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The periodic rotary motions of a rigid body in a new domain of angular velocity

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Abstract

In the previous works, the limiting case for the motion of a rigid body about a fixed point in a Newtonian force field, which comes from a gravity center lies on *Z*-axis, is solved. The authors apply the small parameter technique which is achieved giving the body a sufficiently large angular velocity component r_o about the fixed *z*-axis of the body. The periodic solutions of motion are obtained in neighborhood r_o tends to ∞. In our work, we aim to find periodic solutions to the problem of motion in the neighborhood of r_0 tends to 0. So, we give a new assumption that: r_o is sufficiently small. Under this assumption, we must achieve a large parameter and search for another technique for solving this problem. This technique is named; a large parameter technique instead of the small one well known previously. We see the advantage of the new technique which appears in saving high energy used to begin the motion and give the solution of the problem in another domain. The obtained solutions by the new technique depend on r_o. We consider that the center of mass of this body does not necessarily coincide with the fixed point O. We reduce the six nonlinear differential equations of the body and their three first integrals to a quasilinear autonomous system of two degrees of freedom and one first integral. We solve the rational case when the frequencies of the generating system are rational except ($\omega = 1, 2, 1/2, 3, 1/3, \ldots$) under the condition $\gamma_0'' = \cos\theta_o \approx 0$. We use the fourth-order Runge–Kutta method to find the periodic solutions in the closed interval of the time t and to compare the analytical method with the numerical one.

Keywords: Equations of Euler, Rigid body dynamics, Mathematical techniques, Gyroscopic motions, Numerical methods, Satellite motions, Navigation

Mathematics Subject Classification: 70E05, 70E15, 70E17, 70E20

Introduction

Some asymptotic perturbed techniques [1–3] are widely used by many authors for solving the ordinary linear and nonlinear systems for differential equations in different problems of engineering, mathematical physics, and astronomy. As an extension of this type of problem, we use some perturbation and numerical techniques in the movement of coherent bodies around a fixed point in the presence of new conditions to obtain periodic solutions of different scales to those obtained before.

As in [4], let the fixed *z*-axis of the ellipsoid of inertia of the body makes an angle $\theta_o \approx {}^{\pi}/_{2}$ with the fixed *Z*-axis in space. We assume the new value of r_o, which is sufficiently small instead of sufficiently large value in [4]. We define a large parameter μ proportional to $1/r_o$ instead of the small one in [4]. Consider A, B, C represent the moments of inertia of the body, p, q, r are the components of the angular velocity vector and $\gamma, \gamma', \gamma''$ are the direction cosines of the unit vector in direction of the Z-axis. So, the equations of motion and their three first integrals are derived in the form:

$$\begin{aligned}
&\dot{p}_1 + A_1 q_1 r_1 = \mu^{-1} a^{-1}(\gamma'' y_0' - \gamma' z_0' + kaA_1\gamma'\gamma''),\\
&\dot{q}_1 + B_1 p_1 r_1 = \mu^{-1} b^{-1}(\gamma z_0' - \gamma'' x_0' + kbB_1\gamma''\gamma),\\
&\dot{r}_1 = \mu^{-2}(-C_1 p_1 q_1 + \gamma' x_0' - \gamma y_0' + kC_1\gamma\gamma'),\\
&\dot{\gamma} = r_1\gamma' - \mu^{-1} q_1\gamma'', \quad \dot{\gamma}' = \mu^{-1} p_1\gamma'' - r_1\gamma, \quad \dot{\gamma}'' = \mu^{-1}(q_1\gamma - p_1\gamma').
\end{aligned} \tag{1}$$

$$r_1^2 = 1 + \mu^{-2}s_1, \quad r_1\gamma'' = \gamma_0'' + \mu^{-1}s_2, \quad \gamma^2 + \gamma'^2 + \gamma''^2 = 1. \tag{2}$$

Such that:

$$\begin{aligned}
s_1 &= s_3 - 2z_0'(\gamma_0'' - \gamma'') + k(\gamma_0''^2 - \gamma''^2),\\
s_2 &= a(p_{10}\gamma_0 - p_1\gamma) + b(q_{10}\gamma_0' - q_1\gamma')\\
s_3 &= a(p_{10}^2 - p_1^2) + b(q_{10}^2 - q_1^2) - 2[x_0'(\gamma_0 - \gamma) + y_0'(\gamma_0' - \gamma')]\\
&\quad + k[a(\gamma_0^2 - \gamma^2) + b(\gamma_0'^2 - \gamma'^2)].
\end{aligned} \tag{3}$$

$$p_1 = p/c, \quad (pqr), \quad \gamma_0 > 0, \quad \mu = c/r_0, \quad l^2 = x_0^2 + y_0^2 + z_0^2, \tag{4}$$

$$k = N\Big/c^2, \quad A_1 = (C - B)/A, \quad (ABC). \tag{5}$$

Reduction of the equations of motion to a quasilinear autonomous system

Solving the first and the second equations of (2) for r_1 and γ'', we get:

$$r_1 = 1 + \frac{1}{2}\mu^{-2}s_3 + \ldots, \quad \gamma'' = \gamma_0'' + \mu^{-1}s_2 - \frac{1}{2}\mu^{-2}s_3\gamma_0'' + \ldots. \tag{6}$$

