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Randomly Coordinated Search for a lost target

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Target, Coordinated search, Optimal search plan.

ABSTRACT

This paper considers the coordinated search problem in which the searcher moves randomly, our goal is to detect a randomly situated hole on one of the two intersected lines such as oil pipeline or a cut in one of the two power cable lines beneath the surface of the ocean. The four searchers start together looking for the target from the intersection point. The intersected point is a starting point of the motion of the searchers (the origin point), also the searchers search the target with random distances and velocities across time, there is a known probability distribution for the target's position. The target located on one of the two intersected real lines according to a known un-symmetric distribution except the origin which the point of intersected lines. Where four searchers aim to detect the lost target. The final results determined the expected value of the detection time, also deduced the conditions that can minimize this expected value.

Introduction

In many real-world applications such as looking for bombs in a metropolis or a runaway criminal the study of search issues for any missing target, whether this target is stationary or moving, is deemed essential. The main aim is to locate the missing target at the least possible cost. The so-called linear search problem arises when the lost target is on a line **(Stone, 1989; Beck and Warren, 1973; Beck and Warren, 1973; Balkhi, 1989; Teamah et al., 2011; Teamah et al., 2000; El-Rayes et al., 2003; Reyniers, 1995)**. One of a number of techniques that have been researched on the line is the coordinated search technique, and it was investigated on straight line **(Reyniers, 1996; Teamah, 2019)**.

The problem of four searchers searching for a target located on two intersecting lines was investigated **(Teamah et al., 2013; Abou-Gabal, 2018)**, and the problem of finding a target located on one of n-disjoint lines was described by the quasi-coordinated search problem, in which the motion of every two searchers are independent of the motion of other searchers while the case of moving target was discussed **(Teamah, 2019)**, also a quasi-coordinated search method of a lost target was studied **(Teamah, 2019)**, in this case the target is randomly moving on one of the two intersection lines in accordance with a random walk motion and there are two searchers on each line, moving from the origin at a constant speed, the authors also studied the problem of finding a lost target is random walker on one of two intersection lines, the four searchers start from the origin **(Teamah, 2019)**. In the above discussed literature, the path of the searcher is deterministic, but the case where two searchers looking for a target located on an under-water oil tube with their speed represented by a random variable with a certain probability function was studied **(Teamah et al 2009)**. In the present paper we will assume that there is only one target present on one of the two lines (each line represents an oil pipeline or a power cable) under the surface of the water, but these two lines are disjoint and the speed of searchers represented by a random variable with a certain probability function The present paper is organized as follows. In section 2, we define search plan and calculate the expected value of the time to find the lost target. In section 3, we derive the conditions under which can minimize the expected value. In section 4, we finally provide a summary of our findings.

Search Plan

Let X be a random variable which represented the position of the target which follow a distribution function, the lines are divided into stop points w_{ij}, $r_{ij}$where i = 1,2 for first and second lines respectively, each line divided into two parts, right par w_{ij}, i = 1,2 , j = 0,1,2,⋯ and left part r_{ij}, i = 1,2 , j = 0,1,2,⋯. We have four searchers S_1, S_2, S_3 and S_4, The four searchers S_1, S_2, S_3 and S_4 start together looking for the target from the intersection point (the origin point be ($w_{10} = r_{10} = w_{20} = r_{20} = 0$)) of the two lines L_1 and L_2, the target located on one of the two intersected real lines according to a known unsymmetric distribution except the origin which the point of intersected lines. Searcher S_1 search in the right part of L_1 and searcher S_3 search in the right part of L_2, while S_2 search in the left part of L_2 and searcher S_4 search in the right part of L_2, with random velocity V_k, k = 1,2,3,4 two searchers in each line will connect with one another using today's methods of communication rather than going back to where you started, Consequently, there is no return time or standby.

All the searchers start at the same moment following the search plan denoted by $\emptyset : R \rightarrow R^+$ which completely defined by sequences a = {a_{ij}, i = 1,2 ; j = 1,2,⋯ } and b = {b_{ij} ; i = 1,2 ; j = 1,2,⋯ }.

Let the search plan be defined by $\emptyset$ = (a, b) ∈ Φ where Φ is the set of all search plans. Let

$$\rho_i = \inf \{x; F(x) > 0\} \qquad (1)$$

$$\sigma_i = \sup \{x; F(x) < 1\}, i = 1, 2$$

where ρ_i represents the minimum value of b_{ij} and σ_i represents the maximum value of a_{ij}, and The target's position distribution function is denoted by F(x).

The searchers S_1 and S_2 search on the first line L_1 , S_3 and S_4 search on the second line L_2 , the searchers S_1 and S_3 search in the right part of the two lines while S_2 and S_4 search the other parts.

S_1 will search in the right part of the first line as following:

I- Starts from the origin and goes to the positive part of L_1 with distance $a_{11} = w_{11} - w_{10}$ such that $0 < a_{11} < H^2_{11}$ and sends a report to the ship (located at the starting point and receives communications from searchers which use under water audio signals instead of electromagnetic waves) whether the target is found or not.

