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ORIGINAL ARTICLE

Ideal convergent sequence spaces over *p*-metric spaces defined by Musielak-modulus functions

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Abstract In this paper we introduce the *I*- of χ^2 sequence spaces over *p*-metric spaces defined by Musielak function. We also examine some topological properties and prove some inclusion relation between these spaces.

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1. Introduction

Throughout w, χ and Λ denote the classes of all, gai and analytic scalar valued single sequences, respectively.

We write w^2 for the set of all complex sequences (x_{mn}), where $m, n \in \mathbb{N}$, the set of positive integers. Then, w^2 is a linear space under the coordinate wise addition and scalar multiplication.

Some initial works on double sequence spaces are found in Bromwich [1]. Later on, they were investigated by Hardy [2], Moricz [3], Moricz and Rhoades [4], Basarir and Solankan [5], Tripathy [6], Turkmenoglu [7], and many others.

We procure the following sets of double sequences:

$$\mathcal{M}_u(t) := \{(x_{mn}) \in w^2 : sup_{m,n\in N}|x_{mn}|^{t_{mn}} < \infty\},$$
$$\mathcal{C}_p(t) := \{(x_{mn}) \in w^2 : p - lim_{m,n\to\infty}|x_{mn} - l|^{t_{mn}} = 1 \, for\ some\ l \in \mathbb{C}\},$$
$$\mathcal{C}_{0p}(t) := \{(x_{mn}) \in w^2 : p - lim_{m,n\to\infty}|x_{mn}|^{t_{mn}} = 1\},$$
$$\mathcal{L}_u(t) := \{(x_{mn}) \in w^2 : \sum_{m=1}^{\infty}\sum_{n=1}^{\infty}|x_{mn}|^{t_{mn}} < \infty\},$$
$$\mathcal{C}_{bp}(t) := \mathcal{C}_p(t) \bigcap \mathcal{M}_u(t) \ and\ \mathcal{C}_{0bp}(t) = \mathcal{C}_{0p}(t) \bigcap \mathcal{M}_u(t);$$

where $t = (t_{mn})$ is the sequence of strictly positive reals t_{mn} for all $m, n \in \mathbb{N}$ and $p - lim_{m,n\to\infty}$ denotes the limit in the Pringsheim's sense. In the case $t_{mn} = 1$ for all $m, n \in \mathbb{N}$; $\mathcal{M}_u(t)$, $\mathcal{C}_p(t)$, $\mathcal{C}_{0p}(t)$, $\mathcal{L}_u(t)$, $\mathcal{C}_{bp}(t)$ and $\mathcal{C}_{0bp}(t)$ reduce to the sets $\mathcal{M}_u$, $\mathcal{C}_p$, $\mathcal{C}_{0p}$, $\mathcal{L}_u$, $\mathcal{C}_{bp}$ and $\mathcal{C}_{0bp}$, respectively. Now, we may summarize the knowledge given in some document related to the double sequence spaces. Gökhan and Colak [8,9] have proved that $\mathcal{M}_u(t)$ and $\mathcal{C}_p(t), \mathcal{C}_{bp}(t)$ are complete paranormed spaces of double sequences and gave the $\alpha-$, $\beta-$, $\gamma-$ duals of the spaces $\mathcal{M}_u(t)$ and $\mathcal{C}_{bp}(t)$. Quite recently, in her PhD the-

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sis, Zelter [10] has essentially studied both the theory of topological double sequence spaces and the theory of summability of double sequences. Mursaleen and Edely [11] and Tripathy have independently introduced the statistical convergence and Cauchy for double sequences and given the relation between statistical convergent and strongly Cesàro summable double sequences. Altay and Basar [12] have defined the spaces $\mathcal{BS}$, $\mathcal{BS}(t)$, $\mathcal{CS}_p$, $\mathcal{CS}_{bp}$, $\mathcal{CS}_r$ and $\mathcal{BV}$ of double sequences consisting of all double series whose sequence of partial sums are in the spaces $\mathcal{M}_u$, $\mathcal{M}_u(t)$, $\mathcal{C}_p$, $\mathcal{C}_{bp}$, $\mathcal{C}_r$ and $\mathcal{L}_u$, respectively, and also examined some properties of those sequence spaces and determined the α-duals of the spaces $\mathcal{BS}$, $\mathcal{BV}$, $\mathcal{CS}_{bp}$ and the $\beta(\vartheta)$-duals of the spaces $\mathcal{CS}_{bp}$ and $\mathcal{CS}_r$ of double series. Basar and Sever [13] have introduced the Banach space $\mathcal{L}_q$ of double sequences corresponding to the well-known space ℓ_q of single sequences and examined some properties of the space $\mathcal{L}_q$. Quite recently Subramanian and Misra [14] have studied the space $\chi^2_M(p,q,u)$ of double sequences and gave some inclusion relations.

The class of sequences which are strongly Cesáro summable with respect to a modulus was introduced by Maddox [15] as an extension of the definition of strongly Cesàro summable sequences. Connor [16] further extended this definition to a definition of strong A-summability with respect to a modulus where $A = (a_{n,k})$ is a non-negative regular matrix and established some connections between strong A-summability, strong A-summability with respect to a modulus, and A-statistical convergence. In [17] the notion of convergence of double sequences was presented by A. Pringsheim. Also, in [18–20] the four dimensional matrix transformation $(Ax)_{k,\ell} = \sum_{m=1}^{\infty}\sum_{n=1}^{\infty} a_{k\ell}^{mn} x_{mn}$ was studied extensively by Robison and Hamilton.

We need the following inequality in the sequel of the paper. For $a, b, \geqslant 0$ and $0 < p < 1$, we have

$$(a+b)^p \leqslant a^p + b^p. \tag{1.1}$$

The double series $\sum_{m,n=1}^{\infty} x_{mn}$ is called convergent if and only if the double sequence (s_{mn}) is convergent, where $s_{mn} = \sum_{i,j=1}^{m,n} x_{ij} (m,n \in \mathbb{N})$.

A sequence $x = (x_{mn})$ is said to be double analytic if $sup_{mn}|x_{mn}|^{1/m+n} < \infty$. The vector space of all double analytic sequences will be denoted by Λ^2. A sequence $x = (x_{mn})$ is called double gai sequence if $((m+n)!|x_{mn}|)^{1/m+n} \to 0$ as $m,n \to \infty$. The double gai sequences will be denoted by χ^2. Let $\phi = \{finite\ sequences\}$.

Consider a double sequence $x = (x_{ij})$. The $(m,n)^{th}$ section $x^{[m,n]}$ of the sequence is defined by $x^{[m,n]} = \sum_{i,j=0}^{m,n} x_{ij}\mathfrak{I}_{ij}$ for all $m,n \in \mathbb{N}$; where $\mathfrak{I}_{ij}$ denotes the double sequence whose only non-zero term is a $\frac{1}{(i+j)!}$ in the $(i,j)^{th}$ place for each $i,j \in \mathbb{N}$.

An FK-space(or a metric space) X is said to have AK property if $(\mathfrak{I}_{mn})$ is a Schauder basis for X. Or equivalently $x^{[m,n]} \to x$.

An FDK-space is a double sequence space endowed with a complete metrizable; locally convex topology under which the coordinate mappings $x = (x_k) \to (x_{mn}) (m,n \in \mathbb{N})$ are also continuous.

Let M and Φ are mutually complementary modulus functions. Then, we have:

(i) For all $u, y \geqslant 0$,

$$uy \leqslant M(u) + \Phi(y), (Young's\ inequality)[See[21]]. \tag{1.2}$$

(ii) For all $u \geqslant 0$,

$$u\eta(u) = M(u) + \Phi(\eta(u)). \tag{1.3}$$

(iii) For all $u \geqslant 0$, and $0 < \lambda < 1$,

$$M(\lambda u) \leqslant \lambda M(u). \tag{1.4}$$

Lindenstrauss and Tzafriri [22] used the idea of Orlicz function to construct Orlicz sequence space

$$\ell_M = \left\{ x \in w : \sum_{k=1}^{\infty} M\left(\frac{|x_k|}{\rho}\right) < \infty,\ for\ some\ \rho > 0 \right\}.$$

The space ℓ_M with the norm

$$\|x\| = inf\left\{ \rho > 0 : \sum_{k=1}^{\infty} M\left(\frac{|x_k|}{\rho}\right) \leqslant 1 \right\},$$

becomes a Banach space which is called an Orlicz sequence space. For $M(t) = t^p (1 \leqslant p < \infty)$, the spaces ℓ_M coincide with the classical sequence space ℓ_p.A sequence $f = (f_{mn})$ of modulus function is called a Musielak-modulus function. A sequence $g = (g_{mn})$ defined by

$$g_{mn}(v) = sup\{|v|u - (f_{mn})(u) : u \geqslant 0\}, \quad m,n = 1,2,\ldots$$

is called the complementary function of a Musielak-modulus function f. For a given Musielak modulus function f, the Musielak-modulus sequence space t_f and its subspace h_f are defined as follows

$$t_f = \{x \in w^2 : M_f(|x_{mn}|)^{1/m+n} \to 0\ as\ m,n \to \infty\},$$
$$h_f = \{x \in w^2 : M_f(|x_{mn}|)^{1/m+n} \to 0\ as\ m,n \to \infty\},$$

where M_f is a convex modular defined by

$$M_f(x) = \sum_{m=1}^{\infty}\sum_{n=1}^{\infty} f_{mn}(|x_{mn}|)^{1/m+n}, \quad x = (x_{mn}) \in t_f.$$