Differentiating the first and the fourth equations of (1) and using (6) for reduction in the four remaining equations into two differential equations of the second order, we get:

$$\begin{aligned}
\ddot{p}_1 + \omega^2 p_1 &= \mu^{-1}[z_0'(a^{-1} - A_1 b^{-1})\gamma + A_1 b^{-1} x_0'\gamma_0'' - k(A_1 - \omega^2)\gamma_0''\gamma]\\
&\quad + \mu^{-2}\{A_1 x_0'(b^{-1}s_2 - q_1\gamma') - \omega^2 p_1 s_3 + p_1(A_1 C_1 q_1^2 - a^{-1}z_0'\gamma_0'')\\
&\quad + y_0'[(2A_1 + a^{-1}b)q_1\gamma - a^{-1}p_1\gamma'] + kA_1[p_1(\gamma_0''^2 - \gamma'^2)\\
&\quad + q_1(1 - C_1)\gamma\gamma' - s_2(1 + B_1)\gamma]\} + \mu^{-3}\{0.5(a^{-1} - A_1 b^{-1})z_0' s_3\gamma\\
&\quad - s_2 p_1[z_0'(2\omega^2 + a^{-1}) - 2k(A_1 + \omega^2)\gamma_0'']\} + \ldots,\\
\ddot{\gamma} + \gamma &= \mu^{-1}\gamma_0'' p_1(1 + B_1) + \mu^{-2}\{p_1[(1 + B_1)s_2 + (1 - C_1)q_1\gamma']\\
&\quad - \gamma[s_3 + y_0'\gamma' + z_0' b^{-1}\gamma_0'' + q_1^2 + k(C_1\gamma'^2 - B_1\gamma_0''^2)]\\
&\quad + x_0'(b^{-1}\gamma_0''^2 + \gamma'^2)\} + \mu^{-3}\{2b^{-1}x_0'\gamma_0'' s_2\\
&\quad + [2k(1 - B_1)\gamma_0'' - z_0'(2 + b^{-1})]s_2\gamma_2\} + \ldots.
\end{aligned} \tag{7}$$

Here:

$$\omega^2 = -A_1 B_1 = (A-C)(B-C)/AB = (a-1)(b-1)/ab.$$

Solving the first and the fourth equations of the system (1) and using (6), we get:

$$\begin{aligned} q_1 &= -\dot{p}_1 A_1^{-1} + \mu^{-1} a^{-1} A_1^{-1}(y_0'\gamma_0'' - z_0'\gamma' + a\,k\,A_1\gamma'\gamma_0'') + \dots, \\ \gamma' &= \dot{\gamma} - \mu^{-1} A_1^{-1}\gamma_0''\dot{p}_1 + \dots . \end{aligned} \tag{8}$$

Introducing the new variables p_2 and γ_2 such that:

$$\begin{aligned} p_1 &= p_2 + \mu^{-1}\chi_1\gamma_2 + \mu^{-1}\frac{x_0'}{1-a}\gamma_0'', \\ \gamma &= \gamma_2 + \mu^{-1} a\gamma_0'' p_2, \end{aligned} \tag{9}$$

where:

$$\chi_1 = (1-\omega^2)^{-1}[-z_0'(a^{-1} - A_1\, b^{-1}) + k\gamma_0''(A_1 - \omega^2)].$$

Substituting from (9) into (8), we get:

$$\begin{aligned} q_1 &= -A_1^{-1}\dot{p}_2 + \mu^{-1}A_1^{-1}(y_0' a^{-1}\gamma_0'' - \chi_2\dot{\gamma}_2) + \dots, \\ \gamma' &= \dot{\gamma}_2 - \mu^{-1}\gamma_0'' A_1^{-1} b\dot{p}_2 + \dots, \quad \chi_2 = \chi_1 + z_0' a^{-1} - kA_1\gamma_0''. \end{aligned} \tag{10}$$

Substituting (10) and (9) into (3), we get:

$$s_i = s_{i1} + 2^{2-i}\mu^{-1}s_{i2} + \dots, \quad i = 1, 2, \tag{11}$$

where:

$$\begin{aligned} s_{11} &= a(p_{20}^2 - p_2^2) + b(\dot{p}_{20}^2 - \dot{p}_2^2)\Big/ A_1^2 - 2[x_0'(\gamma_{20} - \gamma_2) \\ &\quad + y_0'(\dot{\gamma}_{20} - \dot{\gamma}_2)] + k[a(\gamma_{20}^2 - \gamma_2^2) + b(\dot{\gamma}_{20}^2 - \dot{\gamma}_2^2)], \\ s_{12} &= a\left[\frac{x_0'}{1-a}\gamma_0''(p_{20} - p_2) + \chi_1(p_{20}\gamma_{20} - p_2\gamma_2)\right] \\ &\quad - bA_1^{-2}[y_0' a^{-1}\gamma_0''(\dot{p}_{20} - \dot{p}_2) - \chi_2(\dot{p}_{20}\dot{\gamma}_{20} - \dot{p}_2\dot{\gamma}_2)] \\ &\quad - ax_0'\gamma_0''(p_{20} - p_2) + y_0' bA_1^{-1}\gamma_0''(\dot{p}_{20} - \dot{p}_2) + (z_0' - k\gamma_0'')s_{21} \\ &\quad + k\gamma_0''[a^2(p_{20}\gamma_{20} - p_2\gamma_2) - b^2A_1^{-1}(\dot{p}_{20}\dot{\gamma}_{20} - \dot{p}_2\dot{\gamma}_2)], \\ s_{21} &= a(p_{20}\gamma_{20} - p_2\gamma_2) - bA_1^{-1}(\dot{p}_{20}\dot{\gamma}_{20} - \dot{p}_2\dot{\gamma}_2), \\ s_{22} &= a\left[a\gamma_0''(p_{20}^2 - p_2^2) + \frac{x_0'}{1-a}\gamma_0''(\gamma_{20} - \gamma_2) + \chi_1(\gamma_{20}^2 - \gamma_2^2)\right] \\ &\quad + bA_1^{-1}[bA_1^{-1}\gamma_0''(\dot{p}_{20}^2 - \dot{p}_2^2) + y_0' a^{-1}\gamma_0''(\dot{\gamma}_{20} - \dot{\gamma}_2) - \chi_2(\dot{\gamma}_{20}^2 - \dot{\gamma}_2^2)], \\ s_3 &= s_{11} + 2\mu^{-1}(s_{12} - z_0' s_{21} + ks_{21}\gamma_0'') + \dots . \end{aligned} \tag{12}$$

