivations of R. If $[d(x^m), g(y^n)] = 0$ holds for all $x, y \in R$, then R is commutative.*

Proof. Since d and g are Jordan derivations on R, d and g also are derivations on R by Herstein's theorem [1]. By the assumption, we have

$$[d(x^m), g(y^n)] = 0 \quad \text{for all } x, y \in R.$$

It is well known that R and Q satisfy the same differential identities [14, Theorem 3]. Therefore

$$[d(x^m), g(y^n)] = 0 \quad \text{for all } x, y \in Q. \tag{3.1}$$

Note that Q has the identity element. Replacing x by $1 + x$ in (3.1), we get

$$\binom{m}{1}[d(x), g(y^n)] + \binom{m}{2}[d(x^2), g(y^n)] + \cdots + \binom{m}{m}[d(x^m), g(y^n)] = 0. \tag{3.2}$$

Substituting px for x in (3.2), where $p = 1, 2, \ldots, m$, we get

$$p\binom{m}{1}[d(x), g(y^n)] + p^2\binom{m}{2}[d(x^2), g(y^n)] + \cdots + p^m\binom{m}{m}[d(x^m), g(y^n)] = 0.$$

Using Lemma 2.3, we obtain $\binom{m}{r}[d(x^r), g(y^n)] = 0$ for all $x, y \in Q$ and $r = 1, 2, \ldots, m$. In particular for $r = 1$, we have $m[d(x), g(y^n)] = 0$ for $x, y \in Q$. By applying torsion free fact of Q, we are forced to conclude that

$$[d(x), g(y^n)] = 0 \quad \text{for all } x, y \in Q.$$

Now, replacing y by $y + 1$ and using similar approach as above, we obtain

$$[d(x), g(y)] = 0 \quad \text{for all } x, y \in Q. \tag{3.3}$$

Again replace y by yz in (3.3) to get

$$[d(x), g(y)]z + g(y)[d(x), z] + y[d(x), g(z)] + [d(x), y]g(z) = 0 \quad \text{for all } x, y, z \in Q. \tag{3.4}$$

Application of (3.3) yields that

$$g(y)[d(x), z] + [d(x), y]g(z) = 0 \quad \text{for all } x, y, z \in Q. \tag{3.5}$$

Substituting rz for z in (3.5) and using it, we get

$$g(y)r[d(x), z] + [d(x), y]rg(z) = 0 \quad \text{for all } r, x, y, z \in Q.$$

In particular, for $y = z$, we have

$$g(y)r[d(x),y]+[d(x),y]rg(y)=0 \quad \text{for all } r,x,y\in Q.$$

By Lemma 2.2, we conclude that

$$g(y)Q[d(x),y]=\{0\} \quad \text{for all } x,y\in Q.$$

The primeness of Q forces that either $g(y)=0$ or $[d(x),y]=0$. For each fixed $y\in Q$, we set $U=\{y\in Q \mid g(y)=0\}$ and $V=\{y\in Q \mid [d(x),y]=0 \text{ for all } x\in Q\}$. Clearly, U and V both are additive subgroups of Q whose union is Q, but a group cannot be the union of its two proper subgroups. Hence, either $Q=U$ or $Q=V$. Suppose $Q=U$, then $g(y)=0$ for all $y\in Q$, which gives a contradiction as g is nonzero. Thus, we have $Q=V$, which implies that $[d(x),y]=0$ for all $x,y\in Q$. That is, $[d(x),x]=0$ for all $x\in Q$. Therefore, in view of Lemma 2.1, Q is commutative and hence R is commutative. This proves the theorem completely. □

As direct corollaries of Theorem 3.1 we immediately get:

Corollary 3.2. *Let R be a $\max\{m,n,2\}$!-torsion free prime ring. If R admits a Jordan derivation d such that $[d(x^m),d(y^n)]=0$ for all $x,y\in R$, then either $d=0$ or R is commutative.*

Corollary 3.3 [6, Theorem 2]. *Let R be a prime ring such that char $R\neq 2$. If R admits a nonzero derivation d such that $[d(x),d(y)]=0$ for all $x,y\in R$, then R is commutative.*

In case of semiprime ring we have the following result:

Theorem 3.4. *Let R be a $\max\{m,n,2\}$!-torsion free semiprime ring. If R admits a nonzero Jordan derivation d such that $[d(x^m),d(y^n)]=0$ for all $x,y\in R$, then R contains a nonzero central ideal.*

Proof. In view of Theorem 1 of [10], we conclude that d is derivation on R. Further we have

$$[d(x^m),d(y^n)]=0 \quad \text{for all } x,y\in R.$$

This implies that

$$[d(x^m),d(y^n)]=0 \quad \text{for all } x,y\in Q.$$

Henceforth, using the similar approach as we have used to get (3.3) from relation (3.1) in Theorem 3.1, we find that

$$[d(x),d(y)]=0 \quad \text{for all } x,y\in Q.$$

Hence, we obtain

$$[d(x),d(y)]=0 \quad \text{for all } x,y\in R.$$

In view of Theorem 2.2 in [7], we conclude that R contains a nonzero central ideal. This completes the proof of the theorem. □

If we replace commutator by anti-commutator in Theorem 3.1, the corresponding result also holds, which is a partial generalization of Theorem 4.3 in [8].

