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Pairwise weakly and pairwise strongly irresolute functions

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ij-Semi-closure;
ij-Semi T_2-space;
ij-Semi-compact

Abstract In this paper we consider a new weak and strong forms of irresolute functions in bitopological spaces, namely, *ij*-quasi-irresolute functions and strongly irresolute functions. Several characterizations and basic properties of these functions are given. We investigate the relationships among some weak forms of continuity and other generalizations of continuous mappings in bitopological spaces.

1. Introduction

The study of bitopological spaces was first initiated by Kelly [3] and thereafter a large number of papers have been done to generalized the topological concepts to bitopological space. Irresolute mappings in bitopological spaces was defined by Mukherjee [11]. In 1991 Khedr [4] introduced and investigate a class of mappings in bitopological spaces called pairwise θ-irresolute mappings. Khedr [7] defined the concept of quasi-irresolute mappings in these spaces and studied some of its properties. The concepts of strongly irresolute mappings in bitopological spaces was defined by Khedr in [6] and he showed that quasi-irresoluteness and semi-continuity are independent of each other.

The aim of this paper is to introduce basic properties of quasi-irresolute and strongly irresolute functions in bitopological spaces. We study these functions and some of results on s-closed spaces and semi-compact spaces in bitopological spaces. Also, we investigate the relationships among some weak forms of continuity, irresoluteness, quasi-irresoluteness and strong irresoluteness.

Throughout this paper (X,τ_1,τ_2), (Y,σ_1,σ_2) and (Z,ν_1,ν_2) (or briefly X, Y and Z) denote bitopological spaces on which no separation axioms are assumed unless otherwise mentioned. For a subset A of X, we shall denote the closure of A and the interior of A with respect to τ_i (or σ_i) by i-cl(A) and i-int(A) respectively for $i = 1,2$. Also $i,j = 1,2$ and $i \neq j$.

A subset A is said to be *ij*-semi-open [1], if there exists a τ_i-open set U of X such that $U \subset A \subset j$-cl(U), or equivalently if $A \subset j$-cl(i-int(A)). The complement of an *ij*-semi-open set is said to be *ij*-semi-closed. An *ij*-semi-interior [1] of A, denoted by *ij*-sint(A), is the union of all *ij*-semi-open sets contained in A. The intersection of all *ij*-semi-closed sets containing A is called the *ij*-semi-closure [1] of A and denoted by *ij*-scl(A).

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A subset A of X is said to be *ij*-semi-regular [7] if it is both *ji*-semi-open and *ij*-semi-closed in X. The family of all *ij*-semi-open (resp. *ij*-semi-closed, *ij*-semi-regular) sets of X is denoted by *ij*-SO(X) (resp. *ij*-SC(X), *ij*-SR(X)) and for $x \in X$, the family of all *ij*-semi-open sets containing x is denoted by *ij*-SO(X, x).

A point $x \in X$ is said to be *ij*-semi θ-adherent point of A [2] if *ji*-scl(U) $\cap A \neq \phi$ for every *ij*-semi-open set U containing x. The set of all *ij*-semi θ-adherent points of A is called the *ij*-semi θ-closure of A and denoted by *ij*-scl$_\theta$(A). A subset A is called *ij*-semi θ-closed if *ij*-scl$_\theta$(A) $= A$. The set $\{x \in X \setminus$ *ji*-scl(U) $\subset A$, for some U is *ij*-semi-open$\}$ is called the *ij*-semi θ-interior of A and is denoted by *ij*-sint$_\theta$(A). A subset A is called *ij*-semi θ-open if $A =$ *ij*-sint$_\theta$(A).

Now, we mention the flowing definitions and results:

Definition 1.1. A bitopological space (X, τ_1, τ_2) is said to be:

(i) Pairwise semi-T_0 [8] (briefly P-semi T_0) if for each distinct points $x, y \in X$, there exists either an *ij*-semi-open set containing x but not y or a *ji*-semi-open set containing y but not x.
(ii) Pairwise semi-T_1 [8] (briefly P-semi T_1) if for every two distinct points x and y in X, there exists an *ij*-semi-open set U containing x but not y and a *ji*-semi-open set V containing y but not x.
(iii) Pairwise semi-T_2 [6] (briefly P-semi T_2) if for every two distinct points x and y in X, there exists either $U \in$ *ij*-SO(X, x) and $V \in$ *ji*-SO(X, y) such that $U \cap V = \phi$.

Definition 1.2. [7] A bitopological space (X, τ_1, τ_2) is said to be *ij*-semi-regular (resp. *ij*-s-regular) if for each *ij*-semi-closed (resp. τ_i-closed) set F and each point $x \notin F$, there exists an *ij*-semi-open set U and a *ji*-semi-open set V such that $x \in U$, $F \subset V$ and $U \cap V = \phi$.

Lemma 1.3. [8] *For every subset A of a space X, we have the following:*

(i) $X \setminus ij\text{-scl}(A) = ij\text{-sint}(X \setminus A)$.
(ii) $X \setminus ij\text{-sint}(A) = ij\text{-scl}(X \setminus A)$.

Lemma 1.4. [7] *Let A be a subset of a space X. Then we have:*

(i) If $U \in$ *ij*-SO(X), then *ji*-scl(U) $\in$ *ji*-SR(X).
(ii) If A $\in$ *ij*-SO(X), then *ji*-scl(A) $=$ *ij*-scl$_\theta$(A).