II- He decides whether or not to finish the search operation after waiting for the ship's response. The searcher S_2 has found the target if the response indicates that the search is not yet complete.

III- Otherwise, S_1 goes to the positive part of L_1 with distance $a_{12} = w_{12} - w_{11}$, then reports back to the ship, and so forth, until the target gets identified. The identical method will be used by searcher S_2 to look in the left portion, but with distance $b_{11} = r_{11} - r_{10}$ such that $0 < b_{11} < \breve{H}^2_{11}$, $b_{12} = r_{12} - r_{11}$ such that $0 < b_{12} < \breve{H}^2_{12}$, $b_{13} = r_{13} - r_{12}$ such that $0 < b_{13} < \breve{H}^2_{13}$.

IV- The searchers S_3 and S_4 also carry out the same action.

We consider any searcher has a random velocity with probability density function

$$p(v) = |v - v_0|\delta(v^2 - v_0), \quad -\infty \le v \le \infty, \qquad (2)$$

Where δ is dirac delta function and v_0 is the initial velocity.

We take into account how far the greatest distance was traveled by S_1 and S_3 before the i^{th} connection in the right part be H_{1j} *and* H_{2j} respectively, (in the left part be $\breve{H}_{1j}$ and $\breve{H}_{2j}$ for the searchers S_2 and S_4 respectively), the distance which passed the searchers by S_1 and S_3 are random and given by

$$0 < a_{ij} < H^2_{ij},\ i = 1,2\ ;\ j = 1,2,3,\cdots,n$$

in the right part where $a_{ij} = w_{ij} - w_{i(j-1)}$ ($0 < b_{ij} \le \tilde{H}^2_{ij}$ in the left part where $b_{ij} = r_{ij} - r_{i(j-1)}$) we consider the p. d. f. of the random distance see **(Teamah and Elbery, 2019; Teamah et al., 2011)** is given by:

$$f_1(a_{ij}) = \frac{1}{H_{ij}\sqrt{a_{ij}}} - \frac{1}{H^2_{ij}},$$

$$\mathbf{0 < a_{ij} \le H^2_{ij},\ i = 1,2;\ j = 1,2,3,\cdots,n} \qquad (3)$$

$$f_2(b_{ij}) = \frac{1}{\breve{H}_{ij}\sqrt{b_{ij}}} - \frac{1}{\breve{H}^2_{ij}},$$

$$\mathbf{0 < b_{ij} \le \breve{H}^2_{ij},\ i = 1,2\ ;\ j = 1,2,3,\cdots,n} \qquad \mathbf{(4)}$$

The searchers desire to minimize the expected time to detect the target. Let X be a random variable that shows the location of the target. Searchers are moving on the line at varying distances based on the probability that the target is on the line because they are moving at a random velocity.

So, if their exist $\Gamma_i \ge 0$, $i = 1,2$ is a random variable has a known distribution with expected value $E(\Gamma_i)$ then we can consider $\breve{H}_{ij} = H_{ij} + E(\Gamma_i)$. So, (4) become

$$f_2(b_{ij}) = \frac{1}{(H_{ij} + E(\Gamma_i))\sqrt{b_{ij}}} - \frac{1}{(H_{ij} + E(\Gamma_i))^2} \qquad (5)$$

Where $0 < \boldsymbol{b_{ij}} \le \mathbf{(H_{ij} + E(\Gamma_i))^2}$. Let γ be a measure of probability caused by the target position and $\gamma(x, y) = F(y) - F(x)$. Also, let $\Omega(\emptyset)$ be the detection time by one of the searchers.

