Journal of the Egyptian Mathematical Society (2014) **22**, 327–329

Egyptian Mathematical Society

Journal of the Egyptian Mathematical Society

www.etms-eg.org
www.elsevier.com/locate/joems

Generalized left derivations acting as homomorphisms and anti-homomorphisms on Lie ideal of rings

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Received 30 August 2013; revised 19 November 2013; accepted 25 December 2013
Available online 31 January 2014

KEYWORDS

Left derivations;
Generalized left derivations;
Prime pings;
Lie ideals;
Martindale ring of quotients

Abstract Let R be a prime ring with characteristic different from 2 and L be a Lie ideal of R. In this paper, we characterize generalized left derivation, which acts as a homomorphisms or an anti-homomorphisms on L.

2000 MATHEMATICS SUBJECT CLASSIFICATION: 16W25; 16N60; 16U80; 16W10

1. Introduction

Throughout the present paper R will denote an associative ring with center $Z(R)$. Recall that R is prime if $aRb = (0)$ implies that either $a = 0$ or $b = 0$, and R is semiprime if $aRa = (0)$ implies $a = 0$. As usual $[x, y]$ will denote the commutator $xy - yx$. We shall make an extensive use of commutator identities; $[x, yz] = [x, y]z + y[x, z]$ and $[xy, z] = [x, z]y + x[y, z]$. An additive subgroup L of R is said to be a Lie ideal of R it $[L, R] \subseteq L$. A Lie ideal L is said to be square closed if $a^2 \in L$ for all $a \in L$. An additive mapping $\delta : R \to R$ is called a derivation (resp. Jordan derivation) if $\delta(xy) = \delta(x)y + x\delta(y)$ (resp. $\delta(x^2) = \delta(x)x + x\delta(x)$) holds for all $x, y \in R$. An additive mapping $H : R \to R$ is called a generalized derivation if there exists a derivation $\delta : R \to R$ such that $H(xy) = H(x)y + x\delta(y)$ holds for all $x, y \in R$. In 1990, Bresar and Vukman [6] introduced the concept of left derivation as follows: An additive mapping $d : R \to R$ is called left derivation (resp. Jordan left derivation) if $d(xy) = xd(y) + yd(x)$ (resp. $d(x^2) = 2xd(x)$) holds for all $x, y \in R$. They proved that a prime ring which admits a nonzero left derivation is commutative. Obviously in commutative ring, derivations (resp. generalized derivations) act as a left derivations (resp. generalized left derivations). However in noncommutative ring, the case is quite different in general. According to [2], an additive mapping $F : R \to R$ is called a generalized left derivation (resp. generalized Jordan left derivation) if there exists a Jordan left derivation $d : R \to R$ such that $F(xy) = xF(y) + yd(x)$ (resp. $F(x^2) = xF(x) + xd(x)$) holds for all $x, y \in R$.

Let S be a nonempty subset of R and $\delta : R \to R$ be a derivation of R. If $\delta(xy) = \delta(x)\delta(y)$ (resp. $(\delta(xy) = \delta(y)\delta(x))$ holds for all $x, y \in S$, then δ is said to act as a homomorphism (resp. anti-homomorphism) on S. In [7] Bell and Kappe proved

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Peer review under responsibility of Egyptian Mathematical Society.

Production and hosting by Elsevier

1110-256X
http://dx.doi.org/10.1016/j.joems.2013.12.015

that, if I is a nonzero right ideal of a prime ring R and $\delta : R \to R$ is a derivation of R such that δ acts as a homomorphism or an ant-homomorphism on I, then $\delta = 0$ on R. In the present paper we study the generalized left derivation of a prime ring which acts either homomorphism or ant-homomorphism on a certain well behaved subset of R.

2. Preliminaries

We collect some known results and review a few important facts about the left Martindale ring of quotients that will be needed in the subsequent discussions of a ring, $Q_l(R)$ will denote the left Martindale ring of quotients of a prime ring R. This ring was introduced by Martindale in [8] as a tool in the study of prime rings satisfying generalized polynomial identities (e.f. [4]). The center of $Q_l(R)$, will be denoted by C, and called the extended centroid of R. It is well known that C is a field. Also, it is easily seen that C is the centralizer of R in $Q_l(R)$. In particular, $Z(R) = C$. The subring of $Q_l(R)$ generated by R and C is called the central closure of R and will be denoted by R_C. Another subring of $Q_l(R)$ is $Q_s(R) = \{q \in Q_l(R) | qI \subseteq R$ for some nonzero ideal $I$of $R\}$. This ring is known as the symmetric Martindale ring of quotients. It is easy to verify that $R \subseteq R_C \subseteq Q_s(R) \subseteq Q_l(R)$. Note that $aRb = \{0\}$ with $a, b \in Q_l(R)$ implies that $a = 0$ or $b = 0$. Whence we can see that $R_C, Q_l(R)$ and $Q_s(R)$ are prime rings.

