



ORIGINAL ARTICLE

Subordination and superordination preserving properties for a family of integral operators involving the Noor integral operator



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Abstract In the present paper, we introduce a family of integral operators $\mathcal{I}_{p,n,\delta}^{\lambda,\mu}(a,b,c)$ associate with the Noor integral operator in the open unit disk $\mathbb{U} = \{z \in \mathbb{C} : |z| < 1\}$, which is defined by the convolution $[f_{p,\delta}^{\mu}(a,b,c)(z)]^{(-1)} * f(z)$, where

$$f_{p,\delta}^{\mu}(a,b,c)(z) = (1 - \mu + \delta)z^p {}_2F_1(a,b;c;z) + (\mu - \delta)z[z^p {}_2F_1(a,b;c;z)]' + \mu\delta z^2 [z^p {}_2F_1(a,b;c;z)]''$$

$(p \in \mathbb{N} = \{1, 2, \dots\}; \mu, \delta \geq 0; z \in \mathbb{U}).$

By using the operator $\mathcal{I}_{p,n,\delta}^{\lambda,\mu}(a,b,c)$, we investigate some subordination and superordination preserving properties for certain classes of analytic and multivalent functions in \mathbb{U} . Various sandwich-type results for these multivalent functions are also obtained.

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1. Introduction

Let $\mathcal{H}(\mathbb{U})$ denote the class of analytic functions in the open unit disk $\mathbb{U} = \{z \in \mathbb{C} : |z| < 1\}$. For $a \in \mathbb{C}$ and $n \in \mathbb{N} = \{1, 2, \dots\}$, let

$$\mathcal{H}[a,n] = \{f \in \mathcal{H}(\mathbb{U}) : f(z) = a + a_n z^n + a_{n+1} z^{n+1} + \dots\}.$$

Let f and g be two members of $\mathcal{H}(\mathbb{U})$. The function f is said to be subordinate to g , or g is said to be superordinate to f , if there exists a Schwarz function ω , analytic in \mathbb{U} with $\omega(0) = 0$ and $|\omega(z)| < 1 (z \in \mathbb{U})$, such that $f(z) = g(\omega(z)) (z \in \mathbb{U})$. In such a case, we write $f \prec g$ or

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$f(z) \prec g(z) (z \in \mathbb{U})$. Furthermore, if the function g is univalent in \mathbb{U} , then we have (see [1,2]):

$$f \prec g (z \in \mathbb{U}) \iff f(0) = g(0) \text{ and } f(\mathbb{U}) \subset g(\mathbb{U}).$$

Definition 1.1 (see [1]). Let $\phi : \mathbb{C}^2 \rightarrow \mathbb{C}$ and let h be univalent in \mathbb{U} . If p is analytic in \mathbb{U} and satisfies the following differential subordination

$$\phi(p(z), zp'(z)) \prec h(z) \quad (z \in \mathbb{U}), \tag{1.1}$$

then p is called a solution of the differential subordination (1.1). The univalent function q is called a dominant of the solutions of the differential subordination (1.1), if $p \prec q$ for all p satisfying (1.1). A dominant \tilde{q} that satisfies $\tilde{q} \prec q$ for all dominants q of (1.1) is said to be the best dominant.

Definition 1.2 (see [3]). Let $\varphi : \mathbb{C}^2 \rightarrow \mathbb{C}$ and let h be univalent in \mathbb{U} . If p and $\varphi(p(z), zp'(z))$ are univalent in \mathbb{U} and satisfy the following differential superordination

$$h(z) \prec \varphi(p(z), zp'(z)) \quad (z \in \mathbb{U}), \tag{1.2}$$

then p is called a solution of the differential superordination (1.2). An analytic function q is called a subordination of the solutions of the differential superordination (1.2), if $q \prec p$ for all p satisfying (1.2). A univalent subordination \tilde{q} that satisfies $q \prec \tilde{q}$ for all subordinations q of (1.2) is said to be the best subordination.

Definition 1.3 (see [3]). We denote by \mathcal{Q} the class of functions f that are analytic and injective on $\overline{\mathbb{U}} \setminus E(f)$, where

$$E(f) = \left\{ \xi : \xi \in \partial\mathbb{U} \text{ and } \lim_{z \rightarrow \xi} f(z) = \infty \right\},$$

and are such that $f'(\xi) \neq 0 (\xi \in \partial\mathbb{U} \setminus E(f))$.

Let $\mathcal{A}_n(p)$ denote the class of all analytic functions of the form

$$f(z) = z^p + \sum_{k=n}^{\infty} a_{p+k} z^{p+k} \quad (p, n \in \mathbb{N}; z \in \mathbb{U}),$$

and let $\mathcal{A}_1(p) = \mathcal{A}(p)$.

For $f \in \mathcal{A}(p)$, we denote by $\mathcal{D}^{n+p-1} : \mathcal{A}(p) \rightarrow \mathcal{A}(p)$ the operator defined by

$$\mathcal{D}^{n+p-1} f(z) = \frac{z^p}{(1-z)^{n+p}} * f(z) \quad (n > -p)$$

or, equivalently, by

$$\mathcal{D}^{n+p-1} f(z) = \frac{z^p (z^{n-1} f(z))^{(n+p-1)}}{(n+p-1)!},$$

where n is any integer greater than $-p$ and the symbol $*$ stands for the Hadamard product (or convolution). The operator \mathcal{D}^{n+p-1} with $p = 1$ was introduced by Ruscheweyh [4], and \mathcal{D}^{n+p-1} was introduced by Goel and Sohi [5]. The operator \mathcal{D}^{n+p-1} is called as the Ruscheweyh derivative of $(n + p - 1)$ th order.

Recently, analogous to \mathcal{D}^{n+p-1} , Liu and Noor [6] introduced an integral operator $\mathcal{I}_{n,p} : \mathcal{A}(p) \rightarrow \mathcal{A}(p)$ as below.

Let $f_{n,p}(z) = z^p/(1-z)^{n+p} (n > -p)$, and let $f_{n,p}^{(\dagger)}(z)$ be defined such that

$$f_{n,p}(z) * f_{n,p}^{(\dagger)}(z) = \frac{z^p}{(1-z)^{p+1}}.$$

Then

$$\mathcal{I}_{n,p} f(z) = f_{n,p}^{(\dagger)}(z) * f(z) = \left(\frac{z^p}{(1-z)^{n+p}} \right)^{(\dagger)} * f(z) \quad (f \in \mathcal{A}(p)). \tag{1.3}$$

We note that $\mathcal{I}_{0,n} f(z) = z^f(z)/p$ and $\mathcal{I}_{1,n} f(z) = f(z)$. Also, the operator $\mathcal{I}_{n,p}$ defined by (1.3) is called the Noor integral operator $(n + p - 1)$ -th order [6]. For $p = 1$, the operator $\mathcal{I}_{n,1} \equiv \mathcal{I}_n$ was introduced by Noor [7] and Noor and Noor [8], which is an important operator in defining several classes of analytic functions. In recent years, it has been shown that Noor integral operator has fundamental and significant applications in analytic function theory. For the properties and applications of the Noor integral operator, see, for example, [9–13].

For real or complex numbers a, b, c other than $0, -1, -2, \dots$, the Gauss hypergeometric function ${}_2F_1(a, b; c; z)$ is defined by

$${}_2F_1(a, b; c; z) = \sum_{k=0}^{\infty} \frac{(a)_k (b)_k}{(c)_k} \frac{z^k}{k!}, \tag{1.4}$$

where $(v)_k$ denotes the Pochhammer symbol defined, in terms of Gamma function, by

$$(v)_k = \frac{\Gamma(v+k)}{\Gamma(v)} = \begin{cases} 1 & (k=0), \\ v(v+1) \cdots (v+k-1) & (k \in \mathbb{N}). \end{cases}$$

Since the series in (1.4) converges absolutely for all $z \in \mathbb{U}$, so that it represents an analytic function in \mathbb{U} .

We now introduce a function $f_{p,\delta}^\mu(a, b, c)(z)$ defined by

$$\begin{aligned} f_{p,\delta}^\mu(a, b, c)(z) &= (1-\mu+\delta)z^p {}_2F_1(a, b; c; z) \\ &+ (\mu-\delta)z[z^p {}_2F_1(a, b; c; z)]' \\ &+ \mu\delta z^2[z^p {}_2F_1(a, b; c; z)]'' \quad (p \in \mathbb{N}; \mu, \delta \geq 0; z \in \mathbb{U}). \end{aligned}$$

In its special case when $p = 1$ and $\delta = 0$, we obtain $f_{1,0}^\mu(a, b, c)(z) = f_\mu(a, b, c)(z)$ studied by Shukla and Shukla [14].