Lemma 1.5. [9] *Let A be a subset of a space X. Then we have if $A \in ij\text{-}SR(X)$, then A is both ij-semi θ-closed and ji-semi θ-open.*

Lemma 1.6. [8] *If a function $f: (X, \tau_1, \tau_2) \to (Y, \sigma_1, \sigma_2)$ is an ij-pre-semi-closed, then for each subset $S \subset Y$ and each $U \in ij\text{-}SO(X)$ containing $f^{-1}(S)$, there exists $V \in ij\text{-}SO(Y)$ such that $S \subset V$ and $f^{-1}(V) \subset U$.*

Lemma 1.7. [7] *A bitopological space (X, τ_1, τ_2)ij-semi-regular (resp. ij-s-regular) if and only if for each ij-semi-open (resp. τ_i-open) set G and each point $x \in G$, there exists an ij-semi-open set U such that $x \in U$, $F \subset V$ and ji-scl$(U) \subset G$.*

Lemma 1.8. [7] *A bitopological space (X, τ_1, τ_2) is ij-s-closed if and only if for every cover of X by ji-semi-regular sets has a finite subcover.*

Lemma 1.9. [9]

(i) *Every ji-semi θ-closed set is ij-θ-sg-closed.*
(ii) *A bitopological space (X, τ_1, τ_2) is an P-semi $T_{1/2}$-space if and only if every ij-θ-sg-closed set is ij-semi-closed.*

Definition 1.10. [9] A function $f: (X, \tau_1, \tau_2) \to (Y, \sigma_1, \sigma_2)$ is called:

(i) *ij*-θ-semigeneralized continuous (briefly *ij*-θ-sg-continuous) if $f^{-1}(V)$ is *ij*-θ-sg-closed in X for every *ji*-semi-closed V of Y.
(ii) *ij*-θ-semigeneralized irresolute (briefly *ij*-θ-sg-irresolute) if $f^{-1}(V)$ is *ij*-θ-sg-closed in X for every *ij*-θ-sg-closed set V of Y.
(iii) *ij*-θ-sg-closed if for every *ji*-semi-closed set U of X, $f(U)$ is an *ij*-θ-sg-closed in Y.
(iv) *ij*-semi-generalized closed (briefly *ij*-sg-closed) if for each τ_j-closed set F of X, $f(F)$ is an *ij*-sg-closed set in Y.

2. Characterization of pairwise quasi-irresolute functions.

Definition 2.1. [7] A function $f: (X, \tau_1, \tau_2) \to (Y, \sigma_1, \sigma_2)$ is said to be *ij*-quasi-irresolute if for each $x \in X$ and each $V \in$ *ij*-SO($Y, f(x)$), there exists $U \in$ *ij*-SO(X, x) such that $f(U) \subset$ *ji*-scl(V). If f is 12-quasi-irresolute and 21-quasi-irresolute, then f is called pairwise quasi-irresolute.

Definition 2.2. A function $f: (X, \tau_1, \tau_2) \to (Y, \sigma_1, \sigma_2)$ is said to be *ij*-irresolute [11] (resp. *ij*-semi-continuous [1]) if $f^{-1}(V)$ is an *ij*-semi-open set of X for every *ij*-semi-open (resp. σ_i-open) set V of Y.

Theorem 2.3. *The following statements are equivalent for a function $f: X \to Y$:*

(i) f is *ij*-quasi-irresolute
(ii) *ij*-scl($f^{-1}(B)$) $\subset f^{-1}$(*ij*-scl$_\theta$(B)) for every subset B of Y.
(iii) f(*ij*-scl(A)) $\subset$ *ij*-scl$_\theta$($f(A)$) for every subset A of X.
(iv) f^{-1}(F) $\in$ *ij*-SC(X) for every *ij*-semi θ-closed set F in Y.
(v) $f^{-1}(V) \in$ *ij*-SO(X) for every *ij*-semi θ-open set V in Y.

Proof. (i) $\Rightarrow$ (ii): Let $B \subset Y$ and $x \notin f^{-1}$(*ij*-scl$_\theta$(B)). Then $f(x) \notin$ *ij*-scl$_\theta$(B) and there exists $V \in$ *ij*-SO($Y, f(x)$) such that *ji*-scl(v) $\cap B = \phi$. By (i), there exists $U \in$ *ij*-SO(X, x) such that $f(U) \subset$ *ji*-scl(v). Hence $f(U) \cap B = \phi$ and $U \cap f^{-1}(B) = \phi$. Consequently, we obtain $x \notin$ *ij*-scl (f^{-1}(B)).

(ii) $\Rightarrow$ (iii): For any subset A of X, the inclusion *ij*-scl(A) $\subset$ *ij*-scl($f^{-1}(f(A))$) holds. By (ii), we have *ij*-scl($f^{-1}(f(A))$) $\subset f^{-1}$(*ij*-scl$_\theta$ ($f(A)$)) and hence f(*ij*-scl(A)) $\subset$ *ij*-scl$_\theta$($f(A)$).

(iii) $\Rightarrow$ (ii): For any subset B of Y, we have *ij*-scl$_\theta$($f(f^{-1}(B))$) $\subset$ *ij*-scl$_\theta$(B). By (iii), we obtain f(*ij*-scl(f^{-1}(B))) $\subset$ *ij*-scl$_\theta$($f(f^{-1}(B))$) and hence *ij*-scl(f^{-1}(B)) $\subset f^{-1}$(*ij*-scl$_\theta$(B)).

(ii) ⇒ (iv): Let F be an *ij*-semi θ-closed set in Y. By (ii), we have *ij*-scl(f^{-1}(F)) $\subset f^{-1}$(*ij*-scl$_\theta$(F)) = f^{-1}(F). Therefore, f^{-1}(F) is *ij*-semi-closed in X.

(iv) ⇒ (v): If V is *ij*-semi θ-open in Y, then $Y\backslash V$ is *ij*-semi θ-closed. By (iv), $f^{-1}(Y\backslash V) = X\backslash f^{-1}(V)$ is *ij*-semi-closed in X. Thus $f^{-1}(V) \in$ *ij*-SO(X).

(v) ⇒ (i): Let $x \in X$ and $V \in$ *ij*-SO(Y,$f(x)$). It follows from Lemmas 1.4 and 1.5, that *ji*-scl(v) is *ji*-semi θ-closed and *ij*-semi θ-open in Y. Set $U = f^{-1}$(*ji*-scl(V)). By (v), we observe that $U \in$ *ij*-SO(X) and $f(U) \subset$ *ji*-scl (V). The proof is complete. □

The next theorem contains an unexpected result.