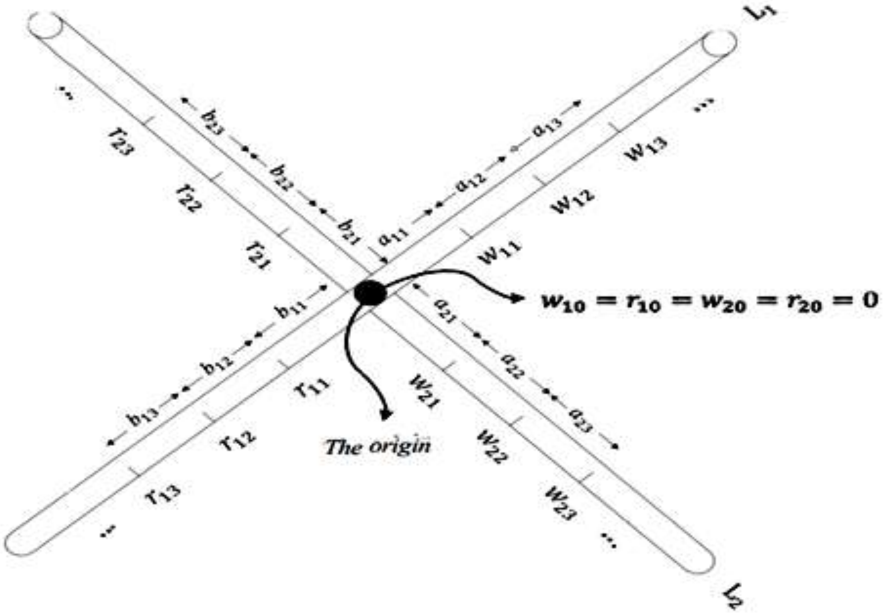

Fig. (1): Coordinated search technique for finding a target

Theorem (1): The expected value of detection time is given by

$$E[\Omega(\emptyset)] = \frac{1}{6} \sum_{j=0}^{\infty} \left[H^2_{1(j+1)}\gamma(\rho_1, r_{1j}) - \left(H_{1(j+1)} + E(\Gamma_1)\right)^2 \gamma(w_{1j}, \sigma_1)\right] + \left[H^2_{2(j+1)}\gamma(\rho_2, r_{2j}) - \left(H_{2(j+1)} + E(\Gamma_2)\right)^2 \gamma(w_{2j}, \sigma_2)\right]. \quad (6)$$

Proof. Since, the searchers S_1 and S_2 each one, moves the first line with different and random distance (have pdfs (3) and (5) in the two parts of first line) based on the target's probability of existing, as shown by the target's probability on the first line. Thus, the distance on the right should have the following predicted

$$E(a_{1j}) = \int_0^{H^2_{1j}} a_{1j}\left(\frac{1}{H_{1j}\sqrt{a_{1j}}} - \frac{1}{H^2_{1j}}\right) da_{1j} = \frac{H^2_{1j}}{6}$$

, and in the left part is

$$E(b_{1j}) = \int_0^{(H_{1j}+E(\Gamma_1))^2} b_{1j}\left(\frac{1}{(H_{1j}+E(\Gamma_1))\sqrt{b_{1j}}} - \frac{1}{\left(H_{1j}+E(\Gamma_1)\right)^2}\right) db_{1j} = \frac{\left(H_{1j}+E(\Gamma_1)\right)^2}{6}$$

Also, we assume that the expected value of the velocity is $E(v) = \pm 1$, then the expected value of detection time in the right part is $\Omega_{11} = \frac{E(a_{1j})}{+1} = \frac{H^2_{1j}}{6}$,

in the left part is

$$\Omega_{12} = \frac{E(b_{1j})}{-1} = -\frac{(H_{1j} + E(\Gamma_1))^2}{6}.$$

If the target lies in $]\, w_{10}, w_{11}]$, then

$$\Omega_{12} = -\frac{(H_{11} + E(\Gamma_1))^2}{6}$$

If the

target lies in] w_{11}, w_{12}], then

$$\Omega_{12} = -\left[\frac{(H_{11}+E(\Gamma_1))^2+(H_{12}+E(\Gamma_1))^2}{6}\right]$$

If the target lies in] w_{12}, w_{13}], then

$$\Omega_{12} = -[(H_{11} + E(\Gamma_1))^2 + (H_{12} + E(\Gamma_1))^2 + (H_{13} + E(\Gamma_1))^2]/6$$

If the target lies in] $w_{1(k-1)}, w_{1k}$], then

$$\Omega_{12} = -[(H_{11} + E(\Gamma_1))^2 + (H_{12} + E(\Gamma_1))^2 + (H_{13} + E(\Gamma_1))^2 + \cdots + (H_{1k} + E(\Gamma_1))^2]/6$$

Similarly, if the target lies in $[r_{1k}, r_{1(k-1)}[$, then

$$\Omega_{11} = -\left[\frac{H^2{}_{11}+H^2{}_{12}+\cdots+H^2{}_{1k}}{6}\right],$$

and so on.

$$E[\Omega_1(\emptyset)] = -\left[\frac{(H_{11}+E(\Gamma_1))^2}{6}\right][\gamma(w_{10},w_{11})] - \left[\frac{(H_{11}+E(\Gamma_1))^2+(H_{12}+E(\Gamma_1))^2}{6}\right][\gamma(w_{11},w_{12})] \cdot [[(H_{11} + E(\Gamma_1))^2 + (H_{12} + E(\Gamma_1))^2 + (H_{13} + E(\Gamma_1))^2 + \cdots + (H_{1k} + E(\Gamma_1))^2]/6][\gamma(w_{1(k-1)},w_{1k})] - \cdots$$

$$+ \frac{H^2{}_{11}}{6}[\gamma(r_{11},r_{10})] + \frac{H^2{}_{11}+H^2{}_{12}}{6}[\gamma(r_{12},r_{11})] + \cdots\left[\frac{H^2{}_{11}+H^2{}_{12}+\cdots+H^2{}_{1k}}{6}\right][\gamma(r_{1k},r_{1(k-1)})]$$

$$= \frac{H^2{}_{11}}{6}[\gamma(r_{11},r_{10}) + \gamma(r_{12},r_{11}) + \cdots + \gamma(r_{1k},r_{1(k-1)})] + \frac{H^2{}_{12}}{6}[\gamma(r_{12},r_{11}) + \cdots + \gamma(r_{1k},r_{1(k-1)})]$$