Substituting (11), and (12) into (6) yields:

$$r_1 = 1 + \frac{1}{2}\mu^{-2}s_{11} + \dots, \quad \gamma'' = \gamma_0'' + \mu^{-1}s_{21} + \mu^{-2}s_{22} - \frac{1}{2}\mu^{-2}s_{11}\gamma_0'' + \dots . \tag{13}$$

Substituting (9), (10), (11), and (12) into (7), we obtain a quasilinear autonomous system [5]:

$$\ddot{p}_2 + \omega^2 p_2 = \mu^{-2} F(p_2, \dot{p}_2, \gamma_2, \dot{\gamma}_2, \varepsilon), \quad \ddot{\gamma}_2 + \gamma_2 = \mu^{-2}\Phi(p_2, \dot{p}_2, \gamma_2, \dot{\gamma}_2, \varepsilon), \tag{14}$$

where:

$$\begin{aligned} &F = F_2 + \mu^{-1}F_3 + \ldots, \qquad \Phi = \Phi_2 + \mu^{-1}\Phi_3 + \ldots, \\ &F_2 = f_2 - a\gamma_0''\chi_1(1-\omega^2)p_2, \; \Phi_2 = \varphi_2 + a\gamma_0''(1-\omega^2)[\chi_1\gamma_2 + \gamma_0''x_0'(1-a)^{-1}], \end{aligned}$$

$$\begin{aligned} F_3 &= f_3 - \chi_1\{\varphi_2 + a\gamma_0''(1-\omega^2)[\chi_1\gamma_2 + \gamma_0''x_0'(1-a)^{-1}]\}, \\ \Phi_3 &= \varphi_3 + a\gamma_0''[a\gamma_0''\chi_1(1-\omega^2)p_2 - f_2], \\ f_2 &= x_0'(A_1b^{-1}s_{21} + \dot{p}_2\dot{\gamma}_2) + A_1^{-1}\dot{p}_2[C_1p_2\dot{p}_2 - y_0'(2A_1 + ba^{-1})\gamma_2] - p_2(\omega^2 s_{11} \\ &\quad + y_0'a^{-1}\dot{\gamma}_2 + z_0'a^{-1}\gamma_0'') - k\{\gamma_2[(1-C_1)\dot{p}_2\dot{\gamma}_2 + A_1(1+B_1)s_{21}] - A_1p_2(\gamma_0''^2 - \dot{\gamma}_2^2)\}, \end{aligned}$$

$$\begin{aligned} \varphi_2 &= (1+B_1)p_2s_{21} + x_0'(\dot{\gamma}_2^2 + b^{-1}\gamma_0'') - A_1^{-1}\dot{p}_2[A_1^{-1}\gamma_2\dot{p}_2 + (1-C_1)p_2\dot{\gamma}_2] \\ &\quad - \gamma_2[s_{11} + z_0'b^{-1}\gamma_0'' + y_0'\dot{\gamma}_2 - k(C_1\dot{\gamma}_2^2 - B_1\gamma_0''^2)], \\ f_3 &= -\omega^2\left[s_{11}\left(\chi_1\gamma_2 + \frac{x_0'}{1-a}\gamma_0''\right) + 2p_2s_{12}\right] + C_1A_1^{-1}\dot{p}_2 \\ &\quad \times \left[\dot{p}_2\left(\chi_1\gamma_2 + \frac{x_0'}{1-a}\gamma_0''\right) + 2p_2\dot{p}_2(\chi_2\dot{\gamma}_2 - y_0'a^{-1}\gamma_0'')\right] \\ &\quad + x_0'[A_1b^{-1}s_{22} - bA_1^{-1}\gamma_0''\dot{p}_2^2 - \gamma_0''\dot{\gamma}_2(y_0'a^{-1} - \chi_2\dot{\gamma}_2)] \\ &\quad - y_0'a^{-1}\gamma_0''\left[\dot{\gamma}_2\left(\chi_1\gamma_2 + \frac{x_0'}{1-a}\right) - bA_1^{-1}p_2\dot{p}_2\right] \\ &\quad + z_0'a^{-1}\left[\frac{1}{2}b^{-1}(2b-1)s_{11}\gamma_2 - \gamma_0''\left(\chi_1\gamma_2 + \frac{x_0'}{1-a}\right) - p_2s_{21}\right] \\ &\quad + y_0'(2-b)(1-b)^{-1}[\gamma_2(y_0'a^{-1}\gamma_0'' - \chi_2\dot{\gamma}_2) - a\gamma_0''p_2\dot{p}_2] \\ &\quad + k\Big\{(1-C_1)[(y_0'a^{-1}\gamma_0'' - \chi_2\dot{\gamma}_2)\gamma_2\dot{\gamma}_2 - a\gamma_0''p_2\dot{p}_2\dot{\gamma}_2 + bA_1^{-1}\gamma_0''\gamma_2\dot{p}_2^2] \\ &\quad + p_2\gamma_0''[2b\dot{p}_2\dot{\gamma}_2 - aA_1(1+B_1)s_{21} + 2A_1s_{21}] \\ &\quad + A_1\left[\left(\chi_1\gamma_2 + \gamma_0''\frac{x_0'}{1-a}\right)(\gamma_0''^2 - \dot{\gamma}_2^2) - (1+B_1)\gamma_2s_{22}\right]\Big\}, \\ \varphi_3 &= -2s_{12}\gamma_2 - a\gamma_0''p_2s_{11} + (1+B_1)\left[p_2s_{22} + \left(\frac{x_0'}{1-a}\gamma_0'' + \chi_1\gamma_2\right)s_{21}\right] \\ &\quad + 2x_0'\gamma_0''(b^{-1}s_{21} - bA_1^{-1}\dot{p}_2\dot{\gamma}_2) - z_0'b^{-1}(a\gamma_0''^2p_2 + s_{21}\gamma_2) + A_1^{-1}(1-C_1) \\ &\quad \times \left[bA_1^{-1}\gamma_0''p_2\dot{p}_2^2 - \left(\frac{x_0'}{1-a}\gamma_0'' + \chi_1\gamma_2\right)\dot{p}_2\dot{\gamma}_2 + b^{-1}\gamma_0''^2x_0' + p_2\dot{\gamma}_2(y_0'a^{-1} - \chi_2\dot{\gamma}_2)\right] \\ &\quad - y_0'\gamma_0''(ap_2\dot{\gamma}_2 - bA_1^{-1}\gamma_2\dot{p}_2) + A_1^{-2}[2\dot{p}_2(y_0'a^{-1}\gamma_0'' - \chi_2\dot{\gamma}_2)\gamma_2 - a\gamma_0''p_2\dot{p}_2^2] \\ &\quad + k\gamma_0''[ap_2(C_1\dot{\gamma}_2^2 - B_1\gamma_0''^2) - 2\gamma_2(bA_1^{-1}C_1\dot{p}_2\dot{\gamma}_2 + B_1s_{21})]. \end{aligned}$$