Theorem 3.5. *Let R be a $\max\{m,n,2\}$!-torsion free prime ring, and d,g be nonzero Jordan derivations of R. If $d(x^m)\circ g(y^n)=0$ holds for all $x,y\in R$, then R is commutative.*

Proof. We are given that d,g are Jordan derivations on R and hence by Theorem 3.1 of [1], d and g are derivations on R. It is well known that R and Q satisfy the same differential identities [14, Theorem 3]. Thus, our assumption implies that

$$d(x^m)\circ g(y^n)=0 \quad \text{for all } x,y\in Q.$$

Now, using the same arguments as we have used to get relation (3.3) from (3.1) in the proof of Theorem 3.1, we conclude that

$$d(x)\circ g(y)=0 \quad \text{for all } x,y\in Q. \tag{3.6}$$

Putting $x=xz$ in (3.6), we get

$$\begin{aligned}0&=d(xz)\circ g(y)\\&=d(x)z\circ g(y)+xd(z)\circ g(y)\\&=(d(x)\circ g(y))z+d(x)[z,g(y)]+x(d(z)\circ g(y))-[x,g(y)]d(z)\end{aligned}$$

for all $x,y,z\in Q$. Application of (3.6) yields that

$$d(x)[z,g(y)]=[x,g(y)]d(z) \quad \text{for all } x,y,z\in Q.$$

Taking $x=g(y)$ in above, we obtain

$$d(g(y))[z,g(y)]=0 \quad \text{for all } y,z\in Q. \tag{3.7}$$

Replacing z by rz in (3.7) and using it, we get

$$dg(y)Q[z,g(y)]=\{0\} \quad \text{for all } y,z\in Q.$$

Since Q is prime, the last relation forces that either $dg(y)=0$ or $[z,g(y)]=0$. For fixed $y\in Q$, we set $U=\{y\in Q \mid dg(y)=0\}$ and $V=\{y\in Q \mid [z,g(y)]=0 \text{ for all } z\in Q\}$. Clearly, U and V both are additive subgroups of Q whose union is Q, but a group cannot be the union of its two proper subgroups. Hence, either $Q=U$ or $Q=V$. First we consider the case when $Q=U$, then $dg=0$. In view of [2, Theorem 1], we conclude that either $d=0$ or $g=0$, which is a contradiction as neither $d=0$ nor $g=0$. Therefore, we have the only case $Q=V$, which implies that $[z,g(y)]=0$ for all $y,z\in Q$. In particular, we have $[y,g(y)]=0$ for all $y\in Q$. Thus, in view of Lemma 2.1, we conclude that Q is commutative and hence R is commutative. □

The following corollary is an immediate consequence of above theorem.

Corollary 3.6. *Let R be a $\max\{m,n,2\}$!-torsion free prime ring. If R admits a Jordan derivation d such that $d(x^m)\circ d(y^n)=0$ for all $x,y\in R$, then either $d=0$ or R is commutative.*

Theorem 3.7. *Let R be a $\max\{m,n,2\}$!-torsion free semiprime ring. If R admits a nonzero Jordan derivation d such that $d(x^m)\circ d(y^n)=0$ for all $x,y\in R$, then R contains a nonzero central ideal.*

Proof. By Theorem 1 of [10], we conclude that d is a derivation on R. From relation (3.6), we obtain $d(x)\circ g(y)=0$ for all $x,y\in R$. Substituting $g=d$ in the last relation, we get $d(x)\circ d(y)=0$ for all $x,y\in R$. Hence, in view of Theorem 3.2 in [9], we conclude that R contains a nonzero central ideal. The theorem is thereby proved. □

If prime ring is replaced by semiprime ring in Theorems 3.1 and 3.5, then results may not be necessarily true. The following example justifies the fact:

Example 3.8. Let R_1, R_2 be noncommutative prime rings and d_1, g_1 be nonzero Jordan derivations of R_1 and R_2, respectively. Consider, $R = R_1 \times R_2$, then R is a semiprime ring. Define mappings $d, g : R \longrightarrow R$ such that $d(r_1, r_2) = (d_1(r_1), 0)$ and $g(r_1, r_2) = (0, g_1(r_2))$ for all $r_1 \in R_1$ and $r_2 \in R_2$. Clearly, d and g are nonzero Jordan derivations of R. For some fixed integers $m, n \geqslant 1$, and d, g satisfying the identities $[d(x^m), g(y^n)] = 0$, and $d(x^m) \circ g(y^n) = 0$ for all $x, y \in R$, but R is not commutative. Thus, the hypothesis of primeness in Theorems 3.1 and 3.5 is crucial.