$$+\ldots+$$

$$\frac{H^2{}_{1k}}{6}[\gamma(r_{1k},r_{1(k-1)})] + \cdots - \frac{(H_{11}+E(\Gamma_1))^2}{6}[\gamma(w_{10},w_{11}) + \gamma(w_{11},w_{12}) + \cdots + \gamma(w_{1(k-1)},w_{1k})] - \frac{(H_{12}+E(\Gamma_1))^2}{6}[\gamma(w_{11},w_{12}) + \gamma(w_{12},w_{13}) + \cdots + \gamma(w_{1(k-1)},w_{1k})] - \cdots - \frac{(H_{1k}+E(\Gamma_1))^2}{6}[\gamma(w_{1(k-1)},w_{1k})] - \cdots$$

$$\frac{H^2{}_{11}}{6}[\gamma(\rho_1,r_{10})] + \frac{H^2{}_{12}}{6}[\gamma(\rho_1,r_{11})] + \cdots + \frac{H^2{}_{1k}}{6}[\gamma(\rho_1,r_{1(k-1)})] + \cdots - \frac{(H_{11}+E(\Gamma_1))^2}{6}[\gamma(w_{10},\sigma_1)] - \frac{(H_{12}+E(\Gamma_1))^2}{6}[\gamma(w_{11},\sigma_1)] - \frac{(H_{13}+E(\Gamma_1))^2}{6}[\gamma(w_{12},\sigma_1)] - \cdots - \frac{(H_{1k}+E(\Gamma_1))^2}{6}[\gamma(w_{1(k-1)},\sigma_1)] - \cdots$$

$$= \frac{1}{6}\sum_{j=0}^{\infty}\left[[H^2{}_{1(j+1)}\gamma(\rho_1,r_{1j})] - [(H_{1(j+1)} + E(\Gamma_1))^2\gamma(w_{1j},\sigma_1)]\right].$$

Now we will search on the second line according to the probability of existence of the target on the second line, then the expected value of the distance in the right is:

$$E(a_{2j}) = \int_0^{H^2{}_{2j}} a_{2j}\left(\frac{1}{H_{2j}\sqrt{a_{2j}}} - \frac{1}{H^2{}_{2j}}\right)da_{2j} = \frac{H^2{}_{2j}}{6}$$

, and in the left part is

$E(b_{2j}) =$

$$\int_0^{(H_{2j}+E(\Gamma_2))^2} b_{2j}\left(\frac{1}{(H_{2j}+E(\Gamma_2))\sqrt{b_{2j}}} - \frac{1}{(H_{2j}+E(\Gamma_2))^2}\right)db_{2j} = \frac{(H_{2j}+E(\Gamma_2))^2}{6}$$

Also we assume that the expected value of the velocity is E(v) = ±1, then the expected value of detection time in the right part is $\Omega_{21} = \frac{E(a_{2j})}{+1} = \frac{H^2{}_{2j}}{6}$,

in the left part is

$$\Omega_{22} = \frac{E(b_{2j})}{-1} = -\frac{(H_{2j}+E(\Gamma_2))^2}{6}.$$

If the target lies in] w_{20}, w_{21}], then

$$\Omega_{22} = -\frac{(H_{21}+E(\Gamma_2))^2}{6}$$

If the target lies in] w_{21}, w_{22}], then

$$\Omega_{22} = -\left[\frac{(H_{21}+E(\Gamma_2))^2+(H_{22}+E(\Gamma_2))^2}{6}\right].$$

If the target lies in] $w_{2(k-1)}, w_{2k}$], then

$$\Omega_{22} = -[(H_{21}+E(\Gamma_2))^2+(H_{22}+E(\Gamma_2))^2+(H_{23}+E(\Gamma_2))^2+\cdots+(H_{2k}+E(\Gamma_2))^2]/6$$

Similarly, if the target lies

in $[r_{2k}, r_{2(k-1)}[$, then

$$\Omega_{21} = -\left[\frac{H^2{}_{21}+H^2{}_{22}+\cdots+H^2{}_{2k}}{6}\right],$$

and so on.

$E[\Omega_2(\emptyset)] =$

$$-\left[\frac{(H_{21}+E(\Gamma_2))^2}{6}\right][\gamma(w_{20},w_{21})] - \left[\frac{(H_{21}+E(\Gamma_2))^2+(H_{22}+E(\Gamma_2))^2}{6}\right][\gamma(w_{21},w_{22})] - [[(H_{21}+E(\Gamma_2))^2+(H_{22}+E(\Gamma_2))^2+(H_{23}+E(\Gamma_2))^2+\cdots+(H_{2k}+E(\Gamma_2))^2]/6][\gamma(w_{2(k-1)},w_{2k})] - \cdots$$

$$+ \frac{H^2{}_{21}}{6}[\gamma(r_{21},r_{20})] + \frac{H^2{}_{21}+H^2{}_{22}}{6}[\gamma(r_{22},r_{21})] + \cdots + \left[\frac{H^2{}_{21}+H^2{}_{22}+\cdots+H^2{}_{2k}}{6}\right][\gamma(r_{2k},r_{2(k-1)})]$$

$$= \frac{H^2{}_{21}}{6}[\gamma(r_{21},r_{20}) + \gamma(r_{22},r_{21}) + \cdots + \gamma(r_{2k},r_{2(k-1)})] + \frac{H^2{}_{22}}{6}[\gamma(r_{22},r_{21}) + \cdots + \gamma(r_{2k},r_{2(k-1)})]$$