Remark 2.1. Let L be a square closed Lie ideal of R. Notice that $xy + yx = (x + y)^2 - x^2 - y^2$ for all $x, y \in L$. Since $x^2 \in L$ for all $x \in L, xy + yx \in L$ for all $x, y \in L$. Hence we find that $2xy \in L$ for all $x, y \in L$. Therefore, for all $r \in R$, we get $2r[x, y] = 2[x, ry] - 2[x, r]y \in L$ and $2[x, y]r = 2[x, yr] - 2[y, r] \in L$, so that $2R[L, L] \subseteq L$ and $2[L, L]R \subseteq L$.

This remark will be freely used in the whole paper without specific mention.

We begin with the following lemmas which are essential for developing the proof of our results.

Lemma 2.1 [5, Lemma 4]. *Let R be a prime ring of characteristic different from 2 and $L \not\subseteq Z(R)$ be a Lie ideal of R and if $aLb = \{0\}$, then $a = 0$ or $b = 0$.*

Lemma 2.2 [3, Theorem 4]. *Let R be a prime ring of characteristic different from 2 and L be a square closed Lie ideal of R. If $d : R \to R$ is an additive mapping such that $d(x^2) = 2xd(x)$ for all $x \in L$, then $d(xy) = xd(y) + yd(x)$ for all $x, y \in L$.*

Lemma 2.3 [1, Proposition 2.10]. *Let R be a prime ring and $F : R \to R_C$ be an additive mapping satisfying $F(rs) = rF(s)$ for all $r, s \in R$. Then there exists $q \in Q_l(R_C)$ such that $F(r) = rq$ for all $r \in R$.*

3. Main result

Let S be a nonempty subset of R and F be a generalized left derivation on R. If $F(xy) = F(x)F(y)$ or $F(xy) = F(y)F(x)$ for all $x, y \in S$, then F is said to be generalized left derivation which acts as a homomorphism or an anti-homomorphism on S, respectively. Of course, derivation which acts as an endomorphism or an anti-endomorphism of a ring R may behave as such on certain subset of R, for example, any derivation δ behaves as zero endomorphism on a subring consisting of all constants (i.e., element x for which $d(x) = 0$). In fact, in a semiprime ring R, δ may behave as an endomorphism on a proper ideal of R. As an example of such R and δ, let R_1 be any semiprime ring with non zero derivation δ_1, take $R = R_1 \oplus R_1$ and define $\delta(r_1, r_2) = (\delta_1(r_1), 0)$. However in the case of prime rings, Bell and Kappe [7] showed that the behavior of δ is somewhat more restricted. By proving that if R is prime ring and δ is a derivation of R which acts as a homomorphism or an anti-homomorphism on a nonzero right ideal of R, then $\delta = 0$ on R. Further, the first author obtained in [9] the above mentioned result for generalized derivation acting on ideals in prime ring. In the present section, our objective is to extend the above result to the setting of generalized left derivations in the case the underlying subset of R is Lie ideal of R.

Theorem 3.1. *Let R be a prime ring of characteristic different from 2 and L be a noncommutative square closed Lie ideal of R. Suppose that $F : R \to R$ be a generalized left derivation with associated Jordan left derivation $d : R \to R$. If F acts as a homomorphism or as an anti-homomorphism on L, then $F(r) = rq$ for all $r \in R$ and $q \in Q_l(R_C)$.*

Proof. Suppose F acts a homomorphisms on L, then we have $F(xy) = F(x)F(y)$ for all $x, y \in L$. Which can be re-written as $F(x)F(y) = F(xy) = xF(y) + yd(x)$ for all $x, y \in L$. Now, our aim is to prove $d = 0$ on R. Let us consider

$$F(xyz) = F(x(yz)) = xF(yz) + yzd(x) = xF(y)F(z) + yzd(x) \tag{3.1}$$

On the other hand,

$$F(xyz) = F(xy(z)) = F(xy)F(z) = xF(y)F(z) + yd(x)F(z) \tag{3.2}$$

Combining (3.1) and (3.2), we obtain $yzd(x) = yd(x)F(z)$ for all $x, y, z \in L$, i.e., $y(zd(x) - d(x)F(z)) = 0$ for all $x, y, z \in L$. On left multiplication by $zd(x) - d(x)F(z)$ to the above relation, we find $(zd(x) - d(x)F(z))y(zd(x) - d(x)F(z)) = 0$ for all $x, y, z \in L$. Then, by Lemma 2.1, we have that