We consider t_f equipped with the Luxemburg metric

$$d(x,y) = sup_{mn}\left\{ inf\left(\sum_{m=1}^{\infty}\sum_{n=1}^{\infty} f_{mn}\left(\frac{|x_{mn}|^{1/m+n}}{mn}\right)\right) \leqslant 1 \right\}.$$

If X is a sequence space, we give the following definitions:

(i) X' = the continuous dual of X;
(ii) $X^{\alpha} = \left\{ a = (a_{mn}) : \sum_{m,n=1}^{\infty} |a_{mn}x_{mn}| < \infty,\ for\ each\ x \in X \right\}$;
(iii) $X^{\beta} = \left\{ a = (a_{mn}) : \sum_{m,n=1}^{\infty} a_{mn}x_{mn}\ is\ convegent,\ for\ each\ x \in X \right\}$;
(iv) $X^{\gamma} = \left\{ a = (a_{mn}) : sup_{mn \geqslant 1}\left|\sum_{m,n=1}^{M,N} a_{mn}x_{mn}\right| < \infty, for\ each\ x \in X \right\}$;
(v) $let\ X\ be\ an\ FK-space \supset \phi;\ then\ X^f = \{f(\mathfrak{I}_{mn}) : f \in X'\}$;
(vi) $X^{\delta} = \{a = (a_{mn}) : sup_{mn}|a_{mn}x_{mn}|^{1/m+n} < \infty,\ for\ each\ x \in X\}$;

$X^{\alpha}, X^{\beta}, X^{\gamma}$ are called α-(*or Köthe-Toeplitz*) dual of $X, \beta-$(*or generalized − Köthe − Toeplitz*) *dual of* X, $\gamma-$*dual of* X, $\delta-$*dual of* X *respectively*. X^{α} is defined by Gupta and Kamptan. It is clear that $X^{\alpha} \subset X^{\beta}$ and $X^{\alpha} \subset X^{\gamma}$, but $X^{\beta} \subset X^{\gamma}$ does not hold, since the sequence of partial sums of a double convergent series need not to be bounded.The notion of difference sequence spaces (for single sequences) was introduced by Kizmaz as follows

$Z(\Delta) = \{x = (x_k) \in w : (\Delta x_k) \in Z\}$

for $Z = c, c_0$ and ℓ_∞, where $\Delta x_k = x_k - x_{k+1}$ for all $k \in \mathbb{N}$.Here c, c_0 and ℓ_∞ denote the classes of convergent, null and bounded scalar valued single sequences respectively. The difference sequence space bv_p of the classical space ℓ_p is introduced and studied in the case $1 \leqslant p \leqslant \infty$ by Başar and Altay and in the case $0 < p < 1$ by Altay and Başar. The spaces $c(\Delta)$, $c_0(\Delta)$, $\ell_\infty(\Delta)$ and bv_p are Banach spaces normed by

$$\|x\| = |x_1| + sup_{k\geqslant 1}|\Delta x_k| \ and \ \|x\|_{bv_p} = \left(\sum_{k=1}^{\infty}|x_k|^p\right)^{1/p},$$
$$(1 \leqslant p < \infty).$$

Later on the notion was further investigated by many others. We now introduce the following difference double sequence spaces defined by

$Z(\Delta) = \{x = (x_{mn}) \in w^2 : (\Delta x_{mn}) \in Z\}$,

where $Z = \Lambda^2$, χ^2 and $\Delta x_{mn} = (x_{mn} - x_{mn+1}) - (x_{m+1n} - x_{m+1n+1}) = x_{mn} - x_{mn+1} - x_{m+1n} + x_{m+1n+1}$ for all $m, n \in \mathbb{N}$.

2. Definition and preliminaries

Let $n \in \mathbb{N}$ and X be a real vector space of dimension w, where $n \leqslant m$. A real valued function $d_p(x_1, \ldots, x_n) = \|(d_1(x_1), \ldots, d_n(x_n))\|_p$ on X satisfying the following four conditions:

(i) $\|(d_1(x_1), \ldots, d_n(x_n))\|_p = 0$ if and only if $d_1(x_1), \ldots, d_n(x_n)$ are linearly dependent,
(ii) $\|(d_1(x_1), \ldots, d_n(x_n))\|_p$ is invariant under permutation,
(iii) $\|(\alpha d_1(x_1), \ldots, d_n(x_n))\|_p = |\alpha|\|(d_1(x_1), \ldots, d_n(x_n))\|_p, \alpha \in \mathbb{R}$
(iv) $d_p((x_1, y_1), (x_2, y_2) \cdots (x_n, y_n)) = (d_X(x_1, x_2, \ldots, x_n)^p + d_Y(y_1, y_2, \ldots, y_n)^p)^{1/p}$ *for* $1 \leqslant p < \infty$; (or)
(v) $d((x_1, y_1), (x_2, y_2), \ldots, (x_n, y_n)) := \sup\{d_X(x_1, x_2, \ldots, x_n), d_Y(y_1, y_2, \ldots, y_n)\}$, for $x_1, x_2, \ldots, x_n \in X, y_1, y_2, \ldots, y_n \in Y$ is called the p product metric of the Cartesian product of n metric spaces is the p norm of the n-vector of the norms of the n subspaces.

A trivial example of p product metric of n metric space is the p norm space is $X = \mathbb{R}$ equipped with the following Euclidean metric in the product space is the p norm:

$$\|(d_1(x_1), \ldots, d_n(x_n))\|_E = sup(|det(d_{mn}(x_{mn}))|)$$
$$= sup\left(\begin{vmatrix} d_{11}(x_{11}) & d_{12}(x_{12}) & \cdots & d_{1n}(x_{1n}) \\ d_{21}(x_{21}) & d_{22}(x_{22}) & \cdots & d_{2n}(x_{1n}) \\ . & & & \\ . & & & \\ . & & & \\ d_{n1}(x_{n1}) & d_{n2}(x_{n2}) & \cdots & d_{nn}(x_{nn}) \end{vmatrix}\right),$$

where $x_i = (x_{i1}, \ldots, x_{in}) \in \mathbb{R}^n$ for each $i = 1, 2, \ldots, n$.

If every Cauchy sequence in X converges to some $L \in X$, then X is said to be complete with respect to the p-metric. Any complete p-metric space is said to be p-Banach metric space.

Let X be a linear metric space. A function $\rho : X \to \mathbb{R}$ is called paranorm, if

(1) $\rho(x) \geqslant 0$, for all $x \in X$;
(2) $\rho(-x) = \rho(x)$, for all $x \in X$;
(3) $\rho(x + y) \leqslant \rho(x) + \rho(y)$, for all $x, y \in X$;
(4) If (σ_{mn}) is a sequence of scalars with $\sigma_{mn} \to \sigma$ as $m, n \to \infty$ and (x_{mn}) is a sequence of vectors with $\rho(x_{mn} - x) \to 0$ as $m, n \to \infty$, then $\rho(\sigma_{mn}x_{mn} - \sigma x) \to 0$ as $m, n \to \infty$.

A paranorm w for which $\rho(x) = 0$ implies $x = 0$ is called total paranorm and the pair (X, w) is called a total paranormed space. It is well known that the metric of any linear metric space is given by some total paranorm (see [23], Theorem 10.4.2, p. 183).

The notion of deal convergence was introduced first by Kostyrko et al. [24] as a generalization of statistical convergence which was further studied in toplogical spaces by Kumar et al. [25,26] and also more applications of ideals can be deals with various authors by B.Hazarika [27–39] and B.C.Tripathy and B. Hazarika [40–43].

A family $I \subset 2^Y$ of subsets of a non-empty set Y is said to be an ideal in Y if

(1) $\phi \in I$
(2) $A, B \in I$ imply $A \bigcup B \in I$
(3) $A \in I$, $B \subset A$ imply $B \in I$.

while an admissible ideal I of Y further satisfies $\{x\} \in I$ for each $x \in Y$. Given $I \subset 2^{\mathbb{N}}$ be a non-trivial ideal in $\mathbb{N}$. A sequence $(x_{mn})_{m,n \in \mathbb{N}}$ in X is said to be I-convergent to $0 \in X$, if for each $\epsilon > 0$ the set $A(\epsilon) = \{m, n \in \mathbb{N} : \|(d_1(x_1), \ldots, d_n(x_n)) - 0\|_p \geqslant \epsilon\}$ belongs to I.

A sequence of positive integers $\theta = (k_{rs})$ is called double lacunary if $k_{00} = 0$, $0 < k_{rs} < k_{r+1,s+1}$ and $\varphi_{rs} = k_{rs} - k_{r-1s-1} \to \infty$ as $r, s \to \infty$. The intervals determined by θ will be denoted by $J_{rs} = (k_{r-1s-1}, k_{rs})$ and $q_{rs} = \frac{k_{rs}}{k_{r-1s-1}}$.