Theorem 2.4. *The following statements are equivalent for a function $f: X \to Y$:*

(i) f is pairwise quasi-irresolute
(ii) For each $x \in X$ and each $V \in$ *ij*-SO(Y,$f(x)$), there exists $U \in$ *ij*-SO(X) such that f(*ij*-scl(U)) $\subset$ *ji*-scl (V).
(iii) $f^{-1}(F) \in$ *ji*-SR(X) for every $F \in$ *ji*-SR(Y).

Proof

(i) ⇒ (ii): Let $x \in X$ and $V \in$ *ij*-SO(Y,$f(x)$). Then by Lemmas 1.4 and 1.5, *ji*-scl (V) is both *ij*-semi θ-open and *ji*-semi θ-closed. Put $U = f^{-1}$(*ji*-scl(v)). Then it follows from Theorem 2.3(v), that $U \in$ *ji*-SR(X). Thus we obtain $U \in$ *ij*-SO(X). U = *ji*-scl(U) and f(*ij*-scl((U)) $\subset$ *ji*-scl(v).
(ii) ⇒ (i): Obvious.
(i) ⇒ (iii): Let $V \in$ *ji*-SR(Y). By Lemma 1.5, V is *ji*-semi θ-closed and *ij*-semi θ-open in Y. It follows from Theorem 2.3 that $f^{-1}(V) \in$ *ji*-SR(X).
(iii) ⇒ (i): Let $x \in X$ and $V \in$ *ij*-SO(Y,$f(x)$). By Lemma 1.4, *ji*-scl(v) $\in$ *ji*-SR(Y,$f(x)$) and by hypothesis f^{-1}(*ji*-scl(v)) $\in$ *ji*-SR(X,x). Put $U = f^{-1}$(*ij*-scl(v)), then $U \in$ *ij*-SO(X,x) and $f(U) \subset$ *ji*-scl(v). This shows that f is *ij*-quasi-irresolute. □

The following Theorem offers several characterizations of *ij*-quasi-irresolute functions.

Theorem 2.5. *The following statements are equivalent for a function $f: X \to Y$:*

(i) f is pairwise quasi-irresolute
(ii) *ij*-scl$_\theta$(f^{-1}(B)) $\subset f^{-1}$(*ij*-scl$_\theta(B)$) for every subset B of Y.
(iii) f(*ij*-scl$_\theta(A)$) $\subset$ *ij*-scl$_\theta$($f(A)$) for every subset A of X.
(iv) $f^{-1}(F)$ is *ij*-semi θ-closed in X for every *ij*-semi θ-closed set F in Y.
(v) $f^{-1}(V)$ is *ij*-semi θ-open in X, for every *ij*-semi θ-open set V in Y.

Proof. By making use of Theorem 2.4, we can prove this Theorem in the similar way to the proof of Theorem 2.3. □

Theorem 2.6. *Let $f: (X, \tau_1, \tau_2) \to (Y, \sigma_1, \sigma_2)$ be ij-θ-sg-irresolute. If (X, τ_1, τ_2) is pairwise semi $T_{1/2}$, then f is ji-quasi-irresolute*

Proof. Suppose that V is a *ji*-semi θ-closed set in Y. By lemma 1.9(i), V is *ij*-θ-sg-closed in Y. Since f is *ij*-θ-sg-irresolute, $f^{-1}(V)$ is *ij*-θ-sg-closed in X. By Lemma 1.9 (ii), $f^{-1}(v)$ is *ij*-semi-closed. This shows that f is *ij*-quasi-irresolute, by Theorem 2.3. □

Theorem 2.7. *If $f: (X, \tau_1, \tau_2) \to (Y, \sigma_1, \sigma_2)$ is an ij-quasi-irresoulte and Y is ij-semi-regular, then f is ij-irresolute.*

Proof. Let $V \in$ *ij*-SO(Y) and $x \in f^{-1}(V)$, there exists W $\in$ *ij*-SO(Y) such that $f(x) \in W \subset$ *ji*-scl(W) $\subset V$, since f is *ij*-quasi-irresolute, then there exists $U \in$ *ij*-SO(X,x) such that $f(U) \subset$ *ji*-scl(W). Therefore, we have $x \in U \subset f^{-1}$(*ij*-scl(W)) $\subset f^{-1}(V)$ and hence $f^{-1}(V) \in$ *ij*-SO(X). This shows that f is *ij*-irresolute. □

Theorem 2.8. *If $f: (X, \tau_1, \tau_2) \to (Y, \sigma_1, \sigma_2)$ is an ij-quasi-irresolute and Y is ij-semi-regular, then f is ij-semi-continuous.*

Proof. Similar to that of Theorem 2.7. □

Lemma 2.9. *Let $f: X \to Y$ and $g: X \to X \times Y$ the graph function of f where $g(x) = (x, f(x))$ for each $x \in X$. If g is ij-quasi-irresolute, then f is ij-quasi-irresolute.*

Proof. Let $x \in X$ and $V \in$ *ij*-SO($f(x)$). Then $X \times V$ is an *ij*-semi-open set in $X \times Y$ containing $g(x)$. Since g is *ij*-quasi-irresolute there exists $U \in$ *ij*-SO(X) such that $g(U)$ *ji*-scl($X \times V$) $X \times$ *ji*-scl (V). Thus we obtain $f(U)$ *ji*-scl (V). □

The converse of Lemma 2.9, is not true as the next example shows.

Example 2.10. Let $X = \{a, b, c\}$, $\tau_1 = \{\phi, \{a\}, \{b\}, \{a, b\}, X\}$ and $\tau_2 = \{\phi, \{b\}, \{a, b\}, X\}$. Define a function $f: (X, \tau_1, \tau_2) \to (X, \tau_1, \tau_2)$ by setting $f(a) = b$, $f(b) = a$ and $f(c) = c$. Then f is 12-irresolute and hence 12-quasi-irresolute but g is not 12-quasi-irresolute. It is apparent that 12-SO(X) = $\{\phi, \{a\}, \{b\}, \{a, b\}, \{b, c\}, \{a, c\}, X\}$. We prove that g is not 12-quasi-irresolute at c. Now, put $V = \{(a, a), (a, c), (c, c)\}$. Then V is 12-semi-open in $X \times X$, $g(c) = (c, f(c)) = (c, c) \in V$ and V = 21-scl(V). Since 12-SO(c) = $\{\{a, c\}, \{b, c\}, X\}$, $g(U)$ 21-scl(V) for every $U \in$ 12-SO(c). Thus show that g not be 12-quasi-irresolute at c.