$$+ \frac{H^2{}_{2k}}{6}[\gamma(r_{2k},r_{2(k-1)})] + \cdots - \frac{(H_{21}+E(\Gamma_2))^2}{6}[\gamma(w_{20},w_{21}) + \gamma(w_{21},w_{22}) + \cdots + \gamma(w_{2(k-1)},w_{2k})] - \frac{(H_{22}+E(\Gamma_2))^2}{6}[\gamma(w_{21},w_{22}) + \gamma(w_{22},w_{23}) + \cdots + \gamma(w_{2(k-1)},w_{2k})] - \cdots - \frac{(H_{2k}+E(\Gamma_2))^2}{6}[\gamma(w_{2(k-1)},w_{2k})] - \cdots$$

$$= \frac{H^2{}_{21}}{6}[\gamma(\rho_2,r_{20})] + \frac{H^2{}_{22}}{6}[\gamma(\rho_2,r_{21})] + \cdots + \frac{H^2{}_{2k}}{6}[\gamma(\rho_2,r_{2(k-1)})] + \cdots - \frac{(H_{21}+E(\Gamma_2))^2}{6}[\gamma(w_{20},\sigma_2)] - \frac{(H_{22}+E(\Gamma_2))^2}{6}[\gamma(w_{21},\sigma_2)] - \frac{(H_{23}+E(\Gamma_2))^2}{6}[\gamma(w_{22},\sigma_2)] - \cdots - \frac{(H_{2k}+E(\Gamma_2))^2}{6}[\gamma(w_{2(k-1)},\sigma_2)] - \cdots$$

$= \frac{1}{6}$
$\sum_{j=0}^{\infty} \Big[[\mathrm{H}^2{}_{2(j+1)} \gamma(\rho_2, r_{2j})] - [\left(\mathrm{H}_{2(j+1)} + \mathrm{E}(\Gamma_2)\right)^2 \gamma(w_{2j}, \sigma_2)] \Big]$

$\therefore \mathrm{E}[\Omega(\emptyset)] = \mathrm{E}[\Omega_1(\emptyset)] + \mathrm{E}[\Omega_2(\emptyset)]$

Since $0 < a_{ij} \leq \mathrm{H}^2{}_{ij}$, then if there exist $\varepsilon_i \geq 0$, one can get

$\mathrm{H}^2{}_{ij} = (a_{ij} + \varepsilon_i)^2$. Hence we can put equation (6) in this form to find the optimal search path

$\mathrm{E}[\Omega(\emptyset)] = \frac{1}{6}$
$\sum_{j=0}^{\infty} [\left(a_{1(j+1)} + \varepsilon_1\right)^2 \{\gamma(\rho_1, r_{1j})\} - \left(a_{1(j+1)} + \varepsilon_1 + E(\Gamma_1)\right)^2 \{\gamma(w_{1j}, \sigma_1)\}]$
$[\left(a_{2(j+1)} + \varepsilon_2\right)^2 \{\gamma(\rho_2, r_{2j})\} - \left(a_{2(j+1)} + \varepsilon_2 + E(\Gamma_2)\right)^2 \{\gamma(w_{2j}, \sigma_2)\}].$
(7)

Also, if their exist $\xi_i \geq 0$ one can get $\breve{\mathrm{H}}^2{}_{ij} = (b_{ij} + \xi_i)^2$.

Compensate for $\mathrm{H}_{ij} = \breve{\mathrm{H}}_{1j} - \mathrm{E}(\Gamma_i)$, $\mathrm{H}_{ij} = b_{ij} - \xi_i - \mathrm{E}(\Gamma_i)$in (6) we get:

$\mathrm{E}[\Omega(\emptyset)] = \frac{1}{6}$
$\sum_{j=0}^{\infty} [(b_{1(j+1)} - \xi_1 - \mathrm{E}(\Gamma_1))^2 \{\gamma(\rho_1, r_{1j})\} - (b_{1(j+1)} + \xi_1)^2 \{\gamma(w_{1j}, \sigma_1)\}] + [(b_{2(j+1)} - \xi_2 - \mathrm{E}(\Gamma_2))^2 \{\gamma(\rho_2, r_{2j})\} - [(b_{2(j+1)} + \xi_2)^2 \{\gamma(w_{2j}, \sigma_2)\}]]$
. (8)

In this work, we don't need to find the necessary and sufficient conditions that explain the existence of optimal search plan because the target has bounded asymmetric distribution, we will get the optimal values of the point w_{ij} and r_{ij} which afford the optimal path to get the target by using (7) and (8).

Optimal search path for bounded asymmetric target distribution.