Let I be an admissible ideal of $\mathbb{N} \times \mathbb{N}$, $f = (f_{mn})$ be a Musielak-modulus function, $(X, \|(d(x_1), d(x_2), \ldots, d(x_{n-1}))\|_p)$ be a $p-$metric space, $q = (q_{mn})$ be double analytic sequence of strictly positive real numbers. By $w^2(p - X)$ we denote the space of all sequences defined over $(X, \|(d(x_1), d(x_2), \ldots, d(x_{n-1}))\|_p)$.

The following inequality will be used throughout the paper. If $0 \leqslant q_{mn} \leqslant supq_{mn} = H, K = max(1, 2^{H-1})$ then

$$|a_{mn} + b_{mn}|^{q_{mn}} \leqslant K\{|a_{mn}|^{q_{mn}} + |b_{mn}|^{q_{mn}}\} \tag{2.1}$$

for all m, n and $a_{mn}, b_{mn} \in \mathbb{C}$. Also $|a|^{q_{mn}} \leqslant max(1, |a|^H)$ for all $a \in \mathbb{C}$.

In the present paper we define the following sequence spaces:

$$\left[\chi^{2qu}_{f\mu}, \|(d(x_1), d(x_2), \ldots, d(x_{n-1}))\|_p^{\varphi}\right]^I_{N_\theta}$$
$$= \left\{x = (x_{mn}) : \left\{(r, s) \in \mathbb{N} \times \mathbb{N} :\right.\right.$$
$$\left.\left.\frac{1}{\varphi_{rs}}\sum_{m \in J_{rs}}\sum_{n \in J_{rs}} u_{mn}[f_{mn}(\|\mu_{mn}(x), (d(x_1), d(x_2), \ldots, d(x_{n-1}))\|_p)]^{q_{mn}} \geqslant \epsilon\right\} \in I\right\},$$

$$\left[\Lambda^{2qu}_{f\mu}, \|(d(x_1), d(x_2), \ldots, d(x_{n-1}))\|^{\varphi}_p\right]^I_{N_0} = \left\{x = (x_{mn}) : \exists K > 0, \left\{(r,s) \in \mathbb{N} \times \mathbb{N} : \frac{1}{\varphi_{rs}} \sum_{m \in J_{rs}} \sum_{n \in J_{rs}} u_{mn}[f_{mn}(\|\mu_{mn}(x), (d(x_1), d(x_2), \ldots, d(x_{n-1}))\|_p)]^{q_{mn}} \geqslant K\right\} \in I\right\}.$$

If we take $f_{mn}(x) = x$, we get

$$\left[\chi^{2qu}_{f\mu}, \|(d(x_1), d(x_2), \ldots, d(x_{n-1}))\|^{\varphi}_p\right]^I_{N_0} = \left\{x = (x_{mn}) : \left\{(r,s) \in \mathbb{N} \times \mathbb{N} : \frac{1}{\varphi_{rs}} \sum_{m \in J_{rs}} \sum_{n \in J_{rs}} u_{mn}[(\|\mu_{mn}(x), (d(x_1), d(x_2), \ldots, d(x_{n-1}))\|_p)]^{q_{mn}} \geqslant \epsilon\right\} \in I\right\},$$

$$\left[\Lambda^{2qu}_{f\mu}, \|(d(x_1), d(x_2), \ldots, d(x_{n-1}))\|^{\varphi}_p\right]^I_{N_0} = \left\{x = (x_{mn}) : \exists K > 0, \left\{(r,s) \in \mathbb{N} \times \mathbb{N} : \frac{1}{\varphi_{rs}} \sum_{m \in J_{rs}} \sum_{n \in J_{rs}} u_{mn}[(\|\mu_{mn}(x), (d(x_1), d(x_2), \ldots, d(x_{n-1}))\|_p)]^{q_{mn}} \geqslant K.\right\} \in I\right\},$$

If we take $q = (q_{mn}) = 1$, we get

$$\left[\chi^{2u}_{f\mu}, \|(d(x_1), d(x_2), \ldots, d(x_{n-1}))\|^{\varphi}_p\right]^I_{N_0} = \left\{x = (x_{mn}) : \left\{(r,s) \in \mathbb{N} \times \mathbb{N} : \frac{1}{\varphi_{rs}} \sum_{m \in J_{rs}} \sum_{n \in J_{rs}} u_{mn}[f_{mn}(\|\mu_{mn}(x), (d(x_1), d(x_2), \ldots, d(x_{n-1}))\|_p)] \geqslant \epsilon\right\} \in I\right\},$$

$$\left[\Lambda^{2u}_{f\mu}, \|(d(x_1), d(x_2), \ldots, d(x_{n-1}))\|^{\varphi}_p\right]^I_{N_0} = \left\{x = (x_{mn}) : \exists K > 0, \left\{(r,s) \in \mathbb{N} \times \mathbb{N} : \frac{1}{\varphi_{rs}} \sum_{m \in J_{rs}} \sum_{n \in J_{rs}} u_{mn}[f_{mn}(\|\mu_{mn}(x), (d(x_1), d(x_2), \ldots, d(x_{n-1}))\|_p)] \geqslant K\right\} \in I\right\}.$$

If we take $q = (q_{mn}) = 1$ and $u = (u_{mn}) = 1$, we get

$$\left[\chi^{2}_{f\mu}, \|(d(x_1), d(x_2), \ldots, d(x_{n-1}))\|^{\varphi}_p\right]^I_{N_0} = \left\{x = (x_{mn}) : \left\{(r,s) \in \mathbb{N} \times \mathbb{N} : \frac{1}{\varphi_{rs}} \sum_{m \in J_{rs}} \sum_{n \in J_{rs}} [f_{mn}(\|\mu_{mn}(x), (d(x_1), d(x_2), \ldots, d(x_{n-1}))\|_p)] \geqslant \epsilon\right\} \in I\right\},$$

$$\left[\Lambda^{2u}_{f\mu}, \|(d(x_1), d(x_2), \ldots, d(x_{n-1}))\|^{\varphi}_p\right]^I_{N_0} = \left\{x = (x_{mn}) : \exists K > 0, \left\{(r,s) \in \mathbb{N} \times \mathbb{N} : \frac{1}{\varphi_{rs}} \sum_{m \in J_{rs}} \sum_{n \in J_{rs}} [f_{mn}(\|\mu_{mn}(x), (d(x_1), d(x_2), \ldots, d(x_{n-1}))\|_p)] \geqslant K\right\} \in I\right\}.$$

In the present paper we plan to study some topological properties and inclusion relation between the above defined sequence spaces. $\left[\chi^{2qu}_{f\mu}, \|(d(x_1), d(x_2), \ldots, d(x_{n-1}))\|^{\varphi}_p\right]^I_{N_0}$ and $\left[\Lambda^{2qu}_{f\mu}, \|(d(x_1), d(x_2), \ldots, d(x_{n-1}))\|^{\varphi}_p\right]^I_{N_0}$ which we shall discuss in this paper.

3. Main results

Theorem 3.1. *Let $f = (f_{mn})$ be a Musielak-modulus function, $q = (q_{mn})$ be a double analytic sequence of strictly positive real numbers, the sequence spaces $\left[\chi^{2qu}_{f\mu}, \|(d(x_1), d(x_2), \ldots, d(x_{n-1}))\|^{\varphi}_p\right]^I_{N_0}$ and $\left[\Lambda^{2qu}_{f\mu}, \|(d(x_1), d(x_2), \ldots, d(x_{n-1}))\|^{\varphi}_p\right]^I_{N_0}$ are linear spaces.*

Proof. It is routine verification. Therefore the proof is omitted. □

Theorem 3.2. *Let $f = (f_{mn})$ be a Musielak-modulus function, $q = (q_{mn})$ be a double analytic sequence of strictly positive real numbers, the sequence space $[\chi^{2qu}_{f\mu}, \|(d(x_1), d(x_2), \ldots, d(x_{n-1}))\|^{\varphi}_p]^I_{N_0}$ is a paranormed space with respect to the paranorm defined by $g(x) = inf\{\frac{1}{\varphi_{rs}} \sum_{m \in J_{rs}} \sum_{n \in J_{rs}} u_{mn}\ [f_{mn}(\|\mu_{mn}(x), (d(x_1), d(x_2), \ldots, d(x_{n-1}))\|_p)]^{q_{mn}} \leqslant 1\}$.*

Proof. Clearly $g(x) \geqslant 0$ for $x = (x_{mn}) \in \left[\chi^{2qu}_{f\mu}, \|(d(x_1), d(x_2), \ldots, d(x_{n-1}))\|^{\varphi}_p\right]^I_{N_0}$ Since $f_{mn}(0) = 0$, we get $g(0) = 0$.