Lemma 2.11. *A space X is pairwise semi-T_2 if and only if for each pair distinct of points x,y of X, there exist $U \in$ ij-SO(X,x) and $V \in$ ij-SO(X,y) such that ij-scl(U) $\cap$ ij-scl(V) = ϕ.*

Proof. This follows immediately from Lemma 1.4. □

Theorem 2.12. *If Y is pairwise semi-T_2-space and $f: (X, \tau_1, \tau_2) \to (Y, \sigma_1, \sigma_2)$ is a parwise quasi-irresolute injection, then X is pariwise semi-T_2.*

Proof. Let $x_1, x_2 \in X$ and $x_1 \neq x_2$. Then there exists $V_1 \in$ *ij*-SO(Y) containing $f(x_1)$ and $V_2 \in$ *ij*-SO(Y) containing $f(x_2)$ such that *ij*-scl(V_1) $\cap$ *ij*-scl(V_2) = ϕ, from Lemma 2.11. Since f is pairwise quasi-irresolute, there exists $U_1 \in$ *ij*-SO(X) containing x_1 and $U_2 \in$ *ij*-SO(X) containing x_2 such that $f(U_1) \subset$ *ij*-scl(V_1) and $f(U_2) \subset$ *ij*-scl(V_2). Since f is injection, then $U_1 \cap U_2 = \phi$. Thus X is pairwise semi-T_2. □

Theorem 2.13. *If a function $f:(X,\tau_1,\tau_2)\to(Y,\sigma_1,\sigma_2)$ is pairwise quasi-irresolute and ij-pre-semi-closed, then for every ij-θ-sg-closed set F of Y, $f^{-1}(F)$ is ij-θ-sg-closed set of X.*

Proof. Suppose that F is an *ij*-θ-sg-closed set of Y. Assume $f^{-1}(F)\subset U$ where $U\in$ *ij*-SO(X). Since f is *ij*-pre-semi-closed and by lemma 1.6, there is an *ij*-semi-open set V such that $F\subset V$ and $f^{-1}(V)\subset U$. Since F is *ij*-θ-sg-closed set and $F\subset V$, then *ij*-scl$_\theta(F)\subset V$. Hence f^{-1}(*ij*-scl$_\theta(F))\subset f^{-1}(V)\subset U$. Since f is pairwise quasi-irresolute and by Theorem 2.5, *ij*-scl$_\theta(f^{-1}(F))\subset U$ and hence $f^{-1}(F)$ is *ij*-θ-sg-cloed set in X. □

Theorem 2.14. *If a space X is pairwise semi-$T_{1/2}$ and $f:(X,\tau_1,\tau_2)\to(Y,\sigma_1,\sigma_2)$ is surjective, ij-quasi-irresolute and ij-pre-semi-closed, then Y is pairwise semi $T_{1/2}$.*

Proof. Assume that A is an *ij*-θ-sg-closed subset of Y. Then by Theorem 2.13, we have f^{-1}(A) is an *ij*-θ-sg-closed subset of X. By Theorem 2.12, f^{-1}(A) is *ij*-semi-closed and hence, A is *ij*-semi-closed. It follows that Y is pairwise semi $T_{1/2}$. □

Definition 2.15. Let X and Y be bitopological spaces. A subset S of $X\times Y$ is called *ij*-strongly semi θ-closed if for each $(x,y)\in(X\times Y)\setminus S$, there exist $U\in$ *ji*-SO(X,x) and $V\in$ *ij*-SO(Y,y) such that [*ij*-scl(U) × *ji*-scl(V) ∩ S = ϕ.

Definition 2.16. A function $f:X\to Y$ is said to be an *ij*-strongly semi θ-closed graph if its graph $G(f)$ is *ij*-strongly semi θ-closed in $X\times Y$ where $G(f)=\{(x,f(x)):x\in X\}$.

Theorem 2.17. *If a function $f:(X,\tau_1,\tau_2)\to(Y,\sigma_1,\sigma_2)$ is a ij-quasi-irresolute and Y is pairwise semi T_2, then G(f) is ij-strongly semi θ-closed.*

Proof. Let $(x,y)\notin G(f)$, then we have $y\neq f(x)$. Since Y is pariwise semi T_2, by Lemma 2.11, there exist $W\in$ *ij*-SO($Y,f(x)$) and $V\in$ *ij*-SO(Y,y) such that *ij*-scl(W) ∩ *ij*-scl(V) = ϕ. Since f is *ij*-quasi-irresolute, by Theorem 2.4, there exists $U\in$ *ji*-SO(X,x) such that f(*ij*-scl(U)) ⊂ *ij*-scl(W). Therefore, we have f(*ij*-scl(U)) ∩ *ji*-scl(V) = ϕ. This shows that $G(f)$ is *ij*-strongly semi θ-closed. □

Theorem 2.18. *If a function $f:X\to Y$ has an ij-strongly semi θ-closed graph and $g:X\to Y$ is an ij-quasi-irresolute function, then the set $A=\{(x_1,x_2)\in X\times X: f(x_1))=g(x_2)\}$ is an ij-strongly semi θ-closed in $X\times X$.*

Proof. Let $(x_1,x_2)\notin(X\times X)\backslash A$. Then we have $f(x_1)\neq g(x_2)$ and hence $(x_1,g(x_2))\in(X\times Y)\setminus G(f)$. Since $G(f)$ is *ij*-strongly semi θ-closed, there exists $U\in$ *ij*-SO(x_1) and $W\in$ *ij*-SO($g(x_2)$) such that f(*ij*-scl(U)) ∩ *ji*-scl(W) = ϕ. Since g is *ij*-quasi-irresolute, then implies that there exists $V\in$ *ij*-SO(x_2) such that g(*ij*-scl (V)) ⊂ *ji*-scl(W). Consequently, we obtain f(*ij*-scl (U)) ∩ g(*ij*-scl (V)) = ϕ and hence [*ij*-scl(U) × *ji*-scl(V) ∩ A = ϕ. Hence A is *ij*-strongly semi θ-closed in $X\times X$. □

Corollary 2.19. If $f:X\to Y$ is an *ij*-quasi-irresolute function and Y is pairwise semi T_2, then the set $A=\{(x_1,x_2)\in X\times X\setminus f(x_1)=f(x_2)\}$ is *ij*-strongly semi θ-closed in $X\times X$.