The last equation of (2) gives the first integral of the system (14) as follows [6]:

$$\gamma_2^2 + \dot{\gamma}_2^2 + 2\mu^{-1}\gamma_0''(ap_2\gamma_2 - bA_1^{-1}\dot{p}_2\dot{\gamma}_2 + s_{21}) + \ldots = 1 - \gamma_0''^2. \tag{15}$$

In the next, we will look for the periodic solutions of the system (14) under the conditions $A > B > C$ or $A < B < C$ (ω^2 is positive). The first condition [7] gives the slow rotation of the body about the major axis of the ellipsoid of inertia and the second gives a slow rotation of the body about the minor axis of the ellipsoid of inertia. We apply the large parameter method [8] to solve the autonomous system (14).

The formal construction of the periodic solutions for a rational value of the natural frequency ω

We achieve the periodic solutions $p_2(\tau, \mu^{-1}),\ \dot{p}_2(\tau, \mu^{-1}),\ \gamma_2(\tau, \mu^{-1}),\ \dot{\gamma}_2(\tau, \mu^{-1})$ of system (14) when:

$$p_2(0, 0) = \dot{p}_2(0, 0) = \dot{\gamma}_2(0, \mu^{-1}) = 0\,. \tag{16}$$

The generating system of (14) is obtained when $\mu \to \infty$ in the form:

$$\ddot{p}_2^{(0)} + \omega^2 p_2^{(0)} = 0, \quad \ddot{\gamma}_2^{(0)} + \gamma_2^{(0)} = 0. \tag{17}$$

So the periodic solutions of system (17), when the period $T_0 = 2\pi n$, become:

$$p_2^{(0)} = M_1 \cos\, \omega\tau + M_2 \sin\, \omega\tau, \quad \gamma_2^{(0)} = M_3 \cos\, \tau, \tag{18}$$

where $M_i,\ \ i = 1, 2, 3$ are constants to be determined.

Assuming the following solutions of system (14) [9]:

$$\begin{aligned} p_2(\tau, \mu) &= \tilde{M}_1 \cos\, \omega\tau + \tilde{M}_2 \sin\, \omega\tau + \sum_{k=2}^{\infty} \mu^{-k} G_k(\tau), \\ \gamma_2(\tau, \mu) &= \tilde{M}_3 \cos\, \tau + \sum_{k=2}^{\infty} \mu^{-k} H_k(\tau)\,, \end{aligned} \tag{19}$$

with a period $T(\mu^{-1}) = T_0 + \alpha(\mu^{-1})$ which reduces to (18) at $\mu \to \infty$. Let us define the quantities $\tilde{M}_i,\ i = 1, 2, 3$ as follow:

$$\tilde{M}_i = M_i + \beta_i(\mu^{-1}), \quad i = 1, 2, 3, \tag{20}$$

where β_i are functions of μ^{-1} which represent the deviations of the initial values of $p_2,\ \dot{p}_2,\ \gamma_2$ for system (14) from their initial values of generating system (17) such that $\beta_i(0) = 0$.