Conversely, suppose that $g(x) = 0$, then $inf\left\{\frac{1}{\varphi_{rs}} \sum_{m \in J_{rs}} \sum_{n \in J_{rs}} u_{mn}[f_{mn}(\|\mu_{mn}(x), (d(x_1), d(x_2), \ldots, d(x_{n-1}))\|_p)]^{q_{mn}} \leqslant 1\right\}$. Suppose that $\mu_{mn}(x) \neq 0$ for each $m, n \in \mathbb{N}$. Then $\|\mu_{mn}(x), (d(x_1), d(x_2), \ldots, d(x_{n-1}))\|^{\varphi}_p \to \infty$. It follows that $(\frac{1}{\varphi_{rs}} \sum_{m \in J_{rs}} \sum_{n \in J_{rs}} u_{mn}[f_{mn}(\|\mu_{mn}(x), (d(x_1), d(x_2), \ldots, d(x_{n-1}))\|_p)]^{q_{mn}})^{1/H} \to \infty$ which is a contradiction. Therefore $\mu_{mn}(x) = 0$. Let $(\frac{1}{\varphi_{rs}} \sum_{m \in J_{rs}} \sum_{n \in J_{rs}} u_{mn}[f_{mn}(\|\mu_{mn}(x), (d(x_1), d(x_2), \ldots, d(x_{n-1}))\|_p)]^{q_{mn}})^{1/H} \leqslant 1$

and

$$\left(\frac{1}{\varphi_{rs}} \sum_{m \in J_{rs}} \sum_{n \in J_{rs}} u_{mn}[f_{mn}(\|\mu_{mn}(y), (d(x_1), d(x_2), \ldots, d(x_{n-1}))\|_p)]^{q_{mn}}\right)^{1/H} \leqslant 1.$$

Then by using Minkowski's inequality, we have

$$\left(\frac{1}{\varphi_{rs}} \sum_{m \in J_{rs}} \sum_{n \in J_{rs}} u_{mn}[f_{mn}(\|\mu_{mn}(x+y), (d(x_1), d(x_2), \ldots, d(x_{n-1}))\|_p)]^{q_{mn}}\right)^{1/H} \leqslant \left(\frac{1}{\varphi_{rs}} \sum_{m \in J_{rs}} \sum_{n \in J_{rs}} u_{mn}[f_{mn}(\|\mu_{mn}(x), (d(x_1), d(x_2), \ldots, d(x_{n-1}))\|_p)]^{q_{mn}}\right)^{1/H} + \left(\frac{1}{\varphi_{rs}} \sum_{m \in J_{rs}} \sum_{n \in J_{rs}} u_{mn}[f_{mn}(\|\mu_{mn}(y), (d(x_1), d(x_2), \ldots, d(x_{n-1}))\|_p)]^{q_{mn}}\right)^{1/H}.$$

So we have

$$g(x+y) = inf\left\{\frac{1}{\varphi_{rs}} \sum_{m \in J_{rs}} \sum_{n \in J_{rs}} u_{mn}[f_{mn}(\|\mu_{mn}(x+y), (d(x_1), d(x_2), \ldots, d(x_{n-1}))\|_p)]^{q_{mn}} \leqslant 1\right\} \leqslant inf\left\{\frac{1}{\varphi_{rs}} \sum_{m \in J_{rs}} \sum_{n \in J_{rs}} u_{mn}[f_{mn}(\|\mu_{mn}(x), (d(x_1), d(x_2), \ldots, d(x_{n-1}))\|_p)]^{q_{mn}} \leqslant 1\right\} + inf\left\{\frac{1}{\varphi_{rs}} \sum_{m \in J_{rs}} \sum_{n \in J_{rs}} u_{mn}[f_{mn}(\|\mu_{mn}(y), (d(x_1), d(x_2), \ldots, d(x_{n-1}))\|_p)]^{q_{mn}} \leqslant 1\right\}.$$

Therefore,
$g(x+y) \leqslant g(x)+g(y)$.

Finally, to prove that the scalar multiplication is continuous. Let λ be any complex number. By definition,

$$g(\lambda x) = inf\left\{\frac{1}{\varphi_{rs}}\sum_{m\in J_{rs}}\sum_{n\in J_{rs}} u_{mn}[f_{mn}(\|\mu_{mn}(\lambda x),(d(x_1), d(x_2),\ldots,d(x_{n-1}))\|_p)]^{q_{mn}} \leqslant 1\right\}.$$

Then

$$g(\lambda\ x) = inf\left\{((|\lambda|t)^{q_{mn}/H}:\frac{1}{\varphi_{rs}}\sum_{m\in J_{rs}}\sum_{n\in J_{rs}} u_{mn}[f_{mn}(\|\mu_{mn}(\lambda x),(d(x_1), d(x_2),\ldots,d(x_{n-1}))\|_p)]^{q_{mn}} \leqslant 1\right\},$$

where $t=\frac{1}{|\lambda|}$. Since $|\lambda|^{q_{mn}} \leqslant max(1,|\lambda|^{supq_{mn}})$, we have

$$g(\lambda x) \leqslant max(1,|\lambda|^{supp_{mn}}) inf\left\{t^{q_{mn}/H}:\frac{1}{\varphi_{rs}}\sum_{m\in J_{rs}}\sum_{n\in J_{rs}} u_{mn}[f_{mn}(\|\mu_{mn}(\lambda x),(d(x_1),d(x_2),\ldots,d(x_{n-1}))\|_p)]^{q_{mn}} \leqslant 1\right\}.$$

This completes the proof. □

Theorem 3.3. *The β-dual space of*

$$\left[\chi^{2qu}_{f\mu},\|(d(x_1),d(x_2),\ldots,d(x_{n-1}))\|^{\varphi}_{p}\right]^{I}_{N_\theta} = \left[\Lambda^{2qu}_{f\mu},\|(d(x_1),d(x_2),\ldots,d(x_{n-1}))\|^{\varphi}_{p}\right]^{I}_{N_\theta}.$$

Proof. First, we observe that

$$\left[\chi^{2qu}_{f\mu},\|(d(x_1),d(x_2),\ldots,d(x_{n-1}))\|^{\varphi}_{p}\right]^{I}_{N_\theta} \subset \left[\Gamma^{2qu}_{f\mu},\|(d(x_1),d(x_2),\ldots,d(x_{n-1}))\|^{\varphi}_{p}\right]^{I}_{N_\theta}.$$

Therefore

$$\left[\Gamma^{2qu}_{f\mu},\|(d(x_1),d(x_2),\ldots,d(x_{n-1}))\|^{\varphi}_{p}\right]^{I\beta}_{N_\theta} \subset \left[\chi^{2qu}_{f\mu},\|(d(x_1),d(x_2),\ldots,d(x_{n-1}))\|^{\varphi}_{p}\right]^{I\beta}_{N_\theta}.$$

Hence

$$\left[\Lambda^{2qu}_{f\mu},\|(d(x_1),d(x_2),\ldots,d(x_{n-1}))\|^{\varphi}_{p}\right]^{I}_{N_\theta} \subset \left[\chi^{2qu}_{f\mu},\|(d(x_1),d(x_2),\ldots,d(x_{n-1}))\|^{\varphi}_{p}\right]^{I\beta}_{N_\theta}. \quad (3.1)$$

Next we show that

$$\left[\chi^{2qu}_{f\mu},\|(d(x_1),d(x_2),\ldots,d(x_{n-1}))\|^{\varphi}_{p}\right]^{I\beta}_{N_\theta} \subset \left[\Lambda^{2qu}_{f\mu},\|(d(x_1),d(x_2),\ldots,d(x_{n-1}))\|^{\varphi}_{p}\right]^{I}_{N_\theta}.$$

Let $y=(y_{mn}) \in \left[\chi^{2qu}_{f\mu},\|(d(x_1),d(x_2),\ldots,d(x_{n-1}))\|^{\varphi}_{p}\right]^{I\beta}_{N_\theta}$.

Consider $f(x)=\sum_{m=1}^{\infty}\sum_{n=1}^{\infty} x_{mn}y_{mn}$ with

$x=(x_{mn}) \in \left[\chi^{2qu}_{f\mu},\|(d(x_1),d(x_2),\ldots,d(x_{n-1}))\|^{\varphi}_{p}\right]^{I}_{N_\theta}$,

$x = [(\lambda_{mn}-\lambda_{mn+1})-(\lambda_{m+1n}-\lambda_{m+1n+1})]$

$$= \begin{pmatrix} 0 & 0 & \cdots 0 & 0 & \cdots & 0 \\ 0 & 0 & \cdots 0 & 0 & \cdots & 0 \\ . \\ . \\ . \\ 0 & 0 & \cdots \frac{\varphi_{rs}}{\Delta\lambda_{mn}(m+n)!} & \frac{-\varphi_{rs}}{\Delta\lambda_{mn}(m+n)!} & \cdots & 0 \\ 0 & 0 & \cdots 0 & 0 & \cdots & 0 \end{pmatrix} - \begin{pmatrix} 0 & 0 & \cdots 0 & 0 & \cdots & 0 \\ 0 & 0 & \cdots 0 & 0 & \cdots & 0 \\ . \\ . \\ . \\ 0 & 0 & \cdots \frac{\varphi_{rs}}{\Delta\lambda_{mn}(m+n)!} & \frac{-\varphi_{rs}}{\Delta\lambda_{mn}(m+n)!} & \cdots & 0 \\ 0 & 0 & \cdots 0 & 0 & \cdots & 0 \end{pmatrix},$$

$u_{mn}[f_{mn}(\|\mu_{mn}(x),(d(x_1),d(x_2),\ldots,d(x_{n-1}))\|^{\varphi}_{p})]$

$$= \begin{pmatrix} 0 & 0 & \cdots 0 & 0 & \cdots & 0 \\ 0 & 0 & \cdots 0 & 0, & \cdots & 0 \\ . \\ . \\ . \\ 0 & 0 & \cdots u_{mn}f_{mn}\left(\frac{\varphi_{rs}}{\Delta\lambda_{mn}(m+n)!}\right) & u_{mn}f_{mn}\left(\frac{-\varphi_{rs}}{\Delta\lambda_{mn}(m+n)!}\right) & \cdots & 0 \\ 0 & 0 & \cdots u_{mn}f_{mn}\left(\frac{-\varphi_{rs}}{\Delta\lambda_{mn}(m+n)!}\right) & u_{mn}f_{mn}\left(\frac{\varphi_{rs}}{\Delta\lambda_{mn}(m+n)!}\right) & \cdots & 0 \\ 0 & 0 & \cdots 0 & 0, & \cdots & 0 \end{pmatrix}.$$

Hence converges to zero.