Proof. This is an immediate consequence of Theorem 2.17 and 2.18. □

Definition 2.20. A space X is to be *ij*-semi-connected [10] if X cannot be expressed as the union of two disjoint non-empty subsets U and V such that $U\in$ *ij*-SO(X) and $V\in$ *ji*-SO(X).

Theorem 2.21. *If $f:X\to Y$ is a P-quasi-irresolute surjection and X is P-semi-connected, then Y is P-semi-connected.*

Proof. Suppose that Y is not P-semi-connected. Then Y is the union of two non-empty disjoint $V_1\in$ *ij*-SO(Y) and $V_2\in$ *ij*-SO(Y) such that $V_1\cap V_2=\phi$ and $V_1\cup V_2=Y$. Let $W_1=$ *ij*-scl(V_1) and $W_2=$ *ij*-scl(V_2). Then $\phi\neq W_1\in$ *ij*-SR(Y) and $\phi\neq W_2\in$ *ij*-SR(Y), by Lemma 1.4, such that $W_1\cap W_2=\phi$ and $W_1\cup W_2=Y$. Therefore, we have $f^{-1}(W_1)\neq\phi$, $f^{-1}(W_2)\neq\phi$, $f^{-1}(W_1)\cap f^{-1}(W_2)=\phi$ and $f^{-1}(W_1)\cup f^{-1}(W_2)$ =X. Moreover by Theorem 2.4, $f^{-1}(W_1)\in$ *ij*-SR(X) and $f^{-1}(W_2)\in$ *ij*-SR(X). This shows that X is not P-semi-connected and this is a contradiction. Therefore, Y is P-semi-connected. □

3. Characterization of *ij*-strongly irresolute functions.

Definition 3.1. [6] A function $f:X\to Y$ is said to be *ij*-strongly irresolute if for each $x\in X$ and each $V\in$ *ij*-SO(Y,$f(x)$), there exists $U\in$ *ij*-SO(X,x) such that f(*ij*-scl(U)) V. If f is 12-strongly irresolute and 21-strongly irresolute, then f is called pairwise strongly irresolute.

Theorem 3.2. *The following statements are equivalent for a function $f:X\to Y$.*

(i) f is *ij*-strongly irresolute.
(ii) *ij*-scl$_\theta(f^{-1}$(B)) ⊂ f^{-1}(*ij*-scl(B)) for every subset B of Y.
(iii) f(*ij*-scl$_\theta(A)$) ⊂ *ij*-scl($f(A)$) for every subset A of X.
(iv) $f^{-1}(F)$ is *ij*-semi-θ-closed in X for every $F\in$ *ij*-SC(Y).
(v) $f^{-1}(V)$ is *ij*-semi θ-open in X, for every $V\in$ *ij*-SO(Y).

Proof. (i) ⇒ (ii): Let B ⊂ Y and $x\notin f^{-1}$(*ij*-scl(B)). Then $f(x)\notin$ *ij*-scl(B) and there exists $V\in$ *ij*-SO(Y,$f(x)$) such that $V\cap B=\phi$. By (i), there exists $U\in$ *ij*-SO(X,x) such that f(*ij*-scl(U)) ⊂ V. Therefore, we have *ij*-scl(U) ∩ f^{-1}(B) = ϕ and hence $x\notin$ *ij*-scl$_\theta(f^{-1}$(B)). □

(ii) ⇒ (iii): For any subset A of X, by (ii) we have *ij*-scl$_\theta$(A) ⊂ $ij-$ scl$_\theta(f^{-1}(f(A)))\subset f^{-1}$(*ij*-scl(f(A))) and hence f(*ij*-scl$_\theta(A)$) ⊂ *ij*-scl($f(A)$).

(iii) ⇒ (iv): For any $F\in$ *ij*-SC(Y), by (iii) we have f(*ij*-scl$_\theta(f^{-1}(F))$) ⊂ *ij*-scl(F) = F and hence *ij*-scl$_\theta$ $(f^{-1}(F))\subset f^{-1}(F)$. This shows that $f^{-1}(F)$ is *ij*-semi θ-closed.

(iv) ⇒ (v): For any $V\in$ *ij*-SO(Y), $Y\backslash V\in$ *ij*-SC(Y) and $X\backslash f^{-1}(v)=f^{-1}(Y\setminus V)$ is *ij*-semi θ-closed. Therefore, f^{-1}(V) is *ij*-semi θ-open in X.

(v) ⇒ (i): Let $x\in X$ and $V\in$ *ij*-SO(Y,$f(x)$). Then by (v), f^{-1}(V) is *ij*-semi θ-open in X. There exists $U\in$ *ij*-SO(X) such that $x\in U\subset$ *ij*-scl(U) ⊂ f^{-1}(V). Therefore, we have f(*ij*-scl (U)) ⊂ V.

Theorem 3.3. *If $f:(X,\tau_1,\tau_2)\to(Y,\sigma_1,\sigma_2)$ is ij-strongly irresolute, then f is ij-θ-sg-continuous.*

Proof. Let V be *ji*-semi-closed set of Y. Since f is *ji*-strongly irresolute, then by Theorem 3.2, f^{-1}(V) is *ij*-semi θ-closed. By lemma 1.9(i), f^{-1}(V) is *ij*-θ-sg-closed. Thus f is *ij*-θ-sg-continuous. □

The converse of above theorem need not be true the following example show that.