On the other hand, we define a function $[f_{p,\delta}^\mu(a, b, c)(z)]^{(-1)}$ by means of Hadamard product (or convolution):

$$f_{p,\delta}^\mu(a, b, c)(z) * [f_{p,\delta}^\mu(a, b, c)(z)]^{(-1)} = \frac{z^p}{(1-z)^{\lambda+p}} \quad (\mu, \delta \geq 0; \lambda > -p),$$

which leads us to the following family of linear operators

$$\mathcal{I}_{p,n,\delta}^{\lambda,\mu}(a, b, c) f(z) = [f_{p,\delta}^\mu(a, b, c)(z)]^{(-1)} * f(z), \tag{1.5}$$

where a, b, c are real numbers other than $0, -1, -2, \dots$, and $f \in \mathcal{A}_n(p)$.

We observe that the operator $\mathcal{I}_{p,n,\delta}^{\lambda,\mu}(a, b, c)$ generalizes several previously studied familiar operators, and we will show some of the interesting particular cases as follows.

- (i) $\mathcal{I}_{1,1,0}^{\lambda,\mu}(a, b, c) = \mathcal{I}_\mu^\lambda(a, b, c)$, where $\mathcal{I}_\mu^\lambda(a, b, c)$ is the Srivastava et al. operator [15];
- (ii) $\mathcal{I}_{p,n,0}^{\lambda,0}(a, b, c) = \mathcal{I}_{p,n}^\lambda(a, b, c)$, where the operator $\mathcal{I}_{p,n}^\lambda(a, b, c)$ was introduced by Fu and Liu [16];

- (iii) $\mathcal{I}_{1,1,0}^{\lambda,0}(a, b, c) = \mathcal{I}_\lambda(a, b, c)$, where the operator $\mathcal{I}_\lambda(a, b, c)$ was introduced by Noor [17];
- (iv) $\mathcal{I}_{p,1,0}^{\lambda,0}(a, 1, c) = \mathcal{I}_p^\lambda(a, c)$, where $\mathcal{I}_p^\lambda(a, c)$ is the Cho et al. operator [18];
- (v) $\mathcal{I}_{p,1,0}^{1,0}(n + p, c, c) = \mathcal{I}_{n,p}$, where the operator $\mathcal{I}_{n,p}$ was introduced by Patel and Cho [19];
- (vi) $\mathcal{I}_{1,1,0}^{n,0}(a, n + 1, a) = \mathcal{I}_n$, where \mathcal{I}_n is the Noor integral operator [7].

It is easily verified from the definition (1.5) that

$$\begin{aligned} \mathcal{I}_{p,n,0}^{\lambda,0}(a, \lambda + p, a)f(z) &= f(z) \text{ and } \mathcal{I}_{p,n,0}^{1,0}(a, p, a)f(z) = \frac{zf'(z)}{p}, \\ z\left(\mathcal{I}_{p,n,\delta}^{\lambda,\mu}(a, b, c)f(z)\right)' &= (\lambda + p)\mathcal{I}_{p,n,\delta}^{\lambda+1,\mu}(a, b, c)f(z) - \lambda\mathcal{I}_{p,n,\delta}^{\lambda,\mu}(a, b, c)f(z), \quad (1.6) \\ z\left(\mathcal{I}_{p,n,\delta}^{\lambda,\mu}(a + 1, b, c)f(z)\right)' &= a\mathcal{I}_{p,n,\delta}^{\lambda,\mu}(a, b, c)f(z) - (a - p)\mathcal{I}_{p,n,\delta}^{\lambda,\mu}(a + 1, b, c)f(z). \quad (1.7) \end{aligned}$$

With the help of the principle of subordination, various subordination preserving properties involving certain integral operators for analytic functions in \mathbb{U} were investigated by Bulboacă [20], Miller et al. [21], and Owa and Srivastava [22]. Moreover, Miller and Mocanu [3] considered differential subordinations, as the dual problem of differential subordinations (see also [23]), while some other interesting results involving differential subordination and subordination, the interested reader may refer to, for example, [24–31]. In the present paper, we obtain some subordination and superordination preserving properties for the operator $\mathcal{I}_{p,n,\delta}^{\lambda,\mu}(a, b, c)$ defined by (1.5). Also, we derive several sandwich-type results for these multivalent functions.

2. Preliminaries

In order to establish our main results, we shall require the following lemmas.

Lemma 2.1 (see [32]). *Suppose that the function $H : \mathbb{C}^2 \rightarrow \mathbb{C}$ satisfies the following condition*

$$\begin{aligned} \operatorname{Re}\{H(is, t)\} &\leq 0 \\ \text{for all real } s \text{ and } t &\leq -\frac{n(1+s^2)}{2} \ (n \in \mathbb{N}). \text{ If the function } \\ p(z) = 1 + p_n z^n + \dots &\text{ is analytic in } \mathbb{U} \text{ and} \\ \operatorname{Re}\{H(p(z), zp'(z))\} &> 0 \ (z \in \mathbb{U}), \\ \text{then } \operatorname{Re}\{p(z)\} &> 0 \text{ for } z \in \mathbb{U}. \end{aligned}$$

Lemma 2.2 (see [33]). *Let $\kappa, \gamma \in \mathbb{C}$ with $\kappa \neq 0$ and let $h \in \mathcal{H}(\mathbb{U})$ with $h(0) = \varrho$. If $\operatorname{Re}\{\kappa h(z) + \gamma\} > 0$ ($z \in \mathbb{U}$), then the solution of the following differential equation*

$$q(z) + \frac{zq'(z)}{\kappa q(z) + \gamma} = h(z) \quad (z \in \mathbb{U}; q(0) = \varrho)$$

is analytic in \mathbb{U} and satisfies the inequality given by $\operatorname{Re}\{\kappa q(z) + \gamma\} > 0$ for $z \in \mathbb{U}$.

Lemma 2.3 (see [1]). *Let $p \in \mathcal{Q}$ with $\phi(0) = a$ and let the function $q(z) = a + a_n z^n + \dots$ be analytic in \mathbb{U} with $q(z) \neq a$ and $n \in \mathbb{N}$. If q is not subordinate to p , then there exist points*

$$z_0 = r_0 e^{i\theta} \in \mathbb{U} \text{ and } \xi_0 \in \partial\mathbb{U} \setminus E(f),$$

for which

$$q(\mathbb{U}_{r_0}) \subset p(\mathbb{U}), q(z_0) = p(z_0) \text{ and } z_0 q'(z_0) = m \xi_0 p'(\xi_0) \quad (m \geq n),$$

where $\mathbb{U}_{r_0} = \{z \in \mathbb{C} : |z| < r_0\}$.

A function $L(z, t) : \mathbb{U} \times [0, \infty) \rightarrow \mathbb{C}$ is called a subordination chain (or Löwner chain) if $L(\cdot, t)$ is analytic and univalent in \mathbb{U} for all $t \geq 0$, and $L(z, t_1) \prec L(z, t_2)$ ($z \in \mathbb{U}; 0 \leq t_1 \leq t_2$).

Lemma 2.4 (see [3]). *Let $q \in \mathcal{H}[a, 1]$ and $\Phi : \mathbb{C}^2 \rightarrow \mathbb{C}$. Also let*

$$\begin{aligned} \Phi(q(z), zq'(z)) &= h(z) \quad (z \in \mathbb{U}). \\ \text{If } L(z, t) = \Phi(q(z), tzq'(z)) &\text{ is a subordination chain and} \\ p \in \mathcal{H}[a, 1] \cap \mathcal{Q}, &\text{ then} \\ h(z) \prec \Phi(p(z), zp'(z)) &\quad (z \in \mathbb{U}), \end{aligned}$$

implies that $q(z) \prec p(z)$. Furthermore, if $\Phi(q(z), zq'(z)) = h(z)$ has a univalent solution $q \in \mathcal{Q}$, then q is the best subordinant.

Lemma 2.5 (see [34]). *Let the function $L(z, t) = a_1(t)z + a_2(t)z^2 + \dots$ with $a_1(t) \neq 0$ for all $t \geq 0$, and $\lim_{t \rightarrow +\infty} |a_1(t)| = +\infty$. Suppose that $L(\cdot, t)$ is analytic in \mathbb{U} for all $t \geq 0$, $L(\cdot, t)$ is continuously differentiable on $[0, \infty)$ for all $z \in \mathbb{U}$. If $L(z, t)$ satisfies*

$$\operatorname{Re}\left\{\frac{z\partial L(z, t)/\partial z}{\partial L(z, t)/\partial t}\right\} > 0 \quad (z \in \mathbb{U}; t \geq 0)$$

and

$$|L(z, t)| \leq K_0 |a_1(t)| \quad (|z| < r_0 < 1; t \geq 0)$$

for some positive constants K_0 and r_0 , then $L(z, t)$ is a subordination chain.