In this section, our primary goal is to minimize E(Ω(∅)), this occurs when we determine the ideal values of $\{a_{ij}; i = 1,2, j \geq 1\}$ and $\{b_{ij}; i = 1,2, j \geq 1\}$ that provide an optimal search path from class Z for the target's location distribution .If $\acute{Z}$ is a subclass of Z for which only one element and if a^* and b^* are optimal values of a and b; respectively, then the optimal search path will be in $\acute{Z}$. It is observed that the target distribution F(x) determines the search path, They are two unknown factors, as well as the values of a and b that the searchers utilized, and because the value of a depend on the value of $\{w_{ij}; i = 1,2, j \geq 0\}$ and $\{r_{ij}; i = 1,2, j \geq 0\}$, we will find the optimal values

$\{w^*_{ij}; i = 1,2, j \geq 0\}$ and

$\{r^*_{ij}; i = 1,2, j \geq 0\}$.

From now the target distribution F(x) is assumed to be known and regular (I.e., F(x) is absolutely continuous with positive density f(x)) and E(X) < ∞. In order to obtain the optimal values

$\{w^*_{ij}; i = 1,2,3, \cdots, j \geq 0\}$ we must solve this non-linear program problem (NLP):

NLP(1)

$$min_{w_{ij}}(a_{1j}+\varepsilon_1)^{2}\{\gamma(\rho_1,r_{1(j-1)})\}-\left(a_{1j}+\varepsilon_1+\mathrm{E}(\Gamma_1)\right)^2\{\gamma(w_{1(j-1)},\sigma_1)\}$$

$$+(a_{1(j+1)}+\varepsilon_1)^{2}\{\gamma(\rho_1,r_{1j})\}-\left(a_{1(j+1)}+\varepsilon_1+\mathrm{E}(\Gamma_1)\right)^2\{\gamma(w_{1j},\sigma_1)\}$$

sub. to:

$$\frac{a_{1j}+\varepsilon_1}{a_{1j}}\geq 1,\qquad \frac{a_{1(j+1)}+\varepsilon_1}{a_{1(j+1)}}\geq 1,$$

$$a_{1j}>0,\qquad a_{1(j+1)}>0\ ,$$

$$\frac{a_{1j}+\varepsilon_1+\mathrm{E}(\Gamma_1)}{b_{1j}}\geq 1,$$

$$\frac{a_{1(j+1)}+\varepsilon_1+\mathrm{E}(\Gamma_1)}{b_{1(j+1)}}\geq 1\ ,$$

$$b_{1j}>0,\qquad b_{1(j+1)}>0\ ,$$

$$\varepsilon_1\geq 0,\qquad \Gamma_1\geq 0.$$

Definition (1): (Optimal solution). For the previous NLP $w^* \in$ R are said to be optimal, if found w∈ R such that g(w^*) ≤ g(w), ∀ w ∈ R. Then the previous NLP (1) take the form (on the first line):

NLP (2)

$$min_{w_{ij}}(a_{1j}+\varepsilon_1)^{2}\{\gamma(\rho_1,r_{1(j-1)})\}-\left(a_{1j}+\varepsilon_1+\mathrm{E}(\Gamma_1)\right)^2\{\gamma(w_{1(j-1)},\sigma_1)\}$$

$$+(a_{1(j+1)}+\varepsilon_1)^{2}\{\gamma(\rho_1,r_{1j})\}-\left(a_{1(j+1)}+\varepsilon_1+\mathrm{E}(\Gamma_1)\right)^2\{\gamma(w_{1j},\sigma_1)\}$$

sub. to:

$$1-\frac{a_{1j}+\varepsilon_1}{a_{1j}}\leq 0\ ,\qquad 1-\frac{a_{1(j+1)}+\varepsilon_1}{a_{1(j+1)}}\leq 0,$$

$$-a_{1j}<0\ ,\qquad -a_{1(j+1)}<0\ ,$$

$$1-\frac{a_{1j}+\varepsilon_1+\mathrm{E}(\Gamma_1)}{b_{1j}}\leq 0\ ,$$

$$1-\frac{a_{1(j+1)}+\varepsilon_1+\mathrm{E}(\Gamma_1)}{b_{1(j+1)}}\leq 0\ ,$$

$$-b_{1j}<0\ ,\qquad -b_{1(j+1)}<0,$$

$$-\varepsilon_1\leq 0,\qquad -\Gamma_1\leq 0.$$

From the Kuhn-Tucker conditions, we get

$$2(a_{1j}+\varepsilon_1)\{\gamma(\rho_1,r_{1(j-1)})\}-2(a_{1j}+\varepsilon_1+\mathrm{E}(\Gamma_1))\{\gamma(w_{1(j-1)},\sigma_1)\}-2(a_{1(j+1)}+\varepsilon_1)\{\gamma(\rho_1,r_{1j})\}+2(a_{1(j+1)}+\varepsilon_1+\mathrm{E}(\Gamma_1))\{\gamma(c_{1j},\sigma_1)\}+\left(a_{1(j+1)}+\varepsilon_1+\mathrm{E}(\Gamma_1)\right)^2 f(w_{1j})+u_1(0-[2(a_{1j}+\varepsilon_1)(a_{1j})-(a_{1j}+\varepsilon_1)]/(a_{1j})^2))+u_2(0-[2(a_{1(j+1)}+\varepsilon_1)(a_{1(j+1)})-(a_{1(j+1)}+\varepsilon_1)]/((a_{1(j+1)})^2))+u_3\left(0-\frac{1}{b_{1j}}\right)+u_4\left(0-\frac{1}{b_{1(j+1)}}\right)+u_5(-1)+u_6(1)=0\ , \tag{9}$$