Therefore

$[(\lambda_{mn}-\lambda_{mn+1})-(\lambda_{m+1n}-\lambda_{m+1n+1})]$

$$\in \left[\chi^{2qu}_{f\mu},\|(d(x_1),d(x_2),\ldots,d(x_{n-1}))\|^{\varphi}_{p}\right]^{I}_{N_\theta}.$$

Hence $d((\lambda_{mn}-\lambda_{mn+1})-(\lambda_{m+1n}-\lambda_{m+1n+1}),0)=1$. But

$$|y_{mn}| \leqslant \|f\| d((\lambda_{mn}-\lambda_{mn+1})-(\lambda_{m+1n}-\lambda_{m+1n+1}),0) \leqslant \|f\|\cdot 1 < \infty$$

for each m,n. Thus (y_{mn}) is a double analytic sequence and hence an p-metric space of Musielak modulus function is a double analytic sequence.

In other words $y \in \left[\Lambda^{2qu}_{f\mu},\|(d(x_1),d(x_2),\ldots,d(x_{n-1}))\|^{\varphi}_{p}\right]^{I}_{N_\theta}$. But $y=(y_{mn})$ is arbitrary in $\left[\chi^{2qu}_{f\mu},\|(d(x_1),d(x_2),\ldots,d(x_{n-1}))\|^{\varphi}_{p}\right]^{I\beta}_{N_\theta}$. Therefore

$$\left[\chi^{2qu}_{f\mu},\|(d(x_1),d(x_2),\ldots,d(x_{n-1}))\|^{\varphi}_{p}\right]^{I\beta}_{N_\theta} \subset \left[\Lambda^{2qu}_{f\mu},\|(d(x_1),d(x_2),\ldots,d(x_{n-1}))\|^{\varphi}_{p}\right]^{I}_{N_\theta}. \quad (3.2)$$

From (3.1) and (3.2) we get

$$\left[\chi^{2qu}_{f\mu},\|(d(x_1),d(x_2),\ldots,d(x_{n-1}))\|^{\varphi}_{p}\right]^{I\beta}_{N_\theta} = \left[\Lambda^{2qu}_{f\mu},\|(d(x_1),d(x_2),\ldots,d(x_{n-1}))\|^{\varphi}_{p}\right]^{I}_{N_\theta}. \quad \square$$

Theorem 3.4. *The dual space of* $\left[\chi^{2qu}_{f\mu},\|\mu_{mn}(x),(d(x_1),d(x_2),\ldots,d(x_{n-1}))\|^{\varphi}_{p}\right]^{I}_{N_\theta}$ *is* $\left[\Lambda^{2qu}_{f\mu},\|\mu_{mn}(x),(d(x_1),d(x_2),\ldots,d(x_{n-1}))\|^{\varphi}_{p}\right]^{I}_{N_\theta}$. *In other words*

$$\left[\chi^{2q}_{f\mu}, \|\mu_{mn}(x), (d(x_1), d(x_2), \ldots, d(x_{n-1}))\|^{\varphi}_{p}\right]^{I*}_{N_\theta}$$
$$= \left[\Lambda^{2q}_{f\mu}, \|\mu_{mn}(x), (d(x_1), d(x_2), \ldots, d(x_{n-1}))\|^{\varphi}_{p}\right]^{I}_{N_\theta}.$$

Proof. We recall that

$$\lambda_{mn} = \begin{pmatrix} 0 & 0 & \cdots 0 & 0 & \cdots \\ 0 & 0 & \cdots 0 & 0 & \cdots \\ . & & & & \\ . & & & & \\ . & & & & \\ 0 & 0 & \cdots \frac{\varphi_{rs}}{\Delta\lambda_{mn}(m+n)!} & 0 & \cdots \\ 0 & 0 & \cdots 0 & 0 & \cdots \end{pmatrix}$$

with $\frac{\varphi_{rs}}{\Delta\lambda_{mn}(m+n)!}$ in the $(m,n)th$ position and zero's elsewhere,

$$\left[\chi^{2qu}_{f\mu}, \|\mu_{mn}(x), (d(x_1), d(x_2), \ldots, d(x_{n-1}))\|^{\varphi}_{p}\right]^{I}_{N_\theta}$$
$$= \begin{pmatrix} 0. & & . & . & 0 \\ . & & & & \\ . & & & & \\ . & & & & \\ 0 & uf\left(\frac{\varphi_{rs}}{\Delta\lambda_{mn}(m+n)!}\right)^{1/m+n} & & . & 0 \\ & (m,n)^{th} & & & \\ 0 & & . & . & 0 \end{pmatrix},$$

which is a p-metric of double gai sequence. Hence,

$$x \in \left[\chi^{2qu}_{f\mu}, \|\mu_{mn}(x), (d(x_1), d(x_2), \ldots, d(x_{n-1}))\|^{\varphi}_{p}\right]^{I}_{N_\theta},$$
$$f(x) = \sum_{m,n=1}^{\infty} x_{mn} y_{mn},$$

with

$$x \in \left[\chi^{2qu}_{f\mu}, \|\mu_{mn}(x), (d(x_1), d(x_2), \ldots, d(x_{n-1}))\|^{\varphi}_{p}\right]^{I}_{N_\theta}$$

and

$$f \in \left[\chi^{2qu}_{f\mu}, \|\mu_{mn}(x), (d(x_1), d(x_2), \ldots, d(x_{n-1}))\|^{\varphi}_{p}\right]^{I*}_{N_\theta},$$

where

$$\left[\chi^{2qu}_{f\mu}, \|\mu_{mn}(x), (d(x_1), d(x_2), \ldots, d(x_{n-1}))\|^{\varphi}_{p}\right]^{I*}_{N_\theta}$$

is the dual space of $\left[\chi^{2qu}_{f\mu}, \|\mu_{mn}(x), (d(x_1), d(x_2), \ldots, d(x_{n-1}))\|^{\varphi}_{p}\right]^{I}_{N_\theta}$. Take $x = (x_{mn}) \in \left[\chi^{2qu}_{f\mu}, \|\mu_{mn}(x), (d(x_1), d(x_2), \ldots, d(x_{n-1}))\|^{\varphi}_{p}\right]^{I}_{N_\theta}$.Then,

$$|y_{mn}| \leqslant \|f\| d(\varphi_{rs}, 0) < \infty \forall m, n. \tag{3.3}$$

Thus, (y_{mn}) is a double analytic sequence and hence an p-metric is a Musielak modulus function of double analytic sequence. In other words, $y \in [\Lambda^{2qu}_{f\mu}, \|\mu_{mn}(x), (d(x_1), d(x_2), \ldots, d(x_{n-1}))\|^{\varphi}_{p}]^{I}_{N_\theta}$. Therefore

$$\left[\chi^{2qu}_{f\mu}, \|\mu_{mn}(x), (d(x_1), d(x_2), \ldots, d(x_{n-1}))\|^{\varphi}_{p}\right]^{I*}_{N_\theta}$$
$$= \left[\Lambda^{2qu}_{f\mu}, \|\mu_{mn}(x), (d(x_1), d(x_2), \ldots, d(x_{n-1}))\|^{\varphi}_{p}\right]^{I}_{N_\theta}.$$

This completes the proof. □

Theorem 3.5.