3. Main results

First of all, we begin by proving the following subordination theorem involving the operator $\mathcal{I}_{p,n,\delta}^{\lambda,\mu}(a, b, c)$. Unless otherwise mentioned, we assume throughout this paper that $a, b, c \in \mathbb{R} \setminus \{0, -1, -2, \dots\}; \lambda > -p; \mu, \delta \geq 0; 0 < \alpha \leq 1; \beta > 0; p, n \in \mathbb{N}$ and $z \in \mathbb{U}$.

Theorem 3.1. *Let $f, g \in \mathcal{A}_n(p)$ and suppose that*

$$\operatorname{Re}\left\{1 + \frac{z\phi''(z)}{\phi'(z)}\right\} > -\sigma, \quad (3.1)$$

where

$$\begin{aligned} \phi(z) &= (1 - \alpha) \left(\frac{\mathcal{I}_{p,n,\delta}^{\lambda,\mu}(a, b, c)g(z)}{z^p}\right)^\beta \\ &\quad + \alpha \left(\frac{\mathcal{I}_{p,n,\delta}^{\lambda+1,\mu}(a, b, c)g(z)}{\mathcal{I}_{p,n,\delta}^{\lambda,\mu}(a, b, c)g(z)}\right) \left(\frac{\mathcal{I}_{p,n,\delta}^{\lambda,\mu}(a, b, c)g(z)}{z^p}\right)^\beta \end{aligned}$$

and

$$\sigma = \frac{\alpha^2 + \beta^2(\lambda + p)^2 - |\alpha^2 - \beta^2(\lambda + p)^2|}{4\alpha\beta(\lambda + p)}. \quad (3.2)$$

Then the following subordination condition

$$(1 - \alpha) \left(\frac{\mathcal{I}_{p,n,\delta}^{\lambda,\mu}(a,b,c)f(z)}{z^p} \right)^\beta + \alpha \left(\frac{\mathcal{I}_{p,n,\delta}^{\lambda+1,\mu}(a,b,c)f(z)}{z^p} \right)^\beta \prec \phi(z) \tag{3.3}$$

implies that

$$\left(\frac{\mathcal{I}_{p,n,\delta}^{\lambda,\mu}(a,b,c)f(z)}{z^p} \right)^\beta \prec \left(\frac{\mathcal{I}_{p,n,\delta}^{\lambda,\mu}(a,b,c)g(z)}{z^p} \right)^\beta.$$

Moreover, the function $\left(\frac{\mathcal{I}_{p,n,\delta}^{\lambda,\mu}(a,b,c)g(z)}{z^p} \right)^\beta$ is the best dominant.

Proof. Let us define the functions F and G , respectively, by

$$F(z) = \left(\frac{\mathcal{I}_{p,n,\delta}^{\lambda,\mu}(a,b,c)f(z)}{z^p} \right)^\beta \text{ and } G(z) = \left(\frac{\mathcal{I}_{p,n,\delta}^{\lambda,\mu}(a,b,c)g(z)}{z^p} \right)^\beta. \tag{3.4}$$

We first prove that, if the function q is defined by

$$q(z) = 1 + \frac{zG''(z)}{G'(z)} \quad (z \in \mathbb{U}), \tag{3.5}$$

then $Re\{q(z)\} > 0$ for $z \in \mathbb{U}$.

Taking the logarithmic differentiation on both sides of the second equation in (3.4) and using (1.6) for $g \in \mathcal{A}_n(p)$, we have

$$\phi(z) = G(z) + \frac{\alpha z G'(z)}{\beta(\lambda + p)}. \tag{3.6}$$

Differentiating both sides of (3.6) with respect to z yields

$$\phi'(z) = \left(1 + \frac{\alpha}{\beta(\lambda + p)} \right) G'(z) + \frac{\alpha z G''(z)}{\beta(\lambda + p)}. \tag{3.7}$$

Combining (3.5) and (3.7), we easily get

$$1 + \frac{z\phi''(z)}{\phi'(z)} = q(z) + \frac{zq'(z)}{q(z) + \beta(\lambda + p)/\alpha} = h(z) \quad (z \in \mathbb{U}). \tag{3.8}$$

Thus, from (3.1) and (3.8), we see that

$$Re\left\{ h(z) + \frac{\beta(\lambda + p)}{\alpha} \right\} > 0 \quad (z \in \mathbb{U}).$$

Also, in view of Lemma 2.2, we conclude that the differential Eq. (3.8) has a solution $q \in \mathcal{H}(\mathbb{U})$ with $q(0) = h(0) = 1$.

Let us put

$$H(u, v) = u + \frac{v}{u + \beta(\lambda + p)/\alpha} + \sigma, \tag{3.9}$$

where σ is given by (3.2). From (3.1) and (3.8), together with (3.9), we obtain

$$Re\{H(q(z), zq'(z))\} > 0 \quad (z \in \mathbb{U}).$$

Now, we proceed to show that

$$Re\{H(is, t)\} \leq 0 \quad \left(s \in \mathbb{R}; t \leq -\frac{1+s^2}{2} \right). \tag{3.10}$$

In fact, from (3.9), we have

$$\begin{aligned} Re\{H(is, t)\} &= Re\left\{ is + \frac{t}{is + \beta(\lambda + p)/\alpha} + \sigma \right\} \\ &= \frac{t\alpha\beta(\lambda + p)}{\alpha^2 s^2 + \beta^2(\lambda + p)^2} + \sigma \leq -\frac{E_\sigma(s)}{2[\alpha^2 s^2 + \beta^2(\lambda + p)^2]}, \end{aligned}$$

where

$$E_\sigma(s) = [\alpha\beta(\lambda + p) - 2\sigma\alpha^2]s^2 - 2\sigma\beta^2(\lambda + p)^2 + \alpha\beta(\lambda + p). \tag{3.11}$$

For σ given by (3.2), we can prove easily that the expression $E_\sigma(s)$ in (3.11) is greater than or equal to zero, which implies that (3.10) holds true. Therefore, by using Lemma 2.1, we conclude that $Re\{q(z)\} > 0$ for $z \in \mathbb{U}$, that is, that the function G defined by (3.4) is convex (univalent) in \mathbb{U} .

Next, we prove that $F \prec G(z \in \mathbb{U})$ holds for the functions F and G defined by (3.4). Without loss of generality, we assume that G is analytic and univalent on $\overline{\mathbb{U}}$ and that $G'(\xi) \neq 0$ for $|\xi| = 1$. Otherwise, we replace F and G by $F(\rho z)$ and $G(\rho z)$, respectively, with $0 < \rho < 1$. These functions satisfy the conditions of the theorem on $\overline{\mathbb{U}}$, and we need to prove that $F(\rho z) \prec G(\rho z)$ for $0 < \rho < 1$, which enables us to obtain $F \prec G$ by letting $\rho \rightarrow 1^-$.

Let us define the function $L(z, t)$ by

$$L(z, t) = G(z) + \frac{\alpha(1+t)}{\beta(\lambda + p)} zG'(z) \quad (t \geq 0; z \in \mathbb{U}). \tag{3.12}$$

Then,

$$\begin{aligned} \frac{\partial L(z, t)}{\partial z} \Big|_{z=0} &= G'(0) \left(1 + \frac{\alpha(1+t)}{\beta(\lambda + p)} \right) = 1 + \frac{\alpha(1+t)}{\beta(\lambda + p)} \neq 0 \\ &(t \geq 0; z \in \mathbb{U}), \end{aligned}$$

and this show that the function $L(z, t) = a_1(t)z + a_2(t)z^2 + \dots$ satisfies the conditions $a_1(t) \neq 0$ for all $t \geq 0$ and $\lim_{t \rightarrow +\infty} |a_1(t)| = +\infty$.