Example 3.4. Let $X=\{a,b,c,d\}$, $Y=\{x,y,z\}$, $\tau_1=\{\phi,\{c\},\{b,c\},X\}$, $\tau_2=\{\phi,\{c,d\},X\}$, $\sigma_1=\{\phi,\{z\},Y\}$ and $\sigma_2=\{\phi,\{y,z\},Y\}$. Define a function $f:(X,\tau_1,\tau_2)\to(Y,\sigma_1,\sigma_2)$ by setting $f(a)=f(b)=f(d)=x$ and $f(c)=z$. Then f is 12-θ-sg-continuous, since $A=\{a,b,d\}=f^{-1}(\{x\})$ is 12-θ-sg-closed. But A is not 21-semi θ-closed. Hence f is not 21-slrongly irresolute, since $\{x\}$ is 21-semi-closed in Y.

Theorem 3.5. *Let $f:(X,\tau_1,\tau_2)\to(Y,\sigma_1,\sigma_2)$ and $g:(Y,\sigma_1,\sigma_2)\to(Z,v_1,v_2)$ are two functions, then:*

(i) If f is *ij*-strongly irresolute and g is *ij*-irresolute, then $g\circ f:X\to Z$ is *ij*-strongly irresolute.
(ii) If f is *ij*-quasi-irresolute and g is *ij*-strongly irresolute, then $g\circ f$ is *ij*-strongly irresolute.

Proof

(i) Let $V\in$ *ij*-SO(Z). Then $g^{-1}(V)\in$ *ij*-SO(Y), since g is *ij*-irresolute. By Theorem 3.2, $f^{-1}(g^{-1}(V))$ = $(gof)^{-1}$ (V) is *ij*-semi θ-open in X. Thus gof is *ij*-strongly irresolute.
(ii) This follows immediately from Theorem 2.5 and 3.2. □

Theorem 3.6. *An ij-irresolute function $f:X\to Y$ is ij-strongly irresolute if and only if X is ij-semi-regular.*

Proof. Let $f:X\to X$ be the identity function. Then f is *ij*-irresolute and *ij*-strongly irresolute by hypothesis. For any $x\in X$ and any $F\in$ *ij*-SC(X) not containing x, $f(x)=x\in X\setminus F\in$ *ij*-SO(X) and there exists $U\in$ *ij*-SO(X) such that f(*ij*-scl(U)) $\subset X\setminus F$. Therefore, we obtain $x\in U\in$ *ij*-SO(X), $F\subset X\setminus$ *ij*-scl(U) $\in$ *ji*-SO(X) and $U\cap(X\setminus$*ij*-scl(U)) $=\phi$. This obvious that X is *ij*-semi-regular. □

Conversely, suppose that $f:X\to Y$ is *ij*-irresolute and X is *ij*-semi-regular. For any $x\in X$ and any $V\in$ *ij*-SO($f(x)$), $f^{-1}(V)\in$ *ij*-SO(X) and there exists $U\in$ *ij*-SO(X) such that $x\in U\in$ *ji*-scl(U) $\subset f^{-1}(V)$, by Lemma 1.7. Therefore, we have f(*ij*-scl(U) $\subset V$. This shows that f is *ij*-strongly irresolute.

Theorem 3.7. *Let $f:X\to Y$ be a function and $g:X\to X\times Y$ the graph function of f. If g is ij-strongly irresolute, then f is ij-strongly irresolute and X is ij-semi-regular.*

Proof. First, we show that f is *ij*-strongly irresolute. Let $x\in X$ and $V\in$ *ij*-SO($f(x)$). Then $X\times V$ is an *ij*-semi-open set of $X\times Y$ containing g(x). Since g is *ij*-strongly irresolute, there exists $U\in$ *ij*-SO(X) such that g(*ij*-scl(U)) $\subset X\times V$. Therefore, we obtain f(*ij*-scl(U)) $\subset V$. Next, let $x\in X$ and $U\in$ *ij*-SO(x). Since $g(x)\in U\times X\in$ *ij*-SO($X\times Y$), there exists $U_0\in$ *ij*-SO(X) such that g(*ij*-scl$(U_0)\subset U\times Y$. Therefore, we obtain $x\in U_0\subset$ *ij*-scl$(U_0)\subset U$ and hence X is *ij*-semi-regular. □

Remark 3.8. The converse to Theorem 3.7, is not true because in Example 2.10, f is 12-strongly irresolute and X is 12-semi-regular but g is not 12-strongly irresolute.

Theorem 3.9. *If $f:X\to Y$ is a P-strongly irresolute injection and Y is P-semi T_0, then X is P-semi T_2.*

Proof. Let x and y be any pair of distinct points of X. Since f is injective it follows that $f(x)\neq f(y)$. Since Y is P-semi T_0, there exists $V\in$ *ij*-SO($f(x)$) not containing $f(y)$ or $W\in$ *ji*-SO($f(y)$) not containing $f(x)$. If it holds that $f(y)\notin V\in$ *ij*-SO($f(x)$) and since f is P-strongly irresolute then there exists $U\in$ *ij*-SO(X) such that f(*ji*-scl(U)) $\subset V$. Therefore, we obtain $f(y)\notin f$(*ji*-scl(U)) and hence $y\in X\setminus$ *ji*-scl(U) $\in$ *ji*-SO(X). If the other case holds, then we obtain the similar result. Therefore, X is P-semi T_2. □

4. *ij*-Semi-compact and *ij*-s-closed spaces.