(i) *If the sequence* (f_{mn}) *satisfies uniform* Δ_2*-condition, then*

$$\left[\chi^{2qu}_{f\mu}, \|\mu_{mn}(x), (d(x_1), d(x_2), \ldots, d(x_{n-1}))\|^{\varphi}_{p}\right]^{I\alpha}_{N_\theta}$$
$$= \left[\chi^{2qu}_{g\mu}, \|\mu_{mn}(x), (d(x_1), d(x_2), \ldots, d(x_{n-1}))\|^{\varphi}_{p}\right]^{I}_{N_\theta}.$$

(ii) *If the sequence* (g_{mn}) *satisfies uniform* Δ_2*-condition, then*

$$\left[\chi^{2qu}_{g\mu}, \|\mu_{mn}(x), (d(x_1), d(x_2), \ldots, d(x_{n-1}))\|^{\varphi}_{p}\right]^{I\alpha}_{N_\theta}$$
$$= \left[\chi^{2qu}_{f\mu}, \|\mu_{mn}(x), (d(x_1), d(x_2), \ldots, d(x_{n-1}))\|^{\varphi}_{p}\right]^{I}_{N_\theta}.$$

Proof. Let the sequence (f_{mn}) satisfies uniform Δ_2-condition, we get

$$\left[\chi^{2qu}_{g\mu}, \|\mu_{mn}(x), (d(x_1), d(x_2), \ldots, d(x_{n-1}))\|^{\varphi}_{p}\right]^{I}_{N_\theta}$$
$$\subset \left[\chi^{2qu}_{f\mu}, \|\mu_{mn}(x), (d(x_1), d(x_2), \ldots, d(x_{n-1}))\|^{\varphi}_{p}\right]^{I\alpha}_{N_\theta}. \tag{3.4}$$

To prove the inclusion

$$\left[\chi^{2qu}_{f\mu}, \|\mu_{mn}(x), (d(x_1), d(x_2), \ldots, d(x_{n-1}))\|^{\varphi}_{p}\right]^{I\alpha}_{N_\theta}$$
$$\subset \left[\chi^{2qu}_{g\mu}, \|\mu_{mn}(x), (d(x_1), d(x_2), \ldots, d(x_{n-1}))\|^{\varphi}_{p}\right]^{I}_{N_\theta},$$

let $a \in \left[\chi^{2qu}_{f\mu}, \|\mu_{mn}(x), (d(x_1), d(x_2), \ldots, d(x_{n-1}))\|^{\varphi}_{p}\right]^{I\alpha}_{N\theta}$. Then for all $\{x_{mn}\}$ with $(x_{mn}) \in \left[\chi^{2qu}_{f\mu}, \|\mu_{mn}(x), (d(x_1), d(x_2), \ldots, d(x_{n-1}))\|^{\varphi}_{p}\right]^{I}_{N_\theta}$ we have

$$\sum_{m=1}^{\infty}\sum_{n=1}^{\infty} |x_{mn} a_{mn}| < \infty. \tag{3.5}$$

Since the sequence (f_{mn}) satisfies uniform Δ_2-condition, then $(y_{mn}) \in \left[\chi^{2qu}_{f\mu}, \|\mu_{mn}(x), (d(x_1), d(x_2), \ldots, d(x_{n-1}))\|^{\varphi}_{p}\right]^{I}_{N_\theta}$, we get $\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}\left|\frac{\varphi_{rs} y_{mn} a_{mn}}{\Delta\lambda_{mn}(m+n)!}\right| < \infty$. by (3.5). Thus $(\varphi_{rs} a_{mn}) \in \left[\chi^{2qu}_{f\mu}, \|\mu_{mn}(x), (d(x_1), d(x_2), \ldots, d(x_{n-1}))\|^{\varphi}_{p}\right]^{I}_{N_\theta} = \left[\chi^{2qu}_{g\mu}, \|\mu_{mn}(x), (d(x_1), d(x_2), \ldots, d(x_{n-1}))\|^{\varphi}_{p}\right]^{I}_{N_\theta}$ and hence $(a_{mn}) \in \left[\chi^{2qu}_{g\mu}, \|\mu_{mn}(x), (d(x_1), d(x_2), \ldots, d(x_{n-1}))\|^{\varphi}_{p}\right]^{I}_{N_\theta}$. This gives that

$$\left[\chi^{2qu}_{f\mu}, \|\mu_{mn}(x), (d(x_1), d(x_2), \ldots, d(x_{n-1}))\|^{\varphi}_{p}\right]^{I\alpha}_{N_\theta}$$
$$\subset \left[\chi^{2qu}_{g\mu}, \|\mu_{mn}(x), (d(x_1), d(x_2), \ldots, d(x_{n-1}))\|^{\varphi}_{p}\right]^{I}_{N_\theta}, \tag{3.6}$$

we are granted with (3.4) and (3.6)

$$\left[\chi^{2qu}_{f\mu}, \|\mu_{mn}(x), (d(x_1), d(x_2), \ldots, d(x_{n-1}))\|^{\varphi}_{p}\right]^{I\alpha}_{N_\theta}$$
$$= \left[\chi^{2qu}_{g\mu}, \|\mu_{mn}(x), (d(x_1), d(x_2), \ldots, d(x_{n-1}))\|^{\varphi}_{p}\right]^{I}_{N_\theta}.$$

(ii) Similarly, one can prove that $\left[\chi^{2qu}_{g\mu}, \|\mu_{mn}(x), (d(x_1), d(x_2), \ldots, d(x_{n-1}))\|^{\varphi}_{p}\right]^{I\alpha}_{N_\theta} \subset \left[\chi^{2qu}_{f\mu}, \|\mu_{mn}(x), (d(x_1), d(x_2), \ldots, d(x_{n-1}))\|^{\varphi}_{p}\right]^{I}_{N_\theta}$ if the sequence (g_{mn}) satisfies uniform Δ_2-condition. □

Proposition 3.6. If $0 < q_{mn} < p_{mn} < \infty$ for each m and m, then

$$\left[\Lambda_{f\mu}^{2qu}, \|\mu_{mn}(x), (d(x_1), d(x_2), \ldots, d(x_{n-1}))\|_p^{\varphi}\right]_{N_\theta}^I \subseteq [\Lambda_{f\mu}^{2pu}, \|\mu_{mn}(x), \left(d(x_1), d(x_2), \ldots, d(x_{n-1}))\|_p^{\varphi}\right]_{N_\theta}^I.$$

Proof. Let $x = (x_{mn}) \in \left[\Lambda_{f\mu}^{2qu}, \|\mu_{mn}(x), (d(x_1), d(x_2), \ldots, d(x_{n-1}))\|_p^{\varphi}\right]_{N_\theta}^I$. We have

$$sup_{mn}\left[\Lambda_{f\mu}^{2qu}, \|\mu_{mn}(x), (d(x_1), d(x_2), \ldots, d(x_{n-1}))\|_p^{\varphi}\right]_{N_\theta}^I < \infty$$

for sufficiently large value of m and n. Since f_{mn}'s are non-decreasing, we get

$$sup_{mn}\left[\Lambda_{f\mu}^{2pu}, \|\mu_{mn}(x), (d(x_1), d(x_2), \ldots, d(x_{n-1}))\|_p^{\varphi}\right]_{N_\theta}^I \leqslant sup_{mn}\left[\Lambda_{f\mu}^{2qu}, \|\mu_{mn}(x), (d(x_1), d(x_2), \ldots, d(x_{n-1}))\|_p^{\varphi}\right]_{N_\theta}^I.$$

Thus, $x = (x_{mn}) \in [\Lambda_{f\mu}^{2pu}, \|\mu_{mn}(x), (d(x_1), d(x_2), \ldots, d(x_{n-1}))\|_p^{\varphi}]_{N_\theta}^I$. This completes the proof. □

Proposition 3.7.

(i) *If* $0 < infq_{mn} \leqslant q_{mn} < 1$ *then*

$$\left[\Lambda_{f\mu}^{2qu}, \|\mu_{mn}(x), (d(x_1), d(x_2), \ldots, d(x_{n-1}))\|_p^{\varphi}\right]_{N_\theta}^I \subset \left[\Lambda_{f\mu}^{2u}, \|\mu_{mn}(x), (d(x_1), d(x_2), \ldots, d(x_{n-1}))\|_p^{\varphi}\right]_{N_\theta}^I.$$

(ii) *If* $1 \leqslant q_{mn} \leqslant supq_{mn} < \infty$, *then*

$$\left[\Lambda_{f\mu}^{2u}, \|\mu_{mn}(x), (d(x_1), d(x_2), \ldots, d(x_{n-1}))\|_p^{\varphi}\right]_{N_\theta}^I \subset [\Lambda_{f\mu}^{2qu}, \|\mu_{mn}(x), (d(x_1), d(x_2), \ldots, d(x_{n-1}))\|_p^{\varphi}]_{N_\theta}^I.$$

Proof. Let $x = (x_{mn}) \in [\Lambda_{f\mu}^{2qu}, \|\mu_{mn}(x), (d(x_1), d(x_2), \ldots, d(x_{n-1}))\|_p^{\varphi}]_{N_\theta}^I$. Since $0 < \inf q_{mn} \leqslant 1$, we have

$$sup_{mn}\left[\Lambda_{f\mu}^{2u}, \|\mu_{mn}(x), (d(x_1), d(x_2), \ldots, d(x_{n-1}))\|_p^{\varphi}\right]_{N_\theta}^I \leqslant \left[\Lambda_{f\mu}^{2qu}, \|\mu_{mn}(x), (d(x_1), d(x_2), \ldots, d(x_{n-1}))\|_p^{\varphi}\right]_{N_\theta}^I,$$

and hence

$$x = (x_{mn}) \in \left[\Lambda_{f\mu}^{2u}, \|\mu_{mn}(x), (d(x_1), d(x_2), \ldots, d(x_{n-1}))\|_p^{\varphi}\right]_{N_\theta}^I.$$

(ii) Let q_{mn} for each (m,n) and $sup_{mn}q_{mn} < \infty$. Let $x = (x_{mn}) \in [\Lambda_{f\mu}^{2u}, \|\mu_{mn}(x), (d(x_1), d(x_2), \ldots, d(x_{n-1}))\|_p^{\varphi}]_{N_\theta}^I$. Then for each $0 < \epsilon < 1$, there exists a positive integer N such that

$$sup_{mn}\left[\Lambda_{f\mu}^{2u}, \|\mu_{mn}(x), (d(x_1), d(x_2), \ldots, d(x_{n-1}))\|_p^{\varphi}\right]_{N_\theta}^I \leqslant \epsilon < 1,$$

for all $m,n \geqslant N$. This implies that

$$sup_{mn}\left[\Lambda_{f\mu}^{2qu}, \|\mu_{mn}(x), (d(x_1), d(x_2), \ldots, d(x_{n-1}))\|_p^{\varphi}\right]_{N_\theta}^I \leqslant sup_{mn}\left[\Lambda_{f\mu}^{2u}, \|\mu_{mn}(x), (d(x_1), d(x_2), \ldots, d(x_{n-1}))\|_p^{\varphi}\right]_{N_\theta}^I.$$

Thus $x = (x_{mn}) \in \left[\Lambda_{f\mu}^{2qu}, \|\mu_{mn}(x), (d(x_1), d(x_2), \ldots, d(x_{n-1}))\|_p^{\varphi}\right]_{N_\theta}^I$. This completes the proof.