From the definition (3.12) and for all $t \geq 0$, we have

$$\begin{aligned} \frac{|L(z, t)|}{|a_1(t)|} &= \frac{\left| G(z) + \frac{\alpha(1+t)}{\beta(\lambda + p)} zG'(z) \right|}{1 + \frac{\alpha(1+t)}{\beta(\lambda + p)}} \\ &\leq \frac{|G(z)| + \frac{\alpha(1+t)}{\beta(\lambda + p)} |zG'(z)|}{1 + \frac{\alpha(1+t)}{\beta(\lambda + p)}}. \end{aligned} \tag{3.13}$$

Since the function G is convex in \mathbb{U} , so the following well-known growth and distortion sharp inequalities (see [35]) are true:

$$\frac{r}{1+r} \leq |G(z)| \leq \frac{r}{1-r} \quad (|z| \leq r), \tag{3.14}$$

$$\frac{1}{(1+r)^2} \leq |G'(z)| \leq \frac{1}{(1-r)^2} \quad (|z| \leq r). \tag{3.15}$$

By using (3.14) and (3.15) in (3.13), we deduce that

$$\begin{aligned} \frac{|L(z, t)|}{|a_1(t)|} &\leq \frac{r}{(1-r)^2} \frac{\alpha(1+t) + \beta(\lambda + p)(1-r)}{\alpha(1+t) + \beta(\lambda + p)} \leq \frac{r}{(1-r)^2} \\ &(|z| \leq r; t \geq 0) \end{aligned}$$

and thus, the second assumption of Lemma 2.5 holds.

Moreover, we have

$$\begin{aligned} Re\left\{ \frac{z\partial L(z, t)/\partial z}{\partial L(z, t)/\partial t} \right\} &= Re\left\{ \beta(\lambda + p) + \alpha(1+t) \left(1 + \frac{zG''(z)}{G'(z)} \right) \right\} \\ &> 0 \quad (t \geq 0), \end{aligned}$$

because G is convex in \mathbb{U} . Hence, by virtue of Lemma 2.5, we see that $L(z, t)$ is a subordination chain. We notice from the definition of subordination chain that

$$\phi(z) = G(z) + \frac{\alpha z G'(z)}{\beta(\lambda + p)} = L(z, 0),$$

and

$$L(z, 0) \prec L(z, t) \quad (t \geq 0),$$

which implies that

$$L(\xi, t) \notin L(\mathbb{U}, 0) = \phi(\mathbb{U}) \quad (\xi \in \partial\mathbb{U}; t \geq 0). \tag{3.16}$$

Now, we suppose that F is not subordinate G , then by Lemma 2.3, there exist two points $z_0 \in \mathbb{U}$ and $\xi_0 \in \partial\mathbb{U}$, such that

$$F(z_0) = G(\xi_0) \text{ and } z_0 F'(z_0) = (1 + t)\xi_0 G'(\xi_0) \quad (t \geq 0).$$

Thus, by means of the subordination condition (3.3), we have

$$\begin{aligned} L(\xi_0, t) &= G(\xi_0) + \frac{\alpha(1+t)\xi_0 G'(\xi_0)}{\beta(\lambda + p)} = F(z_0) + \frac{\alpha z_0 F'(z_0)}{\beta(\lambda + p)} \\ &= (1 - \alpha) \left(\frac{\mathcal{I}_{p,n,\delta}^{\lambda,\mu}(a,b,c)f(z_0)}{z_0^p} \right)^\beta + \alpha \left(\frac{\mathcal{I}_{p,n,\delta}^{\lambda+1,\mu}(a,b,c)f(z_0)}{\mathcal{I}_{p,n,\delta}^{\lambda,\mu}(a,b,c)f(z_0)} \right) \left(\frac{\mathcal{I}_{p,n,\delta}^{\lambda,\mu}(a,b,c)f(z_0)}{z_0^p} \right)^\beta \in \phi(\mathbb{U}), \end{aligned}$$

which contradicts to (3.16). Therefore, we conclude that $F \prec G$. Considering $F = G$, we know that the function G is the best dominant. This completes the proof of Theorem 3.1. \square

By applying the similar method as in the proof of Theorem 3.1 and using (1.7), we easily get the following result.

Corollary 3.1. *Let $f, g \in \mathcal{A}_n(p)$ and suppose that*

$$Re \left\{ 1 + \frac{z\psi''(z)}{\psi'(z)} \right\} > -\tau,$$

where

$$\begin{aligned} \psi(z) &= (1 - \alpha) \left(\frac{\mathcal{I}_{p,n,\delta}^{\lambda,\mu}(a+1,b,c)g(z)}{z^p} \right)^\beta \\ &\quad + \alpha \left(\frac{\mathcal{I}_{p,n,\delta}^{\lambda,\mu}(a,b,c)g(z)}{\mathcal{I}_{p,n,\delta}^{\lambda,\mu}(a+1,b,c)g(z)} \right) \left(\frac{\mathcal{I}_{p,n,\delta}^{\lambda,\mu}(a+1,b,c)g(z)}{z^p} \right)^\beta \end{aligned}$$

and

$$\tau = \frac{\alpha^2 + \beta^2 a^2 - |\alpha^2 - \beta^2 a^2|}{4\alpha\beta a} \quad (a > 0). \tag{3.17}$$

Then the following subordination condition

$$\begin{aligned} (1 - \alpha) \left(\frac{\mathcal{I}_{p,n,\delta}^{\lambda,\mu}(a+1,b,c)f(z)}{z^p} \right)^\beta \\ + \alpha \left(\frac{\mathcal{I}_{p,n,\delta}^{\lambda,\mu}(a,b,c)f(z)}{\mathcal{I}_{p,n,\delta}^{\lambda,\mu}(a+1,b,c)f(z)} \right) \left(\frac{\mathcal{I}_{p,n,\delta}^{\lambda,\mu}(a+1,b,c)f(z)}{z^p} \right)^\beta \\ \prec \psi(z) \end{aligned}$$

implies that

$$\left(\frac{\mathcal{I}_{p,n,\delta}^{\lambda,\mu}(a+1,b,c)f(z)}{z^p} \right)^\beta \prec \left(\frac{\mathcal{I}_{p,n,\delta}^{\lambda,\mu}(a+1,b,c)g(z)}{z^p} \right)^\beta.$$

Moreover, the function $\left(\frac{\mathcal{I}_{p,n,\delta}^{\lambda,\mu}(a+1,b,c)g(z)}{z^p} \right)^\beta$ is the best dominant.

We next derive the dual result of Theorem 3.1, in the sense that subordinations are replaced by superordinations.

Theorem 3.2. *Let $f, g \in \mathcal{A}_n(p)$ and suppose that*

$$Re \left\{ 1 + \frac{z\phi''(z)}{\phi'(z)} \right\} > -\sigma,$$

where

$$\begin{aligned} \phi(z) &= (1 - \alpha) \left(\frac{\mathcal{I}_{p,n,\delta}^{\lambda,\mu}(a,b,c)g(z)}{z^p} \right)^\beta \\ &\quad + \alpha \left(\frac{\mathcal{I}_{p,n,\delta}^{\lambda+1,\mu}(a,b,c)g(z)}{\mathcal{I}_{p,n,\delta}^{\lambda,\mu}(a,b,c)g(z)} \right) \left(\frac{\mathcal{I}_{p,n,\delta}^{\lambda,\mu}(a,b,c)g(z)}{z^p} \right)^\beta \end{aligned}$$

and σ is given by (3.2). If the function

$$\begin{aligned} (1 - \alpha) \left(\frac{\mathcal{I}_{p,n,\delta}^{\lambda,\mu}(a,b,c)f(z)}{z^p} \right)^\beta \\ + \alpha \left(\frac{\mathcal{I}_{p,n,\delta}^{\lambda+1,\mu}(a,b,c)f(z)}{\mathcal{I}_{p,n,\delta}^{\lambda,\mu}(a,b,c)f(z)} \right) \left(\frac{\mathcal{I}_{p,n,\delta}^{\lambda,\mu}(a,b,c)f(z)}{z^p} \right)^\beta \end{aligned}$$

is univalent in \mathbb{U} and $\left(\frac{\mathcal{I}_{p,n,\delta}^{\lambda,\mu}(a,b,c)f(z)}{z^p} \right)^\beta \in \mathcal{H}[1, 1] \cap \mathcal{Q}$. Then the following superordination condition

$$\begin{aligned} \phi(z) \prec (1 - \alpha) \left(\frac{\mathcal{I}_{p,n,\delta}^{\lambda,\mu}(a,b,c)f(z)}{z^p} \right)^\beta \\ + \alpha \left(\frac{\mathcal{I}_{p,n,\delta}^{\lambda+1,\mu}(a,b,c)f(z)}{\mathcal{I}_{p,n,\delta}^{\lambda,\mu}(a,b,c)f(z)} \right) \left(\frac{\mathcal{I}_{p,n,\delta}^{\lambda,\mu}(a,b,c)f(z)}{z^p} \right)^\beta \end{aligned}$$

implies that

$$\left(\frac{\mathcal{I}_{p,n,\delta}^{\lambda,\mu}(a,b,c)g(z)}{z^p} \right)^\beta \prec \left(\frac{\mathcal{I}_{p,n,\delta}^{\lambda,\mu}(a,b,c)f(z)}{z^p} \right)^\beta.$$

Moreover, the function $\left(\frac{\mathcal{I}_{p,n,\delta}^{\lambda,\mu}(a,b,c)g(z)}{z^p} \right)^\beta$ is the best subordination.