$$u_1\left(1-\frac{(a_{1j}+\varepsilon_1)}{a_{1j}}\right)=0 \tag{10}$$

$$u_2\left(1-\frac{(a_{1(j+1)}+\varepsilon_1)}{(a_{1(j+1)})}\right)=0, \tag{11}$$

$$u_3\left(1-\frac{a_{1j}+\varepsilon_1+\mathrm{E}(\Gamma_1)}{b_{1j}}\right)=0\ , \tag{12}$$

$$u_4\left(1-\frac{a_{1(j+1)}+\varepsilon_1+\mathrm{E}(\Gamma_1)}{b_{1(j+1)}}\right)=0, \tag{13}$$

$$u_5(-a_{1j})=0, \tag{14}$$

$$u_6(-a_{1(j+1)})=0, \tag{15}$$

To solve the equations (9)-(15)there are many cases, but he case which gives the optimal value of $\{w_{1j}\ ;\ j\geq 0\}$ is the case: $u_1 = u_2 = \cdots = u_6 = 0$. Consequently, the optimal value of $w_{1(j+1)}$ is provided following the equation's solution,

$$w^2{}_{1(j+1)} f(w_{1j}) - w_{1(j+1)} \left[2\gamma(\rho_1, r_{1j}) - 2\gamma(\sigma_{1j}, \sigma_1) + 2f(w_{1j})\left(w_{1j} + \varepsilon_1 + \mathrm{E}(\Gamma_1)\right)\right] = \vartheta_1 \quad (16)$$

Where

$$\vartheta_1 = -2(a_{1j} + \varepsilon_1)[\gamma(\rho_1, r_{1(j-1)})] + 2(a_{1j} + \varepsilon_1 + \mathrm{E}(\Gamma_1))[\gamma(w_{1(j-1)}, \sigma_1)] + 2(-w_{1j} + \varepsilon_1)$$

$$[\gamma(\rho_1, r_{1j}) - 2\left(-w_{1j} + \varepsilon_1 + \mathrm{E}(\Gamma_1)\right)[\gamma(w_{1j}, \sigma_1)]$$

$$-\left(w_{1j} + \varepsilon_1 + \mathrm{E}(\Gamma_1)\right)^2 f(w_{1j}).$$

Also, by solving the NLP(3) we get the optimal values of r_{1j}

NLP(3)

$$\min_{r_{ij}} (b_{1j} + \xi_1 - \mathrm{E}(\Gamma_1))^2 \{\gamma(\rho_1, r_{1(j-1)})\} - (b_{1j} + \xi_1)^2 \{\gamma(w_{1(j-1)}, \sigma_1)\}$$
$$+ (b_{1(j+1)} + \xi_1 - \mathrm{E}(\Gamma_1))^2 \{\gamma(\rho_1, r_{1j})\} - (b_{1(j+1)} + \xi_1)^2 \{\gamma(w_{1j}, \sigma_1)\}$$

Sub. to:

$$1 - \frac{b_{1j} + \xi_1 - \mathrm{E}(\Gamma_1)}{b_{1j}} \le 0 \quad ,$$

$$1 - \frac{b_{1(j+1)} + \xi_1 - \mathrm{E}(\Gamma_1)}{b_{1(j+1)}} \le 0,$$

$$1 - \frac{b_{1j} + \xi_1}{b_{1j}} \le 0,$$

$$1 - \frac{b_{1(j+1)} + \xi_1}{b_{1(j+1)}} \le 0,$$

$$-a_{1j} < 0 \quad , -a_{1(j+1)} < 0 \ ,$$

$$-b_{1j} < 0 \quad , -b_{1(j+1)} < 0,$$

$$-\xi_1 \le 0 \quad , -\Gamma_1 \le 0 .$$

Applying Kuhn-Tucker conditions, and by soling the following equations, we get the optimal values of $d_{1(j+1)}$,

$$r^2{}_{1(j+1)} f(r_{1j}) + r_{1(j+1)} \begin{bmatrix} -2\gamma(\rho_1, r_{1j}) - 2\gamma(w_{1j}, \sigma_1) \\ -2\left(r_{1j} + \xi_1 + \mathrm{E}(\Gamma_1)\right) \end{bmatrix} f(r_{1j})$$

$$= \Lambda_1$$

where

$$\Lambda_1 = -2\left(r_{1j} - r_{1(j-1)} + \xi_1 - \mathrm{E}(\Gamma_1)\right)[\gamma(\rho_1, r_{1(j-1)})]$$

$$+2(r_{1j} - r_{1(j-1)} + \xi_1)[\gamma(w_{1(j-1)}, \sigma_1)]$$

$$+2(-r_{1j} + \xi_1 - \mathrm{E}(\Gamma_1))[\gamma(\rho_1, r_{1j})$$

$$-2(-r_{1j} + \xi_1)[\gamma(w_{1j}, \sigma_1)]$$

$$-\left(r_{1j} + \xi_1 + \mathrm{E}(\Gamma_1)\right)^2 f(r_{1j}). \quad (17)$$

by solving the equations (16) and (17) we can get the optimal search path on the first line by the same way in order to obtain the optimal values $\{w^*{}_{2j}, \mathrm{j} \ge 0\}$ and $\{r^*{}_{2j}, \mathrm{j} \ge 0\}$ we must solve the NLP(4) and NLP(5)