Example 3.9. Let $R = \left\{ \begin{pmatrix} a & b \\ 0 & c \end{pmatrix} \mid a, b, c \in \mathbb{Z} \right\}$. Clearly, R is a ring under usual matrix operations which is not semiprime. Next, let $d, g : R \longrightarrow R$ be mappings such that

$$d\begin{pmatrix} a & b \\ 0 & c \end{pmatrix} = \begin{pmatrix} 0 & b \\ 0 & 0 \end{pmatrix} \quad \text{and} \quad g\begin{pmatrix} a & b \\ 0 & c \end{pmatrix} = \begin{pmatrix} 0 & c-a \\ 0 & 0 \end{pmatrix} \quad \text{for all} \quad \begin{pmatrix} a & b \\ 0 & c \end{pmatrix} \in R.$$

Then it is straightforward to check that d and g are Jordan derivations of R such that $[d(x^m), d(y^n)] = 0$, and $d(x^m) \circ d(y^n) = 0$ for all $x, y \in R$. However, R contains no nonzero central ideal. Hence in Theorems 3.4 and 3.7 the condition of semiprimeness cannot be omitted.

4. Range inclusion problems

In the present section we will use the previous algebraic results to study the range inclusion problems involving continuous linear Jordan derivations on a Banach algebra. Let us recall some elementary notions for the sake of completeness. A always denotes a Banach algebra which is a complex normed algebra and its underlying vector space is a Banach space. The Jacobson radical of A is the intersection of all primitive ideals of A and is denoted by $\mathrm{rad}(A)$. Throughout the balance of this paper, we assume that all mappings on Banach algebra A are linear mappings.

In 1955 Singer and Wermer [17] proved that a continuous linear derivation on a commutative Banach algebra maps the algebra into its radical. Johnson and Sinclair [18] have proved that any linear (Jordan) derivation on a semisimple Banach algebra is continuous. According to these two results, one can conclude that there are no nonzero linear Jordan derivations on a commutative semisimple Banach algebras. Singer and Wermer conjectured in [17] that the continuity assumption in their result is superfluous. It took more than thirty years until this conjecture was finally proved by Thomas [19]. Obviously, from Thomas's result it follows directly that there are no nonzero linear derivations on a commutative semisimple Banach algebra. By our knowledge the first noncommutative extension of Singer–Wermer theorem has been proved by Yood [20] who showed that if for all pairs $x, y \in A$, where A is a noncommutative Banach algebra, the element $[D(x), y]$ lies in $\mathrm{rad}(A)$, then D maps A into $\mathrm{rad}(A)$. Brešar and Vukman [21] have generalized Yood's result by proving that in case $[D(x), x] \in \mathrm{rad}(A)$ for all $x \in A$, then D maps A into $\mathrm{rad}(A)$. The work of Mathieu and Murphy [22] and Runde [23] should also be mentioned. Recently, Kim [24] has proved that in case $[D(x), x]D(x)[D(x), x] \in \mathrm{rad}(A)$ for any $x \in A$, then a continuous derivation D maps A into $\mathrm{rad}(A)$. Kim's result generalizes a result proved by Vukman [25]. For references concerning range inclusion results of continuous derivations on noncommutative Banach algebras we refer the reader to ([26–29] and reference therein). We proceed with the following theorem.

Theorem 4.1. *Let d be a continuous Jordan derivation of A. If $[d(x^m), d(y^n)] \in \mathrm{rad}(A)$ holds for all $x, y \in A$, then $d(A) \subseteq \mathrm{rad}(A)$.*

Proof. From the hypothesis, we have

$$[d(x^m), d(y^n)] \in \mathrm{rad}(A) \quad \text{for all } x, y \in A.$$

By Lemma 2.4, every continuous linear Jordan derivation of a Banach algebra A leaves the primitive ideals invariant which means that one can introduce for any primitive ideal $P \subset A$, derivation $D : A/P \longrightarrow A/P$, where A/P is the factor algebra, by $D(\bar{x}) = d(x) + P$ for all $x \in A$ and $\bar{x} = x + P$. Since P is a primitive ideal, the quotient Banach algebra A/P is prime and semisimple. Hence, D is derivation by Theorem 3.1 of [1]. When A/P is commutative, then, by Singer–Wermer Theorem, there is no nonzero linear derivation on a commutative semisimple Banach algebra. Hence, we have $D = \bar{0}$. On the other hand, we assume that A/P is noncommutative. Then the assumption of the theorem implies that

$$[D(\bar{x}^m), D(\bar{y}^n)] = \bar{0} \quad \text{for all } \bar{x}, \bar{y} \in A/P.$$

In view of Corollary 3.2, we conclude that $D = \bar{0}$. Thus for any $x \in A$, we are forced to conclude that $d(x) \in P$, where P is any primitive ideal of A. Since $d(x)$, where x is any element from A, is in the intersection of all primitive ideals of A and since the intersection of all primitive ideals of A is the radical, one can conclude that $d(A) \subseteq \mathrm{rad}(A)$. Thereby the proof of theorem is completed. □

Theorem 4.2. *Let d be a continuous Jordan derivation of A. If $d(x^m) \circ d(y^n) \in \mathrm{rad}(A)$ for all $x, y \in A$, then $d(A) \subseteq \mathrm{rad}(A)$.*

Proof. The proof goes through using the same arguments as in the proof of above theorem with the exception that one has to use Corollary 3.6 instead of Corollary 3.2. This completes the proof the theorem. □

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