Proof. Let us define the functions F and G just as (3.4). We first observe that, if the function q is defined by (3.5), then we obtain from (3.6) that

$$\phi(z) = G(z) + \frac{\alpha z G'(z)}{\beta(\lambda + p)} = \Phi(G(z), zG'(z)). \tag{3.18}$$

By using the same method as in the proof of Theorem 3.1, we can prove that $Re\{q(z)\} > 0$ for $z \in \mathbb{U}$. That is, the function G defined by (3.4) is convex (univalent) in \mathbb{U} .

Next, we will show that $G \prec F$. For this purpose, we consider the function $L(z, t)$ defined by

$$L(z, t) = G(z) + \frac{\alpha t}{\beta(\lambda + p)} zG'(z) \quad (t \geq 0; z \in \mathbb{U}).$$

Since the function G is convex in \mathbb{U} , so we can prove easily that $L(z, t)$ is a subordination chain as in the proof of Theorem 3.1. Hence, by Lemma 2.4, we conclude that $G \prec F$. Furthermore, since the differential Eq. (3.18) has the univalent solution G , it is the best subordination of the given differential superordination. We thus complete the proof of Theorem 3.2. \square

By applying the similar method used in the proof of Theorem 3.2, in conjunction with (1.7), we easily obtain the following result.

Corollary 3.2. *Let $f, g \in \mathcal{A}_n(p)$ and suppose that*

$$Re\left\{1 + \frac{z\psi''(z)}{\psi'(z)}\right\} > -\tau,$$

where

$$\psi(z) = (1 - \alpha)\left(\frac{\mathcal{I}_{p,n,\delta}^{\lambda,\mu}(a+1, b, c)g(z)}{z^p}\right)^\beta + \alpha\left(\frac{\mathcal{I}_{p,n,\delta}^{\lambda,\mu}(a, b, c)g(z)}{\mathcal{I}_{p,n,\delta}^{\lambda,\mu}(a+1, b, c)g(z)}\right)\left(\frac{\mathcal{I}_{p,n,\delta}^{\lambda,\mu}(a+1, b, c)g(z)}{z^p}\right)^\beta$$

and τ is given by (3.17). If the function

$$(1 - \alpha)\left(\frac{\mathcal{I}_{p,n,\delta}^{\lambda,\mu}(a+1, b, c)f(z)}{z^p}\right)^\beta + \alpha\left(\frac{\mathcal{I}_{p,n,\delta}^{\lambda,\mu}(a, b, c)f(z)}{\mathcal{I}_{p,n,\delta}^{\lambda,\mu}(a+1, b, c)f(z)}\right)\left(\frac{\mathcal{I}_{p,n,\delta}^{\lambda,\mu}(a+1, b, c)f(z)}{z^p}\right)^\beta$$

is univalent in \mathbb{U} and $\left(\frac{\mathcal{I}_{p,n,\delta}^{\lambda,\mu}(a+1, b, c)f(z)}{z^p}\right)^\beta \in \mathcal{H}[1, 1] \cap \mathcal{Q}$. Then the following superordination condition

$$\psi(z) \prec (1 - \alpha)\left(\frac{\mathcal{I}_{p,n,\delta}^{\lambda,\mu}(a+1, b, c)f(z)}{z^p}\right)^\beta + \alpha\left(\frac{\mathcal{I}_{p,n,\delta}^{\lambda,\mu}(a, b, c)f(z)}{\mathcal{I}_{p,n,\delta}^{\lambda,\mu}(a+1, b, c)f(z)}\right)\left(\frac{\mathcal{I}_{p,n,\delta}^{\lambda,\mu}(a+1, b, c)f(z)}{z^p}\right)^\beta$$

implies that

$$\left(\frac{\mathcal{I}_{p,n,\delta}^{\lambda,\mu}(a+1, b, c)g(z)}{z^p}\right)^\beta \prec \left(\frac{\mathcal{I}_{p,n,\delta}^{\lambda,\mu}(a+1, b, c)f(z)}{z^p}\right)^\beta.$$

Moreover, the function $\left(\frac{\mathcal{I}_{p,n,\delta}^{\lambda,\mu}(a+1, b, c)g(z)}{z^p}\right)^\beta$ is the best subordination.

Combing Theorems 3.1 and 3.2, and Corollaries 3.1 and 3.2, respectively, we derive the following two sandwich-type results.

Theorem 3.3. *Let $f, g_j \in \mathcal{A}_n(p)$ ($j = 1, 2$) and suppose that*

$$Re\left\{1 + \frac{z\phi_j''(z)}{\phi_j'(z)}\right\} > -\sigma, \tag{3.19}$$

where

$$\phi_j(z) = (1 - \alpha)\left(\frac{\mathcal{I}_{p,n,\delta}^{\lambda,\mu}(a, b, c)g_j(z)}{z^p}\right)^\beta + \alpha\left(\frac{\mathcal{I}_{p,n,\delta}^{\lambda+1,\mu}(a, b, c)g_j(z)}{\mathcal{I}_{p,n,\delta}^{\lambda,\mu}(a, b, c)g_j(z)}\right)\left(\frac{\mathcal{I}_{p,n,\delta}^{\lambda,\mu}(a, b, c)g_j(z)}{z^p}\right)^\beta$$

and σ is given by (3.2). If the function

$$(1 - \alpha)\left(\frac{\mathcal{I}_{p,n,\delta}^{\lambda,\mu}(a, b, c)f(z)}{z^p}\right)^\beta + \alpha\left(\frac{\mathcal{I}_{p,n,\delta}^{\lambda+1,\mu}(a, b, c)f(z)}{\mathcal{I}_{p,n,\delta}^{\lambda,\mu}(a, b, c)f(z)}\right)\left(\frac{\mathcal{I}_{p,n,\delta}^{\lambda,\mu}(a, b, c)f(z)}{z^p}\right)^\beta$$

is univalent in \mathbb{U} and $\left(\frac{\mathcal{I}_{p,n,\delta}^{\lambda,\mu}(a, b, c)f(z)}{z^p}\right)^\beta \in \mathcal{H}[1, 1] \cap \mathcal{Q}$. Then the following subordination relationship

$$\phi_1(z) \prec (1 - \alpha)\left(\frac{\mathcal{I}_{p,n,\delta}^{\lambda,\mu}(a, b, c)f(z)}{z^p}\right)^\beta + \alpha\left(\frac{\mathcal{I}_{p,n,\delta}^{\lambda+1,\mu}(a, b, c)f(z)}{\mathcal{I}_{p,n,\delta}^{\lambda,\mu}(a, b, c)f(z)}\right)\left(\frac{\mathcal{I}_{p,n,\delta}^{\lambda,\mu}(a, b, c)f(z)}{z^p}\right)^\beta \prec \phi_2(z)$$

implies that

$$\left(\frac{\mathcal{I}_{p,n,\delta}^{\lambda,\mu}(a, b, c)g_1(z)}{z^p}\right)^\beta \prec \left(\frac{\mathcal{I}_{p,n,\delta}^{\lambda,\mu}(a, b, c)f(z)}{z^p}\right)^\beta \prec \left(\frac{\mathcal{I}_{p,n,\delta}^{\lambda,\mu}(a, b, c)g_2(z)}{z^p}\right)^\beta.$$

Moreover, the functions $\left(\frac{\mathcal{I}_{p,n,\delta}^{\lambda,\mu}(a, b, c)g_1(z)}{z^p}\right)^\beta$ and $\left(\frac{\mathcal{I}_{p,n,\delta}^{\lambda,\mu}(a, b, c)g_2(z)}{z^p}\right)^\beta$ are, respectively, the best subordination and the best dominant.