Let us express the initial conditions (16) by the relations:

$$p_2(0, \mu^{-1}) = \tilde{M}_1, \quad \dot{p}_2(0, \mu^{-1}) = \omega\tilde{M}_2, \quad \gamma_2(0, \mu^{-1}) = \tilde{M}_3, \quad \dot{\gamma}_2(0, \mu^{-1}) = 0. \tag{21}$$

We rewrite the periodic solutions (18) in the form:

$$p_2^{(0)} = E \cos\, (\omega\tau - \varepsilon), \quad \gamma_2^{(0)} = M_3 \cos\, \tau, \tag{22}$$

where $E = \sqrt{M_1^2 + M_2^2}$ and $\varepsilon = \tan^{-1} M_2 / M_1$. Using (22) and (12), we get:

$$
\begin{aligned}
s_{11}^{(0)} &= E^2[(a\cos^2\varepsilon - 0.5) + b\omega^2 A_1^{-2}(\sin^2\varepsilon - 0.5) + 0.5(b\omega^2 A_1^{-2} - a)\cos 2(\omega\tau - \varepsilon)] \\
&\quad - 2M_3[x_0'(1-\cos\tau) + y_0'\sin\tau] - 0.5kM_3^2 C_1(1-\cos 2\tau), \\
s_{12}^{(0)} &= \mathrm{a}^2 E\gamma_0'' x_0'(1-a)^{-1}[\cos\varepsilon - \cos(\omega\tau - \varepsilon)] \\
&\quad + Ey_0' b\gamma_0'' A_1^{-1}\omega(1 - a^{-1}A_1^{-1})[\sin\varepsilon + \sin(\omega\tau - \varepsilon)] \\
&\quad + aEM_3(\chi_1 + a\gamma_0'' k)\{\cos\varepsilon - 0.5\cos[(\omega+1)\tau - \varepsilon] - 0.5\cos[(\omega-1)\tau - \varepsilon]\} \\
&\quad + \omega b A_1^{-1} EM_3(\chi_2 A_1^{-1} - bk\gamma_0'')\{0.5\cos[(\omega+1)\tau - \varepsilon] - 0.5\cos[(\omega-1)\tau - \varepsilon]\} \\
&\quad + EM_3(z_0' - \gamma_0'' k)\{a\cos\varepsilon + 0.5(b\omega A_1^{-1} - a)\cos[(\omega-1)\tau - \varepsilon] \\
&\quad - 0.5(b\omega A_1^{-1} + a)\cos[(\omega+1)\tau - \varepsilon]\}, \\
s_{21}^{(0)} &= EM_3\{a\cos\varepsilon + 0.5(b\omega A_1^{-1} - a)\cos[(\omega-1)\tau - \varepsilon] \\
&\quad - 0.5(b\omega A_1^{-1} + a)\cos[(\omega+1)\tau - \varepsilon]\}, \\
s_{22}^{(0)} &= E^2\gamma_0''[a^2(\cos^2\varepsilon - 0.5) + b^2\omega^2 A_1^{-2}(\sin^2\varepsilon - 0.5) \\
&\quad - 0.5(a^2 - b^2\omega^2 A_1^{-2})\cos 2(\omega\tau - \varepsilon)] + 0.5M_3^2(a\chi_1 + bA_1^{-1}\chi_2)(1-\cos 2\tau) \\
&\quad + \gamma_0'' M_3[a x_0'(1-a)^{-1}(1-\cos\tau) + by_0' a^{-1}A_1^{-1}\sin\tau].
\end{aligned} \tag{23}
$$

Substituting (22) and (23) into (16), we get:

$$
\begin{aligned}
F_2^{(0)} &= M_1 L(\omega)\cos\omega\tau + M_2 L(\omega)\sin\omega\tau + \ldots, \\
\Phi_2^{(0)} &= M_3 N(\omega)\cos\tau + \ldots, \\
F_3^{(0)} &= M_1 K(\omega)\cos\omega\tau + M_2 K(\omega)\sin\omega\tau + \ldots,
\end{aligned} \tag{24}
$$

where:

$$
\begin{aligned}
L(\omega) &= \omega^2[-(aM_1^2 + b\omega^2 A_1^{-2}M_2^2) + 0.25(M_1^2 + M_2^2)(C_1 A_1^{-1} + 3a + b\omega^2 A_1^{-2})] \\
&\quad + 2\omega^2 M_3 x_0' - \gamma_0''[z_0' a^{-1} + a\chi_1(1-\omega^2)] + k\{A_1(\gamma_0''^2 - 0.5M_3^2) \\
&\quad + 0.5M_3^2[a(A_1 - 2\omega^2) + \omega^2 b]\}, \\
N(\omega) &= -(aM_1^2 + b\omega^2 A_1^{-2}M_2^2) - 0.5(M_1^2 + M_2^2)[aB_1 + \omega^2 A_1^{-2}(1-b)] \\
&\quad + 2M_3 x_0' - \gamma_0''[z_0' b^{-1} - a\chi_1(1-\omega^2)] + k[M_3^2(b-a) - B_1\gamma_0''^2], \\
K(\omega) &= -2\omega^2\gamma_0''[a^2 x_0' M_1(1-a)^{-1} + \omega y_0' b M_2(1 - a^{-1}A_1^{-1})A_1^{-1} \\
&\quad + 0.25a^{-1}A_1^{-1}C_1 y_0'(M_1^2 + M_2^2)] - 2\omega^2 a M_1 M_3[\chi_1 + a\gamma_0'' k + z_0' - \gamma_0'' k] \\
&\quad + aM_1 M_3[2A_1 k\gamma_0'' - z_0' a^{-1} - aA_1 k\gamma_0''(1+B_1) - \chi_1(1+B_1)].
\end{aligned} \tag{25}
$$

Using (24) and (25), the following functions are obtained:

$$
\begin{aligned}
&g_2(T_0) = -\pi n\omega^{-1}M_2 L(\omega), && \dot{g}_2(T_0) = \pi n M_1 L(\omega), \\
&h_2(T_0) = 0, && \dot{h}_2(T_0) = \pi n M_3 N(\omega), \\
&g_3(T_0) = -\pi n\omega^{-1}M_2 K(\omega), && \dot{g}_3(T_0) = \pi n M_1 K(\omega).
\end{aligned} \tag{26}
$$