NLP(4)

$$\min_{w_{2j}} (\rho_{2j} + \varepsilon_2)^2 \{\gamma(\rho_2, r_{2(j-1)})\} - \left(a_{2j} + \varepsilon_2 + \mathrm{E}(\Gamma_2)\right)^2 \{\gamma(w_{2(j-1)}, \sigma_2)\}$$
$$+ (a_{2(j+1)} + \varepsilon_2)^2 \{\gamma(\rho_2, r_{2j})\} - \left(a_{2(j+1)} + \varepsilon_2 + \mathrm{E}(\Gamma_2)\right)^2 \{\gamma(w_{2j}, \sigma_2)\}$$

sub. to:

$$1-\frac{a_{2j}+\varepsilon_2}{a_{2j}} \leq 0,$$

$$1-\frac{a_{2(j+1)}+\varepsilon_2}{a_{2(j+1)}} \leq 0,$$

$$-a_{2j} < 0 \quad , \qquad -a_{2(j+1)} < 0 \quad ,$$

$$1-\frac{a_{2j}+\varepsilon_2+\mathrm{E}(\Gamma_2)}{b_{2j}} \leq 0,$$

$$1-\frac{a_{2(j+1)}+\varepsilon_2+\mathrm{E}(\Gamma_2)}{b_{2(j+1)}} \leq 0,$$

$$-b_{2j} < 0 \quad , \qquad -b_{2(j+1)} < 0,$$

$$-\varepsilon_2 \leq 0, \qquad -\Gamma_2 \leq 0 .$$

NLP(5)

$$min_{r_{2j}} \left(b_{2j}+\xi_2-\mathrm{E}(\Gamma_2)\right)^2$$

$$\{\gamma(\rho_2,r_{2(j-1)})\}-(b_{2j}+\xi_2)^2\{\gamma(w_{2(j-1)},\sigma_2)\}$$

$$+\left(b_{2(j+1)}+\xi_2-\mathrm{E}(\Gamma_2)\right)^2\{\gamma(\rho_2,r_{2j})\}$$

$$-(b_{2(j+1)}+\xi_2)^2\{\gamma(w_{2j},\sigma_2)\}$$

Sub. to:

$$1-\frac{b_{2j}+\xi_2-\mathrm{E}(\Gamma_2)}{b_{2j}} \leq 0,$$

$$1-\frac{b_{2(j+1)}+\xi_2-\mathrm{E}(\Gamma_2)}{b_{2(j+1)}} \leq 0,$$

$$1-\frac{b_{2j}+\xi_2}{b_{2j}} \leq 0,$$

$$1-\frac{b_{2(j+1)}+\xi_2}{b_{2(j+1)}} \leq 0,$$

$$-a_{2j} < 0 \quad , -c_{2(j+1)} < 0 \; ,$$

$$-b_{2j} < 0 \quad , -b_{2(j+1)} < 0,$$

$$-\xi_2 \leq 0 \quad , -\Gamma_2 \leq 0 .$$

Since, we have two lines that do intersect, then

min $(\Omega(\emptyset)) =$

min $(\Omega_1(\emptyset))$+ min $(\Omega_2(\emptyset))$.

Conclusion and future work

The Coordinated search technique is analyzed in order to locate a target at random on one of two intersection lines. The searchers search the target with random distances and velocities across time, there is a known probability distribution for the target's position. We obtain the expected value of the detection time for finding the target. Moreover, we present the optimal search path that minimizes this expected value. In the future work, this technique will be extended on the plane when the target's distribution is bounded in each searching step.

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البحث المنسق عشوائياً عن الهدف المفقود

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يتناول هذا البحث مشكلة البحث المنسق التي يتحرك فيها الباحث بشكل عشوائي، وهدفنا هو اكتشاف ثقب يقع بشكل عشوائي على أحد الخطين المتقاطعين مثل خط أنابيب النفط أو قطع في أحد خطي كابلات الكهرباء تحت سطح محيط. يبدأ الباحثون الأربعة معًا بالبحث عن الهدف من نقطة التقاطع. نقطة التقاطع هي نقطة بداية لحركة الباحثين (نقطة الأصل)، كما يقوم الباحثون بالبحث عن الهدف بمسافات وسرعات عشوائية عبر الزمن، وهناك توزيع احتمالي معروف لموقع الهدف. الهدف الواقع على أحد الخطين الحقيقيين المتقاطعين وفق توزيع غير متماثل معروف باستثناء نقطة الأصل التي هي نقطة تقاطع الخطوط. حيث يهدف أربعة باحثين إلى اكتشاف الهدف المفقود. وقد حددت النتائج النهائية القيمة المتوقعة لزمن الكشف، كما استنتجت الشروط التي يمكن أن تقلل من هذه القيمة المتوقعة.