Theorem 3.4. *Let $f, g_j \in \mathcal{A}_n(p)$ ($j = 1, 2$) and suppose that*

$$Re\left\{1 + \frac{z\psi_j''(z)}{\psi_j'(z)}\right\} > -\tau, \tag{3.20}$$

where

$$\psi_j(z) = (1 - \alpha)\left(\frac{\mathcal{I}_{p,n,\delta}^{\lambda,\mu}(a+1, b, c)g_j(z)}{z^p}\right)^\beta + \alpha\left(\frac{\mathcal{I}_{p,n,\delta}^{\lambda,\mu}(a, b, c)g_j(z)}{\mathcal{I}_{p,n,\delta}^{\lambda,\mu}(a+1, b, c)g_j(z)}\right)\left(\frac{\mathcal{I}_{p,n,\delta}^{\lambda,\mu}(a+1, b, c)g_j(z)}{z^p}\right)^\beta$$

and τ is given by (3.17). If the function

$$(1 - \alpha)\left(\frac{\mathcal{I}_{p,n,\delta}^{\lambda,\mu}(a+1, b, c)f(z)}{z^p}\right)^\beta + \alpha\left(\frac{\mathcal{I}_{p,n,\delta}^{\lambda,\mu}(a, b, c)f(z)}{\mathcal{I}_{p,n,\delta}^{\lambda,\mu}(a+1, b, c)f(z)}\right)\left(\frac{\mathcal{I}_{p,n,\delta}^{\lambda,\mu}(a+1, b, c)f(z)}{z^p}\right)^\beta$$

is univalent in \mathbb{U} and $\left(\frac{\mathcal{I}_{p,n,\delta}^{\lambda,\mu}(a+1, b, c)f(z)}{z^p}\right)^\beta \in \mathcal{H}[1, 1] \cap \mathcal{Q}$. Then the following subordination relationship

$$\psi_1(z) \prec (1 - \alpha)\left(\frac{\mathcal{I}_{p,n,\delta}^{\lambda,\mu}(a+1, b, c)f(z)}{z^p}\right)^\beta + \alpha\left(\frac{\mathcal{I}_{p,n,\delta}^{\lambda,\mu}(a, b, c)f(z)}{\mathcal{I}_{p,n,\delta}^{\lambda,\mu}(a+1, b, c)f(z)}\right)\left(\frac{\mathcal{I}_{p,n,\delta}^{\lambda,\mu}(a+1, b, c)f(z)}{z^p}\right)^\beta \prec \psi_2(z)$$

implies that

$$\left(\frac{\mathcal{I}_{p,n,\delta}^{\lambda,\mu}(a+1,b,c)g_1(z)}{z^p}\right)^\beta \prec \left(\frac{\mathcal{I}_{p,n,\delta}^{\lambda,\mu}(a+1,b,c)f(z)}{z^p}\right)^\beta \prec \left(\frac{\mathcal{I}_{p,n,\delta}^{\lambda,\mu}(a+1,b,c)g_2(z)}{z^p}\right)^\beta.$$

Moreover, the functions $\left(\frac{\mathcal{I}_{p,n,\delta}^{\lambda,\mu}(a+1,b,c)g_1(z)}{z^p}\right)^\beta$ and $\left(\frac{\mathcal{I}_{p,n,\delta}^{\lambda,\mu}(a+1,b,c)g_2(z)}{z^p}\right)^\beta$ are, respectively, the best subordination and the best dominant.

Since the assumption of Theorem 3.3 that the functions

$$(1-\alpha)\left(\frac{\mathcal{I}_{p,n,\delta}^{\lambda,\mu}(a,b,c)f(z)}{z^p}\right)^\beta + \alpha\left(\frac{\mathcal{I}_{p,n,\delta}^{\lambda+1,\mu}(a,b,c)f(z)}{\mathcal{I}_{p,n,\delta}^{\lambda,\mu}(a,b,c)f(z)}\right)\left(\frac{\mathcal{I}_{p,n,\delta}^{\lambda,\mu}(a,b,c)f(z)}{z^p}\right)^\beta$$

and

$$\left(\frac{\mathcal{I}_{p,n,\delta}^{\lambda,\mu}(a,b,c)f(z)}{z^p}\right)^\beta$$

need to be univalent in \mathbb{U} , is not so easy to check, we will replace these conditions by another simple condition in the following result.

Corollary 3.3. Let $f, g_j \in \mathcal{A}_n(p) (j = 1, 2)$. Suppose that the condition (3.19) is satisfied and

$$Re\left\{1 + \frac{z\varphi''(z)}{\varphi'(z)}\right\} > -\sigma, \tag{3.21}$$

where

$$\varphi(z) = (1-\alpha)\left(\frac{\mathcal{I}_{p,n,\delta}^{\lambda,\mu}(a,b,c)f(z)}{z^p}\right)^\beta + \alpha\left(\frac{\mathcal{I}_{p,n,\delta}^{\lambda+1,\mu}(a,b,c)f(z)}{\mathcal{I}_{p,n,\delta}^{\lambda,\mu}(a,b,c)f(z)}\right)\left(\frac{\mathcal{I}_{p,n,\delta}^{\lambda,\mu}(a,b,c)f(z)}{z^p}\right)^\beta$$

and σ is given by (3.2). Then the following subordination relationship

$$\phi_1(z) \prec (1-\alpha)\left(\frac{\mathcal{I}_{p,n,\delta}^{\lambda,\mu}(a,b,c)f(z)}{z^p}\right)^\beta + \alpha\left(\frac{\mathcal{I}_{p,n,\delta}^{\lambda+1,\mu}(a,b,c)f(z)}{\mathcal{I}_{p,n,\delta}^{\lambda,\mu}(a,b,c)f(z)}\right)\left(\frac{\mathcal{I}_{p,n,\delta}^{\lambda,\mu}(a,b,c)f(z)}{z^p}\right)^\beta \prec \phi_2(z)$$

implies that

$$\left(\frac{\mathcal{I}_{p,n,\delta}^{\lambda,\mu}(a,b,c)g_1(z)}{z^p}\right)^\beta \prec \left(\frac{\mathcal{I}_{p,n,\delta}^{\lambda,\mu}(a,b,c)f(z)}{z^p}\right)^\beta \prec \left(\frac{\mathcal{I}_{p,n,\delta}^{\lambda,\mu}(a,b,c)g_2(z)}{z^p}\right)^\beta.$$

Moreover, the functions $\left(\frac{\mathcal{I}_{p,n,\delta}^{\lambda,\mu}(a,b,c)g_1(z)}{z^p}\right)^\beta$ and $\left(\frac{\mathcal{I}_{p,n,\delta}^{\lambda,\mu}(a,b,c)g_2(z)}{z^p}\right)^\beta$ are, respectively, the best subordination and the best dominant.

Proof. To prove our result, it suffices to show that the condition (3.21) implies the univalence of φ and $F(z) = \left(\frac{\mathcal{I}_{p,n,\delta}^{\lambda,\mu}(a,b,c)f(z)}{z^p}\right)^\beta$. Since σ given by (3.2) in Theorem 3.1 satisfies the inequality $0 < \sigma \leq \frac{1}{2}$, the condition (3.21) means that φ is a close-to-convex function in \mathbb{U} (see [36]) and hence φ is univalent in \mathbb{U} . Also, by using the same techniques as in the proof of Theorem 3.1, we can prove that F is convex (univalent) in \mathbb{U} , and so the details may be omitted. Therefore, by applying Theorem 3.3, we obtain the desired result. \square

Using the same method as in the proof of Corollary 3.3, as well as Theorem 3.4, we have the following result.

Corollary 3.4. Let $f, g_j \in \mathcal{A}_n(p) (j = 1, 2)$. Suppose that the condition (3.20) is satisfied and

$$Re\left\{1 + \frac{z\chi''(z)}{\chi'(z)}\right\} > -\tau,$$

where

$$\chi(z) = (1-\alpha)\left(\frac{\mathcal{I}_{p,n,\delta}^{\lambda,\mu}(a+1,b,c)f(z)}{z^p}\right)^\beta + \alpha\left(\frac{\mathcal{I}_{p,n,\delta}^{\lambda,\mu}(a,b,c)f(z)}{\mathcal{I}_{p,n,\delta}^{\lambda,\mu}(a+1,b,c)f(z)}\right)\left(\frac{\mathcal{I}_{p,n,\delta}^{\lambda,\mu}(a+1,b,c)f(z)}{z^p}\right)^\beta$$

and τ is given by (3.17). Then the following subordination relationship

$$\psi_1(z) \prec (1-\alpha)\left(\frac{\mathcal{I}_{p,n,\delta}^{\lambda,\mu}(a+1,b,c)f(z)}{z^p}\right)^\beta + \alpha\left(\frac{\mathcal{I}_{p,n,\delta}^{\lambda,\mu}(a,b,c)f(z)}{\mathcal{I}_{p,n,\delta}^{\lambda,\mu}(a+1,b,c)f(z)}\right)\left(\frac{\mathcal{I}_{p,n,\delta}^{\lambda,\mu}(a+1,b,c)f(z)}{z^p}\right)^\beta \prec \psi_2(z)$$

implies that

$$\left(\frac{\mathcal{I}_{p,n,\delta}^{\lambda,\mu}(a+1,b,c)g_1(z)}{z^p}\right)^\beta \prec \left(\frac{\mathcal{I}_{p,n,\delta}^{\lambda,\mu}(a+1,b,c)f(z)}{z^p}\right)^\beta \prec \left(\frac{\mathcal{I}_{p,n,\delta}^{\lambda,\mu}(a+1,b,c)g_2(z)}{z^p}\right)^\beta.$$

Moreover, the functions $\left(\frac{\mathcal{I}_{p,n,\delta}^{\lambda,\mu}(a+1,b,c)g_1(z)}{z^p}\right)^\beta$ and $\left(\frac{\mathcal{I}_{p,n,\delta}^{\lambda,\mu}(a+1,b,c)g_2(z)}{z^p}\right)^\beta$ are, respectively, the best subordination and the best dominant.