Substituting by the initial conditions (21) into the first integration (17) when $\tau = 0$, we get:

$$
M_3^2 + 2M_3\beta_3 + \beta_3^2 + 2\mu^{-1}a\gamma_0'' M_3(M_1 + \beta_1) = 1 - \gamma_0''^2. \tag{27}
$$

Let γ_0'' depends on μ^{-1}, we get:

$$\gamma_0'' = \mu^{-1}\Gamma, \quad 0 < \Gamma < 1. \tag{28}$$

Taking into consideration, Eqs. (27) and (28), we get $M_3,\ \beta_3$ as follows:

$$M_3 = 1, \quad \beta_3 = -a\Gamma\mu^{-2}\tilde{M}_1 - \frac{1}{2}\mu^{-2}\Gamma^2 + \ldots. \tag{29}$$

The independent conditions for periodicity are:

$$\begin{aligned} &-(L_1(\omega) - \omega^2 N_1(\omega))\pi n\omega^{-1}\tilde{M}_2 + \mu^{-1}\, G_3(T_0) + \ldots = 0, \\ &(L_1(\omega) - \omega^2 N_1(\omega))\pi n\tilde{M}_1 + \mu^{-1}\, \dot{G}_3(T_0) + \ldots = 0, \\ &\mu^{-2}(\dot{H}_2(T_0) + \mu^{-1}\dot{H}_3(T_0))\tilde{M}_3^{-1} + \ldots = \alpha(\mu^{-1}), \end{aligned} \tag{30}$$

where $L_1(\omega),\ N_1(\omega)$ are obtained from $L(\omega),\ N(\omega)$ replacing M_i by $(M_i + \beta_i),\ i = 1, 2, 3$ to get:

$$L_1(\omega) - \omega^2 N_1(\omega) = W_0(\omega)(\tilde{M}_1^2 + \tilde{M}_2^2) - \gamma_0''[z_0'\, W_1(\omega) + k\gamma_0''\, W_2(\omega)] - kW_3(\omega)\tilde{M}_3^2, \tag{31}$$

where:

$$\begin{aligned} &W_0(\omega) = (a-1)(a+b-2)/2b, \ W_1(\omega) = [3(a+b) - 2(2ab+1)]/ab, \\ &W_2(\omega) = 2\omega^2[1 - (a+b)], \qquad W_3(\omega) = \omega^2 b. \end{aligned} \tag{32}$$

For zeros approximation for power series of $1/\mu$, Eq. (30) give:

$$M_1 = M_2 = 0\,. \tag{33}$$

Since the z-axis is directed along with the major or the minor axis of the ellipsoid of inertia of the body, we get: $W_0(\omega) > 0$ for all ω under consideration.

Assume that:

$$\gamma_0''\,[z_0'\, W_1 + k\gamma_0''\, W_2] + kW_3(\omega)M_3^2 \neq 0\,. \tag{34}$$

Using (30), we get $\beta_1,\ \beta_2$ in power series expansions of powers less than μ^{-2}. Then for the rational values of the natural frequency ω does not equal to $(1, 2, 1/2, 3, 1/3, \ldots)$, we get the required periodic solutions and the correction of the period $\alpha(\mu^{-1})$ as:

$$\begin{aligned} &p_1(\tau, \mu^{-1}) = \mu^{-1}[x_0'(1-a)^{-1}\gamma_0'' + \chi_1 M_3\cos\tau] + \ldots, \\ &q_1(\tau, \mu^{-1}) = \mu^{-1}[y_0'(1-b)^{-1}\gamma_0'' + A_1^{-1}M_3\chi_2\sin\tau] + \ldots, \\ &r_1(\tau, \mu^{-1}) = 1 - 0.25\mu^{-2}M_3[kM_3C_1 + 4x_0'(1-\cos\tau) + y_0'\sin\tau - kM_3C_1\cos 2\tau] + \ldots, \\ &\gamma(\tau, \mu^{-1}) = M_3\cos\tau - 0.5\mu^{-2}\Gamma^2\cos\tau + \ldots, \\ &\gamma'(\tau, \mu^{-1}) = -M_3\sin\tau + 0.5\mu^{-2}\Gamma^2\sin\tau + \ldots, \\ &\gamma''(\tau, \mu^{-1}) = \gamma_0'' + \mu^{-2}M_3[x_0'(1-a)^{-1}\gamma_0'' - 0.5M_3C_1\left(\frac{z_0'}{a+b-1} + 0.5k\gamma_0''\right) \\ &\quad - x_0'(1-a)^{-1}\gamma_0''\cos\tau + y_0'(1-b)^{-1}\gamma_0''\sin\tau + 0.25M_3C_1(\frac{2z_0'}{a+b-1} + k\gamma_0'')\cos 2\tau] + \ldots, \end{aligned} \tag{35}$$

$$\alpha(\mu^{-1}) = 2\mu^{-2}\pi n\Big\{M_3x_0' - z_0'\gamma_0'' - 0.5k\Big[\gamma_0''^2(bB_1 - aA_1) + B_1\gamma_0''(1-\gamma_0'') - 0.125C_1M_3^2\Big]\Big\} + \dots. \tag{36}$$