Proposition 3.8. *Let* $f' = (f'_{mn})$ *and* $f'' = (f''_{mn})$ *are sequences of Musielak functions, we have*

$$\left[\Lambda_{f'\mu}^{2qu}, \|\mu_{mn}(x), (d(x_1), d(x_2), \ldots, d(x_{n-1}))\|_p^{\varphi}\right]_{N_\theta}^I \bigcap \left[\Lambda_{f''\mu}^{2qu}, \|\mu_{mn}(x), \left(d(x_1), d(x_2), \ldots, d(x_{n-1}))\|_p^{\varphi}\right]_{N_\theta}^I \subseteq [\Lambda_{f'+f''\mu}^{2qu}, \|\mu_{mn}(x), (d(x_1), d(x_2), \ldots, d(x_{n-1}))\|_p^{\varphi}]_{N_\theta}^I.$$

Proof. The proof is easy so we omit it. □

Proposition 3.9. *For any sequence of Musielak functions* $f = (f_{mn})$ *and* $q = (q_{mn})$ *be double analytic sequence of strictly positive real numbers. Then*

$$\left[\chi_{f\mu}^{2qu}, \|\mu_{mn}(x), (d(x_1), d(x_2), \ldots, d(x_{n-1}))\|_p^{\varphi}\right]_{N_\theta}^I \subset \left[\Lambda_{f\mu}^{2qu}, \|\mu_{mn}(x), (d(x_1), d(x_2), \ldots, d(x_{n-1}))\|_p^{\varphi}\right]_{N_\theta}^I.$$

Proof. The proof is easy so we omit it. □

Proposition 3.10. *The sequence space* $[\Lambda_{f\mu}^{2qu}, \|\mu_{mn}(x), (d(x_1), d(x_2), \ldots, d(x_{n-1}))\|_p^{\varphi}]_{N_\theta}^I$ *is solid*

Proof. Let $x = (x_{mn}) \in [\Lambda_{f\mu}^{2qu}, \|\mu_{mn}(x), (d(x_1), d(x_2), \ldots, d(x_{n-1}))\|_p^{\varphi}]_{N_\theta}^I$. (i.e)

$$sup_{mn}\left[\Lambda_{f\mu}^{2qu}, \|\mu_{mn}(x), (d(x_1), d(x_2), \ldots, d(x_{n-1}))\|_p^{\varphi}\right]_{N_\theta}^I < \infty.$$

Let (α_{mn}) be double sequence of scalars such that $|\alpha_{mn}| \leqslant 1$ for all $m,n \in N \times N$. Then, we get

$$sup_{mn}\left[\Lambda_{f\mu}^{2qu}, \|\mu_{mn}(\alpha x), (d(x_1), d(x_2), \ldots, d(x_{n-1}))\|_p^{\varphi}\right]_{N_\theta}^I \leqslant \sup_{mn}\left[\Lambda_{f\mu}^{2qu}, \|\mu_{mn}(x), (d(x_1), d(x_2), \ldots, d(x_{n-1}))\|_p^{\varphi}\right]_{N_\theta}^I.$$

This completes the proof. □

Proposition 3.11. *The sequence space* $[\Lambda_{f\mu}^{2qu}, \|\mu_{mn}(x), (d(x_1), d(x_2), \ldots, d(x_{n-1}))\|_p^{\varphi}]_{N_\theta}^I$ *is monotone*

Proof. The proof follows from Proposition 3.10. □

Proposition 3.12. *If* $f = (f_{mn})$ *be any Musielak function. Then*

$$\left[\Lambda_{f\mu}^{2qu}, \|\mu_{mn}(x), (d(x_1), d(x_2), \ldots, d(x_{n-1}))\|_p^{\varphi^*}\right]_{N_\theta}^I \subset \left[\Lambda_{f\mu}^{2qu}, \|\mu_{mn}(x), (d(x_1), d(x_2), \ldots, d(x_{n-1}))\|_p^{\varphi^{**}}\right]_{N_\theta}^I$$

if and only if $sup_{r,s \geqslant 1} \frac{\varphi_{rs}^*}{\varphi_{rs}^{**}} < \infty$.

Proof. Let $x \in \left[\Lambda^{2qu}_{f\mu}, \|\mu_{mn}(x), (d(x_1), d(x_2), \ldots, d(x_{n-1}))\|^{\varphi^*}_p\right]^I_{N_\theta}$ and $N = sup_{r,s\geqslant 1}\frac{\varphi^*_{rs}}{\varphi^{**}_{rs}} < \infty$. Then, we get

$$\left[\Lambda^{2qu}_{f\mu}, \|\mu_{mn}(x), (d(x_1), d(x_2), \ldots, d(x_{n-1}))\|^{\varphi^{**}_{rs}}_p\right]^I_{N_\theta} = N\left[\Lambda^{2qu}_{f\mu}, \|\mu_{mn}(x), (d(x_1), d(x_2), \ldots, d(x_{n-1}))\|^{\varphi^*_{rs}}_p\right]^I_{N_\theta} = 0.$$

Thus, $x \in \left[\Lambda^{2qu}_{f\mu}, \|\mu_{mn}(x), (d(x_1), d(x_2), \ldots, d(x_{n-1}))\|^{\varphi^{**}}_p\right]^I_{N_\theta}$. Conversely, suppose that

$$\left[\Lambda^{2qu}_{f\mu}, \|\mu_{mn}(x), (d(x_1), d(x_2), \ldots, d(x_{n-1}))\|^{\varphi^*}_p\right]^I_{N_\theta} \subset \left[\Lambda^{2qu}_{f\mu}, \|\mu_{mn}(x), (d(x_1), d(x_2), \ldots, d(x_{n-1}))\|^{\varphi^{**}}_p\right]^I_{N_\theta}$$ and $x \in [\Lambda^{2qu}_{f\mu}, \|\mu_{mn}(x), (d(x_1), d(x_2), \ldots, d(x_{n-1}))\|^{\varphi^*}_p]^I_{N_\theta}$.

Then, $\left[\Lambda^{2qu}_{f\mu}, \|\mu_{mn}(x), (d(x_1), d(x_2), \ldots, d(x_{n-1}))\|^{\varphi^*}_p\right]^I_{N_\theta} < \epsilon$, for every $\epsilon > 0$. Suppose that $sup_{r,s\geqslant 1}\frac{\varphi^*_{rs}}{\varphi^{**}_{rs}} = \infty$, then there exists a sequence of members (rs_{jk}) such that $lim_{j,k arrow \infty}\frac{\varphi^*_{jk}}{\varphi^{**}_{jk}} = \infty$. Hence, we have $\left[\Lambda^{2qu}_{f\mu}, \|\mu_{mn}(x), (d(x_1), d(x_2), \ldots, d(x_{n-1}))\|^{\varphi^*_{rs}}_p\right]^I_{N_\theta} = \infty$. Therefore, $x \notin \left[\Lambda^{2qu}_{f\mu}, \|\mu_{mn}(x), (d(x_1), d(x_2), \ldots, d(x_{n-1}))\|^{\varphi^{**}}_p\right]^I_{N_\theta}$, which is a contradiction. This completes the proof. □

Proposition 3.13. *If $f = (f_{mn})$ be any Musielak function. Then*

$$\left[\Lambda^{2qu}_{f\mu}, \|\mu_{mn}(x), (d(x_1), d(x_2), \ldots, d(x_{n-1}))\|^{\varphi^*}_p\right]^I_{N_\theta} = \left[\Lambda^{2qu}_{f\mu}, \|\mu_{mn}(x), (d(x_1), d(x_2), \ldots, d(x_{n-1}))\|^{\varphi^{**}}_p\right]^I_{N_\theta}$$

if and only if $sup_{r,s\geqslant 1}\frac{\varphi^*_{rs}}{\varphi^{**}_{rs}} < \infty, sup_{r,s\geqslant 1}\frac{\varphi^{**}_{rs}}{\varphi^*_{rs}} > \infty$.