$$zd(x) - d(x)F(z) = 0 \quad \text{for all } x, y, z \in L. \tag{3.3}$$

By Lemma 2.2, d will be a left derivation on L. Now, replacing x by $2xy$ in the above relation and using $charR \neq 2$, we find that $zxd(y) + zyd(x) - xd(y)F(z) - yd(x)F(z) = 0$ for all $x, y, z \in L$. Using relation (3.3), we arrive at $[x, z]d(y) + [y, z]d(x) = 0$ for all $x, y, z \in L$. In particular, putting $z = y$, we get

$$[x, y]d(y) = 0 \quad \text{for all } x, y \in L. \tag{3.4}$$

for all $x, y \in L$. Then, linearizing the above relation, we obtain $d(x)[z, y] + d(z)[x, y] = 0$ for all $x, y, z \in L$, and hence

$$d(x)[z, y] = -d(z)[x, y] \quad \text{for all } x, y, z \in L. \tag{3.5}$$

Replacing y by $2uy$ in (3.4) and using (3.4), we get $2d(x)u[x, y] = 0$ for all $x, y, u \in L$. Since $charR \neq 2$, we find that

$d(x)u[x,y] = 0$. Now, replace u by $2[z,y]r$ and use the fact that $charR \neq 2$, to get $d(x)[z,y]r[x,y] = 0$ for all $x,y,z \in L$ and $r \in R$ and hence application of (3.5), we obtain $d(z)[x,y]r[x,y] = 0$ for all $x,y,z \in L$ and $r \in R$. Again replacing r by $rd(z)$ in the above expression, we get $d(z)[x,y]rd(z)[x,y] = 0$ for all $x,y,z \in L$ and $r \in R$, that is, $d(z)[x,y]Rd(z)[x,y] = \{0\}$ for all $x,y,z \in L$. Thus primeness of R forces that $d(z)[x,y] = 0$ for all $x,y,z \in L$. Again, replacing x by $2tx$ and using the fact that $charR \neq 2$, we get $d(z)t[x,y] = 0$ for all $x,y,z,t \in L$. Since L is a noncommutative Lie ideal of R and hence by Lemma 2.1, we get $d(z) = 0$ for all $z \in L$. Replacing z by $2r[y,z]$ and using $charR \neq 2$, we obtain $[y,z]d(r) = 0$ for all $y,z \in L$ and $r \in R$. Again, replacing y by $2yx$ and using $charR \neq 2$, we get $[y,z]xd(r) = 0$ for all $x,y,z \in L$ and $r \in R$. Therefore, by Lemma 2.1, we get $d = 0$ on R. Hence, there exists $q \in Q_l(R_C)$ such that $F(r) = rq$ for all $r \in R$ by Lemma 2.3.

If F acts as an anti-homomorphism on L, then, $F(xy) = F(y)F(x)$ for all $x,y \in L$. This can written as $xF(y) + yd(x) = F(y)F(x)$ for all $x,y \in L$. Replacing y by $2xy$ in above expression and using $charR \neq 2$, we find that

$$xyd(x) = yd(x)F(x) \quad \text{holds for all } x,y \in L. \tag{3.6}$$

Again, replacing y by $2zy$ in (3.6) and using $charR \neq 2$, we get

$$xzyd(x) = zyd(x)F(x) \quad \text{holds for all } x,y,z \in L. \tag{3.7}$$

Multiplying left side by z to the relation (3.6), we obtain

$$zxyd(x) = zyd(x)F(x) \quad \text{holds for all } x,y,z \in L. \tag{3.8}$$

Now, combining relation (3.7) and (3.8), we get $[x,z]yd(x) = 0$ for all $x,y,z \in L$. By Lemma 2.1, we get $d(x) = 0$ for all $x \in L$. Now, using the same argument as we have used the above, we get the required result. This completes the proof of the theorem. □

Corollary 3.1. *Let R be a prime ring of characteristic different from two and L be a noncommutative square closed Lie ideal of R. Suppose that d is a Jordan left derivation on R. If d acts as a homomorphism or as an anti-homomorphism on L, then $d = 0$ on R.*

Acknowledgment

The authors are thankful to the referee for his/her valuable suggestions.

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