The obtained solutions (35) and (36) are considered as the generalization of the corresponding problem in gravity field which studied in previous works [10] (when k = 0), the deviations between them are given by:

$$\begin{aligned}
\Delta p_1 &= \mu^{-1}(1-\omega^2)^{-1}kM_3\gamma_0''(A_1-\omega^2)\cos\tau + \dots,\\
\Delta q_1 &= \mu^{-1}A_1^{-1}M_3\Big\{(1-\omega^2)^{-1}[k(A_1-\omega^2)\gamma_0''] - kA_1\gamma_0''\Big\}\sin\tau + \dots,\\
\Delta r_1 &= -0.25\mu^{-2}M_3kC_1(1-\cos 2\tau) + \dots,\\
\Delta\gamma &= \mu^{-2}[0] + \dots, \quad \Delta\gamma' = \mu^{-2}[0] + \dots,\\
\Delta\gamma'' &= -0.25\mu^{-2}M_3^2C_1k\gamma_0''(1-\cos 2\tau) + \dots,
\end{aligned}$$

$$\Delta\alpha(\mu^{-1}) = -\mu^{-2}\pi nk\Big[\gamma_0''^2(bB_1 - aA_1) - 0.5M_3^2(b-a) + B_1\gamma_0''(1-\gamma_0'')\Big] + \dots. \tag{37}$$

Geometric interpretation of the motion

The geometric interpretation for the motion of the body at any instant of time to Euler's angles definitions $\theta,\ \psi,\ \phi$ is given by [11]:

$$\begin{aligned}
\theta &= \theta_0 + \mu^{-2}\operatorname{cosec}\theta_0[\theta_2(t+t_0) - \theta_2(t_0)] + \dots,\\
\psi &= \psi_0 + 0.5Mg\ell\, C^{-1}r_0^{-1}(\chi_1 - \chi_2A_1^{-1})t + \mu^{-1}\sqrt{Mg\ell/C}\,[\psi_1(t+t_0) - \psi_1(t_0)] + \dots,\\
\phi &= \phi_0 + [r_0^{-1} - 0.5Mg\ell C^{-1}r_0^{-1}\cos\theta_0(\chi_1 - \chi_2A_1^{-1})]t + \dots,
\end{aligned} \tag{38}$$

where:

$$\begin{aligned}
\theta_2(t) &= M_3\Big[y_0'\gamma_0''(1-b)^{-1}\sin r_0^{-1}t - x_0'\gamma_0''(1-a)^{-1}\cos r_0^{-1}t\\
&\quad + 0.5C_1\left(\frac{z_0'}{a+b-1} - 0.667k\gamma_0''\right)\cos 2r_0^{-1}t\Big],\\
\psi_1(t) &= 0.25(\chi_1 + \chi_2A_1^{-1}) + \sin 2r_0^{-1}t + y_0'\gamma_0''(1-b)^{-1}\cos r_0^{-1}t + x_0'(1-a)^{-1}\gamma_0''\sin r_0^{-1}t.
\end{aligned} \tag{39}$$

Numerical solutions

In this section, we use a computer program to determine the obtained solutions (19) and their derivatives for the time in the interval t ∈ [0, 300]. On the other hand, we use the fourth-order Runge–Kutta method [12] through another program to obtain numerical solutions for the autonomous system (14). In the end, we compare both solutions to check the accuracy of the method of solutions. These results are obtained through Tables 1 and 2. From these Tables, we deduce that the numerical solutions are in agreement with the analytical ones which prove the accuracy of considered methods.

Conclusions

We conclude that the problem of the motion of a rigid body about a fixed point is studied in many works [13–18] in both the uniform and gravity fields. We study our problem in case of a right angle of nutation θ_0 when its center of mass does not necessarily coincide with the fixed point. The equations of motions of the problem are obtained and reduced to a quasilinear autonomous system. The obtained system is solved by assuming a large parameter achieved from an angular velocity component tends to zero. The obtained solutions are treated through computer programs in a bounded interval of time. The autonomous system is treated with the Runge–Kutta method in the same interval of time to obtain the numerical solutions of the motion. Both obtained solutions are in full agreement with others which prove the accuracy of both numerical and analytical techniques used in solving the problem. For the geometric interpretation obtained, we note that:

The precession angle ψ

From (38) when $\mu \to \infty$, we deduce that the precession angle ψ is sufficiently large because r_0 is sufficiently small, that is, we obtain a case of large precession $\psi = \psi_0 + 0.5Mg\ell\, C^{-1} r_0^{-1}(\chi_1 - \chi_2 A_1^{-1})\, t$.

The nutation angle θ

We obtain a case of steady regular permutation: $\theta = \theta_0$.

The pure rotation angle ϕ

The case of a large pure rotation is obtained which depends on $1/r_0$ in the form:

$$\phi = \phi_0 + [r_0 - 0.5Mg\ell C^{-1} r_0^{-1} \cos\theta_0(\chi_1 - \chi_2 A_1^{-1})]t.$$

The large parameter technique used here is considered as the only one suitable for this problem in the origin domain of r_o tends to zero. Poincaré–Lindstedt method or Krylov Boboliubov Mitropolski one is failed to solve this problem because they depend on achieving a small parameter in domain r_0 tends to infinity. We conduct a comparison of the results of this manuscript with the results of the previous work. The results were obtained in [19] deals with the disk problem which satisfies the symmetry moments of inertia about two principal axes of the ellipsoid of inertia but our results here treat the general rigid problem in a limiting value of the Euler's angle $\theta_o \approx \pi/2$. The advantage of our used technique [20] depends on a large parameter $\mu \to \infty$. The obtained solutions are checked using two programs to assert their accuracy through Tables 1 and 2. The main results in our work are the obtained analytical solutions in Eq. (35) which is represented through computerized digital data in Table 1. The secondary results are proving the validity of these solutions which are given through the Runge–Kutta method in