Definition 4.1. Let A be a subset of a space X, then:

(i) A subset A is said to be *ij*-semi-compact relative to X (resp. *ij*-s-closed relative to X [7]) if for every cover $\{V_\alpha:\alpha\in\nabla\}$ of A by *ij*-semi-open sets of X, there exists a finite subset ∇_0 of ∇ such $A\subset\cup\{V_\alpha:\alpha\in\nabla_0\}$ (resp. $A\subset\cup\{ij\text{-scl}(V_\alpha):\alpha\in\nabla_0\}$.
(ii) A space X is said to be *ij*-semi-compact [5] (resp. *ij*-s-closed [7]) if X is *ij*-semi-compact relative to X (resp. *ij*-s-closed relative to X).
(iii) A subset A is called *ij*-semi-compact if the subspace A is *ij*-semi-compact.

Theorem 4.2. *Let $f:X\to Y$ be an ij-strongly irresolute function. If A is ij-s-closed relative to X, then f(A) is ij-semi-compact.*

Proof. Let A be *ij*-s-closed relative to X and $\{V_\alpha:\alpha\in\nabla\}$ any cover of $f(A)$ by *ij*-semi-open sets of Y. For each $x\in A$, there exists $\alpha(x)\in$ \$ such that $f(x)\in V_{\alpha(x)}$. Since f is *ij*-strongly irresolute, there exists $U_x\in$*ij*-SO(X) such that $f(ij\text{-scl}(U_x))\subset V_{\alpha(x)}$. The family $\{U_x:x\in A\}$ form an *ij*-semi-open cover of A and there exists a finite number of points $x_1,x_2,\dots,x_n$ in A such that $A\subset\cup\{ij\text{-scl}(U_{x_i}):\ i=1,2,\dots,n\}$. Therefore, we obtain $f(A)\subset\cup\{V_{\alpha(x_i)}:\ i=1,2,\dots,n\}$. Thus $f(A)$ is *ij*-semi-compact relative to Y. □

Corollary 4.3. If X is *ij*-s-closed and $f:X\to Y$ is an *ij*-quasi-irresoulte (resp. *ij*-strongly irresolute) surjection, then Y is *ij*-s-closed (resp. *ij*-semi-compact).

Proof. The second case follows from Theorem 4.2. We shall shows the first. Let $\{V_\alpha:\alpha\in\nabla\}$be an *ij*-semi-open cover of Y. By Lemma 1.4, the family $\{ij\text{-scl}(V_\alpha):\alpha\in\nabla\}$ is a cover of Y by *ij*-semi-regular sets of Y. It follows from Theorem 2.4, that the family $\{f^{-1}(ij - scl(V_\alpha)) : \alpha \in \nabla\}$ is a cover of X by *ij*-semi-regular sets of X. Since X is *ij*-s-closed, then there exists a finite subset ∇_0 of ∇ such that $X = \cup\{f^{-1}(ij\text{-scl}(V_\alpha)) : \alpha \in \nabla_0\}$ by Lemma 1.8. Since f is surjective, we have $Y = \cup\{ij\text{-scl}(V_\alpha) : \alpha \in \nabla_0\}$. This shows that Y is *ij*-s-closed. □

A function $f : X \to Y$ is said to be *ij*-pre-semi-closed [8] if $f(F) \in ij\text{-SC}(Y)$ for every $F \in ij\text{-SC}(X)$.

Lemma 4.4. *A surjection $f : X \to Y$ is ij-pre-semi-closed if and only if for each point $y \in Y$ and each $U \in ij\text{-}SO(X)$ containing $f^{-1}(y)$, there exists $V \in ij\text{-}SO(Y)$ such that $f^{-1}(V) \subset U$.*

Proof. The first side follows from Lemma 1.6. On the other hand, let A be an *ij*-semi-open set of X. Suppose that $y \in Y \setminus f(A)$ where $X \setminus A$ is *ij*-semi-closed set of X. By hypostasis, there exists an *ij*-semi-open set $V \subset Y$ such that $f^{-1}(V) \subset X \setminus A$. Thus $A \subset f^{-1}(Y \setminus V)$, this implies $f(A) \subset Y \setminus V$. Hence $y \in V \subset Y \setminus f(A)$ and $Y \setminus f(A)$ is *ij*-semi-open set of Y. It follows that $f(A)$ is *ij*-semi-closed set in Y and hence f is an *ij*-pre-semi-closed. □

Theorem 4.5. *Let $f : X \to Y$ be an ij-pre-semi-closed surjection and $f^{-1}(y)$ be ij-s-closed relative to Y (resp. ij-semi-compact relative to Y) for each $y \in Y$. If K is ij-semi-compact relative to Y, then $f^{-1}(K)$ is ij-s-closed relative to X (resp. ij-semi-compact relative to Y).*

Proof. Suppose that for each $y \in Y$, $f^{-1}(y)$ is *ij*-s-closed relative to Y and K is *ij*-semi-compact relative to Y. Let $\{U_\alpha : \alpha \in \nabla\}$ be a cover of f^{-1}(K) by *ij*-semi-open sets of X. For each $y \in K$, there exists a finite subset $\nabla(y)$ of ∇ such that $f^{-1}(y) \subset \cup\{ij\text{-scl}(U_\alpha) : \alpha \in \nabla(y)\}$. By Lemma 1.4, *ij*-scl $(U_\alpha) \in ij\text{-SO}(X)$ for each $\alpha \in \nabla$ and hence $\cup\{ij - scl(U_\alpha) : \alpha \in \nabla(y)\} \in ij\text{-SO}(X)$. By Lemma 4.4, there exists $V_y \in ij\text{-SO}(y)$ such that $f^{-1}(V_y) \subset \cup\{ij - scl(U_\alpha) : \alpha \in \nabla(y)\}$. Since $\{V_y : y \in K\}$ is an *ij*-semi-open cover of K, for a finite number of points $y_1, y_2, \ldots, y_n$ in K, we have $K \subset \cup\{V_{y_i} : i = 1, 2, \ldots, n\}$ and hence $f^{-1}(K) \subset \cup_{i=1}^{n} f^{-1}(V_{y_i}) \subset \cup_{i=1}^{n} \cup_{\alpha\in\nabla(y_i)} (ij\text{-scl}(U_\alpha))$. Therefore, $f^{-1}(K)$ is *ij*-s-closed relative to X. The proof of the case *ij*-semi-compact relative to Y is similar. □

Corollary 4.6. Let $f : X \to Y$ be an *ij*-pre-semi-closed surjection and $f^{-1}(y)$ be *ij*-s-closed relative to Y (resp. *ij*-semi-compact relative to Y) for each y ∈ Y. If Y is *ij*-semi-compact, then X is *ij*-s-closed (resp. *ij*-semi-compact).