Upon setting $\beta = 1$ in Theorems 3.3 and 3.4, we are easily led to the following results.

Corollary 3.5. Let $f, g_j \in \mathcal{A}_n(p) (j = 1, 2)$ and suppose that

$$Re\left\{1 + \frac{z\phi_j''(z)}{\phi_j'(z)}\right\} > -\sigma \left(\phi_j(z) = \frac{(1-\alpha)\mathcal{I}_{p,n,\delta}^{\lambda,\mu}(a,b,c)g_j(z) + \alpha\mathcal{I}_{p,n,\delta}^{\lambda+1,\mu}(a,b,c)g_j(z)}{z^p}\right),$$

where σ is given by (3.2) with $\beta = 1$. If the function

$$\frac{(1 - \alpha)\mathcal{I}_{p,n,\delta}^{\lambda,\mu}(a, b, c)f(z) + \alpha\mathcal{I}_{p,n,\delta}^{\lambda+1,\mu}(a, b, c)f(z)}{z^p}$$

is univalent in \mathbb{U} and $\frac{\mathcal{I}_{p,n,\delta}^{\lambda,\mu}(a,b,c)f(z)}{z^p} \in \mathcal{H}[1, 1] \cap \mathcal{Q}$. Then the following subordination relationship

$$\phi_1(z) \prec \frac{(1 - \alpha)\mathcal{I}_{p,n,\delta}^{\lambda,\mu}(a, b, c)f(z) + \alpha\mathcal{I}_{p,n,\delta}^{\lambda+1,\mu}(a, b, c)f(z)}{z^p} \prec \phi_2(z)$$

implies that

$$\frac{\mathcal{I}_{p,n,\delta}^{\lambda,\mu}(a, b, c)g_1(z)}{z^p} \prec \frac{\mathcal{I}_{p,n,\delta}^{\lambda,\mu}(a, b, c)f(z)}{z^p} \prec \frac{\mathcal{I}_{p,n,\delta}^{\lambda,\mu}(a, b, c)g_2(z)}{z^p}.$$

Moreover, the functions $\frac{\mathcal{I}_{p,n,\delta}^{\lambda,\mu}(a,b,c)g_1(z)}{z^p}$ and $\frac{\mathcal{I}_{p,n,\delta}^{\lambda,\mu}(a,b,c)g_2(z)}{z^p}$ are, respectively, the best subordination and the best dominant.

Corollary 3.6. Let $f, g_j \in \mathcal{A}_n(p)$ ($j = 1, 2$) and suppose that

$$Re\left\{1 + \frac{z\psi_j''(z)}{\psi_j'(z)}\right\} > -\tau \left(\psi_j(z) = \frac{(1 - \alpha)\mathcal{I}_{p,n,\delta}^{\lambda,\mu}(a + 1, b, c)g_j(z) + \alpha\mathcal{I}_{p,n,\delta}^{\lambda,\mu}(a, b, c)g_j(z)}{z^p}\right),$$

where τ is given by (3.17) with $\beta = 1$. If the function

$$\frac{(1 - \alpha)\mathcal{I}_{p,n,\delta}^{\lambda,\mu}(a + 1, b, c)f(z) + \alpha\mathcal{I}_{p,n,\delta}^{\lambda,\mu}(a, b, c)f(z)}{z^p}$$

is univalent in \mathbb{U} and $\frac{\mathcal{I}_{p,n,\delta}^{\lambda,\mu}(a+1,b,c)f(z)}{z^p} \in \mathcal{H}[1, 1] \cap \mathcal{Q}$. Then the following subordination relationship

$$\psi_1(z) \prec \frac{(1 - \alpha)\mathcal{I}_{p,n,\delta}^{\lambda,\mu}(a + 1, b, c)f(z) + \alpha\mathcal{I}_{p,n,\delta}^{\lambda,\mu}(a, b, c)f(z)}{z^p} \prec \psi_2(z)$$

implies that

$$\frac{\mathcal{I}_{p,n,\delta}^{\lambda,\mu}(a + 1, b, c)g_1(z)}{z^p} \prec \frac{\mathcal{I}_{p,n,\delta}^{\lambda,\mu}(a + 1, b, c)f(z)}{z^p} \prec \frac{\mathcal{I}_{p,n,\delta}^{\lambda,\mu}(a + 1, b, c)g_2(z)}{z^p}.$$

Moreover, the functions $\frac{\mathcal{I}_{p,n,\delta}^{\lambda,\mu}(a+1,b,c)g_1(z)}{z^p}$ and $\frac{\mathcal{I}_{p,n,\delta}^{\lambda,\mu}(a+1,b,c)g_2(z)}{z^p}$ are, respectively, the best subordination and the best dominant.

Finally, we consider the generalized Libera operator F_m ($m > -p$) defined by (see [37,38]; also [5,39])

$$F_m(f)(z) = \frac{m+p}{z^m} \int_0^z t^{m-1}f(t)dt \quad (m > -p; f \in \mathcal{A}_n(p)), \quad (3.22)$$

which satisfies the following relationship

$$z\left(\mathcal{I}_{p,n,\delta}^{\lambda,\mu}(a, b, c)F_m(f)(z)\right)' = (m+p)\mathcal{I}_{p,n,\delta}^{\lambda,\mu}(a, b, c)f(z) - m\mathcal{I}_{p,n,\delta}^{\lambda,\mu}(a, b, c)F_m(f)(z). \quad (3.23)$$

We now derive the following sandwich-type result involving the integral operator F_m defined by (3.22).

Theorem 3.5. Let $f, g_j \in \mathcal{A}_n(p)$ ($j = 1, 2$) and suppose that

$$Re\left\{1 + \frac{z\phi_j''(z)}{\phi_j'(z)}\right\} > -\sigma, \quad (3.24)$$

where

$$\begin{aligned} \phi_j(z) &= (1 - \alpha) \left(\frac{\mathcal{I}_{p,n,\delta}^{\lambda,\mu}(a, b, c)F_m(g_j)(z)}{z^p}\right)^\beta \\ &+ \alpha \left(\frac{\mathcal{I}_{p,n,\delta}^{\lambda,\mu}(a, b, c)g_j(z)}{\mathcal{I}_{p,n,\delta}^{\lambda,\mu}(a, b, c)F_m(g_j)(z)}\right) \left(\frac{\mathcal{I}_{p,n,\delta}^{\lambda,\mu}(a, b, c)F_m(g_j)(z)}{z^p}\right)^\beta \end{aligned}$$

and

$$\sigma = \frac{\alpha^2 + \beta^2(m+p)^2 - |\alpha^2 - \beta^2(m+p)^2|}{4\alpha\beta(m+p)} \quad (m > -p). \quad (3.25)$$

If the function

$$\begin{aligned} (1 - \alpha) \left(\frac{\mathcal{I}_{p,n,\delta}^{\lambda,\mu}(a, b, c)F_m(f)(z)}{z^p}\right)^\beta \\ + \alpha \left(\frac{\mathcal{I}_{p,n,\delta}^{\lambda,\mu}(a, b, c)f(z)}{\mathcal{I}_{p,n,\delta}^{\lambda,\mu}(a, b, c)F_m(f)(z)}\right) \left(\frac{\mathcal{I}_{p,n,\delta}^{\lambda,\mu}(a, b, c)F_m(f)(z)}{z^p}\right)^\beta \end{aligned}$$

is univalent in \mathbb{U} and $\left(\frac{\mathcal{I}_{p,n,\delta}^{\lambda,\mu}(a,b,c)F_m(f)(z)}{z^p}\right)^\beta \in \mathcal{H}[1, 1] \cap \mathcal{Q}$. Then the following subordination relationship

$$\begin{aligned} \phi_1(z) &\prec (1 - \alpha) \left(\frac{\mathcal{I}_{p,n,\delta}^{\lambda,\mu}(a, b, c)F_m(f)(z)}{z^p}\right)^\beta \\ &+ \alpha \left(\frac{\mathcal{I}_{p,n,\delta}^{\lambda,\mu}(a, b, c)f(z)}{\mathcal{I}_{p,n,\delta}^{\lambda,\mu}(a, b, c)F_m(f)(z)}\right) \left(\frac{\mathcal{I}_{p,n,\delta}^{\lambda,\mu}(a, b, c)F_m(f)(z)}{z^p}\right)^\beta \\ &\prec \phi_2(z) \end{aligned}$$

implies that

$$\begin{aligned} \left(\frac{\mathcal{I}_{p,n,\delta}^{\lambda,\mu}(a, b, c)F_m(g_1)(z)}{z^p}\right)^\beta &\prec \left(\frac{\mathcal{I}_{p,n,\delta}^{\lambda,\mu}(a, b, c)F_m(f)(z)}{z^p}\right)^\beta \\ &\prec \left(\frac{\mathcal{I}_{p,n,\delta}^{\lambda,\mu}(a, b, c)F_m(g_2)(z)}{z^p}\right)^\beta. \end{aligned}$$