Table 1 Represents the values of the obtained analytical solutions using the large parameter method in the interval $t \in [0, 300]$

t	p_{2a}	γ_{2a}	$x_a = dp_{2a}/dt$	$y_a = d\gamma_{2a}/dt$
0	1.88117E−15	1	0	0
10	1.52189E−15	0.809017	− 1.10572E−15	− 0.587785
20	5.81312E−16	0.309017	− 1.78909E−15	− 0.951057
30	− 5.81312E−16	− 0.309017	− 1.78909E−15	− 0.951056
40	− 1.5219E−15	− 0.809017	− 1.10572E−15	− 0.587785
50	− 1.88117E−15	− 1	− 2.84048E−22	− 1.50996E−07
60	− 1.52189E−15	− 0.809017	1.10572E−15	0.587785
70	− 5.81312E−16	− 0.309017	1.78909E−15	0.951056
80	5.81312E−16	0.309017	1.78909E−15	0.951056
90	1.5219E−15	0.809017	1.10572E−15	0.587785
100	1.88117E−15	1	5.68096E−22	3.01992E−07
110	1.52189E−15	0.809017	− 1.10572E−15	− 0.587785
120	5.81312E−16	0.309017	− 1.78909E−15	− 0.951057
130	− 5.81313E−16	− 0.309017	− 1.78909E−15	− 0.951056
140	− 1.52189E−15	− 0.809017	− 1.10572E−15	− 0.587785
150	− 1.88117E−15	− 1	4.48654E−23	2.38498E−08
160	− 1.52189E−15	− 0.809017	1.10572E−15	0.587785
170	− 5.81311E−16	− 0.309016	1.78909E−15	0.951057
180	5.81314E−16	0.309018	1.78909E−15	0.951056
190	1.5219E−15	0.809017	1.10572E−15	0.587785
200	1.88117E−15	1	1.13619E−21	6.03983E−07
210	1.5219E−15	0.809017	− 1.10572E−15	− 0.587785
220	5.81312E−16	0.309017	− 1.78909E−15	− 0.951056
230	− 5.81312E−16	− 0.309017	− 1.78909E−15	− 0.951056
240	− 1.5219E−15	− 0.809017	− 1.10572E−15	− 0.587785
250	− 1.88117E−15	− 1	1.27079E−21	6.75532E−07
260	− 1.52189E−15	− 0.809016	1.10572E−15	0.587786
270	− 5.81313E−16	− 0.309018	1.78909E−15	0.951056
280	5.81311E−16	0.309017	1.78909E−15	0.951057
290	1.52189E−15	0.809017	1.10572E−15	0.587785
300	1.88117E−15	1	− 8.97307E−23	− 4.76995E−08

Table 2. So we confirm that the presented numerical results would correlate to results obtained by other distinguished numerical techniques (except the Runge–Kutta method) will reveal the same results obtained in Table 1.

Table 2 represents the values of the numerical solutions using fourth-order Runge–Kutta method in the interval t ∈ [0, 300]

t	p_{2n}	γ_{2n}	$x_n = dp_{2n}/dt$	$y_n = d\gamma_{2n}/dt$
0	1.8812E−15	1	0	0
10	1.5247E−15	0.809035	− 1.09756E−15	− 0.587738
20	5.9028E−16	0.309102	− 1.77911E−15	− 0.951001
30	− 5.6779E−16	− 0.308864	− 1.78633E−15	− 0.951064
40	− 1.5106E−15	− 0.808859	− 1.11651E−15	− 0.587913
50	− 1.8809E−15	− 0.999934	− 2.35325E−17	− 0.000245929
60	− 1.5383E−15	− 0.809126	1.07833E−15	0.5875
70	− 6.1263E−16	− 0.309316	1.77147E−15	0.950862
80	5.4519E−16	0.30861	1.79318E−15	0.951078
90	1.4963E−15	0.808661	1.13524E−15	0.588073
100	1.8803E−15	0.999868	4.70583E−17	0.000491738
110	1.5517E−15	0.809217	− 1.05894E−15	− 0.587262
120	6.3487E−16	0.309529	− 1.76355E−15	− 0.950724
130	− 5.225E−16	− 0.308356	− 1.79974E−15	− 0.951091
140	− 1.4818E−15	− 0.808463	− 1.1538E−15	− 0.588234
150	− 1.8795E−15	− 0.999802	− 7.05736E−17	− 0.000737578
160	− 1.5648E−15	− 0.809309	1.03937E−15	0.587025
170	− 6.5702E−16	− 0.309743	1.75536E−15	0.950585
180	4.9974E−16	0.308102	1.80602E−15	0.951104
190	1.4671E−15	0.808265	1.17217E−15	0.588394
200	1.8783E−15	0.999736	9.40748E−17	0.000983447
210	1.5777E−15	0.809399	− 1.01965E−15	− 0.586787
220	6.7906E−16	0.309956	− 1.74689E−15	− 0.950446
230	− 4.7689E−16	− 0.307848	− 1.81201E−15	− 0.951117
240	− 1.4521E−15	− 0.808067	− 1.19036E−15	− 0.588553
250	− 1.8769E−15	− 0.99967	− 1.17558E−16	− 0.00122914
260	− 1.5903E−15	− 0.80949	9.99769E−16	0.586549
270	− 7.0098E−16	− 0.310169	1.73814E−15	0.950307
280	4.5398E−16	0.307593	1.81772E−15	0.95113
290	1.4369E−15	0.807869	1.20835E−15	0.588713
300	1.8751E−15	0.999603	1.41019E−16	0.00147495

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