Proof. This follows immediately from Theorem 4.5. □

5. Comparisons.

Definition 5.1. A function $f : (X, \tau_1, \tau_2) \to (Y, \sigma_1, \sigma_2)$ is said to be *ij*-semi-weakly continuous if for each $x \in X$ and each σ_i-open neighborhood V of $f(x)$, there exists $U \in ij\text{-SO}(X)$ such that $f(U) \subset$ j-cl(V).

Remark 5.2. *ij*-strongly irresolute implies *ij*-irresolute and *ij*-irresolute implies *ij*-quasi-irresolute. However, *ij*-strongly irresolute and i-continuous are independent of each other as the following two examples show.

Example 5.3. Let $X = Y = \{a, b, c\}$, $\tau_1 = \{\phi, \{a\}, \{b\}, \{a, b\}, X\}$, $\tau_2 = \{\phi, \{a\}, \{c\}, \{a, c\}, \{b, c\}, X\}$, $\sigma_1 = \{\phi, \{a\}, \{b, c\}, Y\}$ and $\sigma_2 = \{\phi, \{b\}, \{b, c\}, Y\}$. Then the identity function $f : (X, \tau_1, \tau_2) \to (Y, \sigma_1, \sigma_2)$ is 12-strongly irresolute but not 1-continuous.

Example 5.4. Let $X = Y = \{a, b, c\}$, $\tau_1 = \{\phi, \{b\}, \{a, b\}, \{b, c\}, X\}$, $\tau_2 = \{\phi, \{a\}, \{a, c\}, X\}$, $\sigma_1 = \{\phi, \{a\}, \{b, c\}, Y\}$ and $\sigma_2 = \{\phi, \{a\}, \{c\}, \{a, c\}, Y\}$. Then the function $f : (X, \tau_1, \tau_2) \to (Y, \sigma_1, \sigma_2)$ defined by $f(a) = f(b) = b$ and $f(c) = c$. It is evident that f is 1-continuous. Since $b \in f(U)$ and $12\text{-SO}(a) = \{\{a, b\}, X\}$, then for each $U \in 12\text{-SO}(a)$, we have $f(U)a = 21\text{-scl}(a)$ for every $U \in$ 12-SO(a). This shows that f is not 12-quasi-irresolute.

Theorem 5.5. *An ij-irresoluteness implies both ij-quasi-irresolute and ij-semi-continuous.*

Proof. Straightforward from the fact that every i-open set is *ij*-semi-open and [[6], Remark 5.1]. □

The converse of Theorem 5.5 is not true as Example 5.4 and the following example show.

Example 5.6. Let $X = \{a, b, c\}$, $\tau_1 = \{\phi, \{a, b\}, X\}$, $\tau_2 = \{\phi, \{a, c\}, X\}$, $\sigma_1 = \{\phi, \{b, c\}, X\}$ and $\sigma_2 = \{\phi, X\}$. Then the identity function $f : (X, \tau_1, \tau_2) \to (X, \sigma_1, \sigma_2)$ is 12-quasi-irresolute. However, f is not 12-semi-continuous and hence not 12-irresolute.

Theorem 5.7. *An ij-quasi-irresolute implies ij-semi-weak continuity.*

Proof. It follows from definition. □

The converse of the above theorem is not true, since in Example 5.4, f is 12-semi-weakly continuous but not 12-quasi-irresolute.

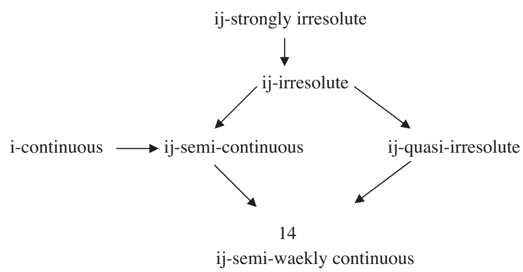
Remark 5.8. Every *ij*-semi-continuous function is *ij*-semi-weakly continuous but the converse is not true, the following example shows that.

Example 5.9. Let $X = \{a, b\}$, $\tau_1 = \{\phi, \{a\}, X\}$, $\tau_2 = \{\phi, X\}$, $\sigma_1 = \{\phi, \{b\}, X\}$ and $\sigma_2 = \{\phi, \{a\}, \{b\}, X\}$. Then the identity function $f : (X, \tau_1, \tau_2) \to (X, \sigma_1, \sigma_2)$ is 12-semi-weakly continuous but not 12-semi-continuous.

Remark 5.10. Every *ij*-strongly irresolute function is *ij*-irresolute. The converse need not be true, the following example shows that.

Example 5.11. Let $X = \{a, b, c\}$, $\tau_1 = \{\phi, \{a, b\}, X\}$, $\tau_2 = \{\phi, \{b\}, \{a, b\}, X\}$, $\sigma_1 = \{\phi, \{b, c\}, X\}$ and $\sigma_2 = \{\phi, X\}$. Then the function $f : (X, \tau_1, \tau_2) \to (X, \sigma_1, \sigma_2)$ defined by $f(a) = c$, $f(b) = b$ and $f(c) = a$. Then f is 12-irresolute but f is not 12-strongly irresolute, since $\{b, c\} \in 12\text{-SO}(X)$ and $12\text{-SO}(X) = \{\phi, \{a, b\}, X\}$ such that $21\text{-scl}(\{a, b\}) = X$. Thus $f(X)\{b, c\}$ and hence f is not 12-strongly irresolute.

By [[6], Remark 5.1] and for remarks in this section, we obtain the following diagram, where none of the implication is reversible.



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