Moreover, the functions $\left(\frac{\mathcal{I}_{p,n,\delta}^{\lambda,\mu}(a,b,c)F_m(g_1)(z)}{z^p}\right)^\beta$ and $\left(\frac{\mathcal{I}_{p,n,\delta}^{\lambda,\mu}(a,b,c)F_m(g_2)(z)}{z^p}\right)^\beta$ are, respectively, the best subordination and the best dominant.

Proof. Let us define the functions F and G_j ($j = 1, 2$), respectively, by

$$\begin{aligned} F(z) &= \left(\frac{\mathcal{I}_{p,n,\delta}^{\lambda,\mu}(a, b, c)F_m(f)(z)}{z^p}\right)^\beta \quad \text{and} \quad G_j(z) \\ &= \left(\frac{\mathcal{I}_{p,n,\delta}^{\lambda,\mu}(a, b, c)F_m(g_j)(z)}{z^p}\right)^\beta. \end{aligned}$$

Without loss of generality, as in the proof of Theorem 3.1, we assume that G_j is analytic and univalent on $\overline{\mathbb{U}}$ and that $G_j'(\xi) \neq 0$ ($\xi \in \partial\mathbb{U}$). Then, from (3.23) and (3.24), we know that

$$\phi_j(z) = G_j(z) + \frac{\alpha z G_j'(z)}{\beta(m+p)}. \quad (3.26)$$

Setting

$$q_j(z) = 1 + \frac{zG_j''(z)}{G_j'(z)} \quad (j = 1, 2),$$

and differentiating both sides of (3.26) with respect to z , we obtain

$$1 + \frac{z\phi_j''(z)}{\phi_j'(z)} = q_j(z) + \frac{zq_j'(z)}{q_j(z) + \beta(m+p)/\alpha} \quad (j = 1, 2).$$

The remaining part of the proof is similar to that of Theorem 3.3 (a combined proof of Theorems 3.1 and 3.2), and is thus omitted. \square

Applying the same method as in the proof of Corollary 3.3, from Theorem 3.5, we can derive the following result.

Corollary 3.7. *Let $f, g_j \in \mathcal{A}_n(p)$ ($j = 1, 2$). Suppose that the condition (3.24) is satisfied and*

$$\operatorname{Re} \left\{ 1 + \frac{z\varphi''(z)}{\varphi'(z)} \right\} > -\sigma,$$

where

$$\begin{aligned} \varphi(z) = & (1 - \alpha) \left(\frac{\mathcal{I}_{p,n,\delta}^{\lambda,\mu}(a,b,c)F_m(f)(z)}{z^p} \right)^\beta \\ & + \alpha \left(\frac{\mathcal{I}_{p,n,\delta}^{\lambda,\mu}(a,b,c)f(z)}{\mathcal{I}_{p,n,\delta}^{\lambda,\mu}(a,b,c)F_m(f)(z)} \right) \left(\frac{\mathcal{I}_{p,n,\delta}^{\lambda,\mu}(a,b,c)F_m(f)(z)}{z^p} \right)^\beta \end{aligned}$$

and σ is given by (3.25). Then the following subordination relationship

$$\begin{aligned} \phi_1(z) \prec & (1 - \alpha) \left(\frac{\mathcal{I}_{p,n,\delta}^{\lambda,\mu}(a,b,c)F_m(f)(z)}{z^p} \right)^\beta \\ & + \alpha \left(\frac{\mathcal{I}_{p,n,\delta}^{\lambda,\mu}(a,b,c)f(z)}{\mathcal{I}_{p,n,\delta}^{\lambda,\mu}(a,b,c)F_m(f)(z)} \right) \left(\frac{\mathcal{I}_{p,n,\delta}^{\lambda,\mu}(a,b,c)F_m(f)(z)}{z^p} \right)^\beta \\ \prec & \phi_2(z) \end{aligned}$$

implies that

$$\begin{aligned} \left(\frac{\mathcal{I}_{p,n,\delta}^{\lambda,\mu}(a,b,c)F_m(g_1)(z)}{z^p} \right)^\beta \prec & \left(\frac{\mathcal{I}_{p,n,\delta}^{\lambda,\mu}(a,b,c)F_m(f)(z)}{z^p} \right)^\beta \\ \prec & \left(\frac{\mathcal{I}_{p,n,\delta}^{\lambda,\mu}(a,b,c)F_m(g_2)(z)}{z^p} \right)^\beta. \end{aligned}$$

Moreover, the functions $\left(\frac{\mathcal{I}_{p,n,\delta}^{\lambda,\mu}(a,b,c)F_m(g_1)(z)}{z^p} \right)^\beta$ and $\left(\frac{\mathcal{I}_{p,n,\delta}^{\lambda,\mu}(a,b,c)F_m(g_2)(z)}{z^p} \right)^\beta$ are, respectively, the best subordination and the best dominant.

By putting $\alpha = \beta = 1$ in Theorem 3.5, we have the following result.

Corollary 3.8. *Let $f, g_j \in \mathcal{A}_n(p)$ ($j = 1, 2$) and suppose that*

$$\operatorname{Re} \left\{ 1 + \frac{z\phi_j''(z)}{\phi_j'(z)} \right\} > -\sigma \quad \left(\phi_j(z) = \frac{\mathcal{I}_{p,n,\delta}^{\lambda,\mu}(a,b,c)g_j(z)}{z^p} \quad (j = 1, 2); z \in \mathbb{U} \right),$$

where

$$\sigma = \frac{1 + (m+p)^2 - |1 - (m+p)^2|}{4(m+p)} \quad (m > -p).$$

If the function $\frac{\mathcal{I}_{p,n,\delta}^{\lambda,\mu}(a,b,c)f(z)}{z^p}$ is univalent in \mathbb{U} and $\frac{\mathcal{I}_{p,n,\delta}^{\lambda,\mu}(a,b,c)F_m(f)(z)}{z^p} \in \mathcal{H}[1, 1] \cap \mathcal{Q}$. Then the following subordination relationship

$$\frac{\mathcal{I}_{p,n,\delta}^{\lambda,\mu}(a,b,c)g_1(z)}{z^p} \prec \frac{\mathcal{I}_{p,n,\delta}^{\lambda,\mu}(a,b,c)f(z)}{z^p} \prec \frac{\mathcal{I}_{p,n,\delta}^{\lambda,\mu}(a,b,c)g_2(z)}{z^p}$$

implies that

$$\begin{aligned} \frac{\mathcal{I}_{p,n,\delta}^{\lambda,\mu}(a,b,c)F_m(g_1)(z)}{z^p} \prec & \frac{\mathcal{I}_{p,n,\delta}^{\lambda,\mu}(a,b,c)F_m(f)(z)}{z^p} \\ \prec & \frac{\mathcal{I}_{p,n,\delta}^{\lambda,\mu}(a,b,c)F_m(g_2)(z)}{z^p}. \end{aligned}$$

Moreover, the functions $\frac{\mathcal{I}_{p,n,\delta}^{\lambda,\mu}(a,b,c)F_m(g_1)(z)}{z^p}$ and $\frac{\mathcal{I}_{p,n,\delta}^{\lambda,\mu}(a,b,c)F_m(g_2)(z)}{z^p}$ are, respectively, the best subordination and the best dominant.

Remark 3.1. By taking $n = 1$, $\mu = \delta = 0$, $b = \lambda + p$ and $c = a$ in Corollary 3.8, we obtain Corollary 5 in [40], which contains, as its special case, the result obtained earlier by Pommerenke [34].

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