Proof. It is easy to prove so we omit. □

Proposition 3.14. *The sequence space* $\left[\chi^{2qu}_{f\mu}, \|\mu_{mn}(x), (d(x_1), d(x_2), \ldots, d(x_{n-1}))\|^{\varphi}_p\right]^I_{N_\theta}$ *is not solid*

Proof. The result follows from the following example. □

Example 1. Consider

$$x = (x_{mn}) = \begin{pmatrix} 1 & 1 & \cdots & 1 \\ 1 & 1 & \cdots & 1 \\ . & & & \\ . & & & \\ . & & & \\ 1 & 1 & \cdots & 1 \end{pmatrix} \in \left[\chi^{2qu}_{f\mu}, \|\mu_{mn}(x), (d(x_1), d(x_2), \ldots, d(x_{n-1}))\|^{\varphi}_p\right]^I_{N_\theta}.$$

Let

$$\alpha_{mn} = \begin{pmatrix} -1^{m+n} & -1^{m+n} & \cdots & -1^{m+n} \\ -1^{m+n} & -1^{m+n} & \cdots & -1^{m+n} \\ . & & & \\ . & & & \\ . & & & \\ -1^{m+n} & -1^{m+n} & \cdots & -1^{m+n} \end{pmatrix},$$

for all $m, n \in \mathbb{N}$.

Then $\alpha_{mn}x_{mn} \notin [\chi^{2qu}_{f\mu}, \|\mu_{mn}(x), (d(x_1), d(x_2), \ldots, d(x_{n-1}))\|^{\varphi}_p]^I_{N_\theta}$. Hence

$$\left[\chi^{2qu}_{f\mu}, \|\mu_{mn}(x), (d(x_1), d(x_2), \ldots, d(x_{n-1}))\|^{\varphi}_p\right]^I_{N_\theta}$$

is not solid.

Proposition 3.15. *The sequence space* $\left[\chi^{2qu}_{f\mu}, \|\mu_{mn}(x), (d(x_1), d(x_2), \ldots, d(x_{n-1}))\|^{\varphi}_p\right]^I_{N_\theta}$ *is not monotone*

Proof. The proof follows from Proposition 3.14. □

4. Generalized four dimensional infinite matrix sequence spaces

Let $A = (a^{mn}_{k\ell})$ be an four dimensional infinite matrix of complex numbers. Then, we have $A(x) = (Ax)_{k\ell} = \sum_{m=1}^{\infty}\sum_{n=1}^{\infty} a^{mn}_{k\ell}x_{mn}$ converges for each k, ℓ. In this section we introduce the following sequence spaces:

$$\left[\chi^{2quA}_{f\mu}, \|(d(x_1), d(x_2), \ldots, d(x_{n-1}))\|^{\varphi}_p\right]^I_{N_\theta} = \left\{x = (x_{mn}) : \left\{(r,s) \in \mathbb{N}\times\mathbb{N} : \frac{1}{\varphi_{rs}}\sum_{m\in J_{rs}}\sum_{n\in J_{rs}}\{u_{mn}[f_{mn}(\|A_{mn}\mu_{mn}(x), (d(x_1), d(x_2), \ldots, d(x_{n-1}))\|_p)]^{q_{mn}} \geqslant \epsilon\} \in I\right\}\right\},$$

$$\left[\Lambda^{2quA}_{f\mu}, \|(d(x_1), d(x_2), \ldots, d(x_{n-1}))\|^{\varphi}_p\right]^I_{N_\theta} = \left\{x = (x_{mn}) : \exists K > 0, \left\{(r,s) \in \mathbb{N}\times\mathbb{N} : \frac{1}{\varphi_{rs}}\sum_{m\in J_{rs}}\sum_{n\in J_{rs}}\{u_{mn}[f_{mn}(\|A_{mn}\mu_{mn}(x), (d(x_1), d(x_2), \ldots, d(x_{n-1}))\|_p)]^{q_{mn}} \geqslant K\} \in I\right\}\right\}.$$

If we take $f_{mn}(x) = x$, we get

$$\left[\chi^{2quA}_{\mu}, \|(d(x_1), d(x_2), \ldots, d(x_{n-1}))\|^{\varphi}_p\right]^I_{N_\theta} = \left\{x = (x_{mn}) : \left\{(r,s) \in \mathbb{N}\times\mathbb{N} : \frac{1}{\varphi_{rs}}\sum_{m\in J_{rs}}\sum_{n\in J_{rs}}\{u_{mn}[(\|A_{mn}\mu_{mn}(x), (d(x_1), d(x_2), \ldots, d(x_{n-1}))\|_p)]^{q_{mn}} \geqslant \epsilon\} \in I\right\}\right\},$$

$$[\Lambda^{2quA}_{\mu}, \|(d(x_1), d(x_2), \ldots, d(x_{n-1}))\|^{\varphi}_p]^I_{N_\theta} = \left\{x = (x_{mn}) : \exists K > 0, \left\{(r,s) \in \mathbb{N}\times\mathbb{N} : \frac{1}{\varphi_{rs}}\sum_{m\in J_{rs}}\sum_{n\in J_{rs}}\{u_{mn}[(\|A_{mn}\mu_{mn}(x), (d(x_1), d(x_2), \ldots, d(x_{n-1}))\|_p)]^{q_{mn}} \geqslant K\} \in I\right\}\right\}.$$

If we take $q = (q_{mn}) = 1$

$$\left[\chi^{2uA}_{f\mu}, \|(d(x_1), d(x_2), \ldots, d(x_{n-1}))\|^{\varphi}_p\right]^I_{N_\theta} = \left\{x = (x_{mn}) : \left\{(r,s) \in \mathbb{N}\times\mathbb{N} : \frac{1}{\varphi_{rs}}\sum_{m\in J_{rs}}\sum_{n\in J_{rs}}\left\{[f_{mn}(\|A_{mn}\mu_{mn}(x), (d(x_1), d(x_2), \ldots, d(x_{n-1}))\|_p)] \geqslant \epsilon\right\} \in I\right\}\right\},$$

$$\left[\Lambda^{2uA}_{f\mu}, \|(d(x_1), d(x_2), \ldots, d(x_{n-1}))\|^{\varphi}_p\right]^I_{N_\theta} = \left\{x = (x_{mn}) : \exists K > 0, \left\{(r,s) \in \mathbb{N}\times\mathbb{N} : \frac{1}{\varphi_{rs}}\sum_{m\in J_{rs}}\sum_{n\in J_{rs}}\{u_{mn}[f_{mn}(\|A_{mn}\mu_{mn}(x), (d(x_1), d(x_2), \ldots, d(x_{n-1}))\|_p)] \geqslant K\} \in I\right\}\right\}.$$

Theorem 4.1. *For a Musielak-modulus function, $f=(f_{mn})$. Then the sequence spaces $\left[\chi^{2quA}_{f\mu},\|(d(x_1),d(x_2),\ldots,d(x_{n-1}))\|^{\varphi}_{p}\right]^{I}_{N_\theta}$ and $\left[\Lambda^{2quA}_{f\mu},\|(d(x_1),d(x_2),\ldots,d(x_{n-1}))\|^{\varphi}_{p}\right]^{I}_{N_\theta}$ are linear spaces over the set of complex numbers $\mathbb{C}$.*

Proof. It is routine verification. Therefore the proof is omitted. □

Theorem 4.2. *For any Musielak-modulus function $f=(f_{mn})$ and a double analytic sequence $q=(q_{mn})$ of strictly positive real numbers, the space $\left[\chi^{2quA}_{f\mu},\|(d(x_1),d(x_2),\ldots,d(x_{n-1}))\|^{\varphi}_{p}\right]^{I}_{N_\theta}$ is a topological linear space paranormed by*

$$g(x)=inf\left\{\frac{1}{\varphi_{rs}}\sum_{m\in J_{rs}}\sum_{n\in J_{rs}}(u_{mn}[f_{mn}(\|A_{mn}\mu_{mn}(x),(d(x_1),d(x_2),\ldots,d(x_{n-1}))\|^{\varphi}_{p})]^{q_{mn}})^{1/H}\leqslant 1\right\},$$

where $H=max(1,sup_{mn}q_{mn}<\infty)$.

Proof. Clearly $g(x)\geqslant 0$ for $x=(x_{mn})\in\left[\chi^{2quA}_{f\mu},\|(d(x_1),d(x_2),\ldots,d(x_{n-1}))\|^{V_2}_{p}\right]^{I}_{N_\theta}$. Since $f_{mn}(0)=0$, we get $g(0)=0$.

Conversely, suppose that $g(x)=0$, then

$$inf\left\{\frac{1}{\varphi_{rs}}\sum_{m\in J_{rs}}\sum_{n\in J_{rs}}(u_{mn}[f_{mn}(\|A_{mn}\mu_{mn}(x),(d(x_1),d(x_2),\ldots,d(x_{n-1}))\|^{\varphi}_{p})]^{q_{mn}})^{1/H}\leqslant 1\right\}.$$

Suppose that $A_{mn}\mu_{mn}(x)\neq 0$ for each $m,n\in\mathbb{N}$. Then $\|A_{mn}\mu_{mn}(x),(d(x_1),d(x_2),\ldots,d(x_{n-1}))\|^{\varphi}_{p}\to\infty$.

It follows that

$$\frac{1}{\varphi_{rs}}\sum_{m\in J_{rs}}\sum_{n\in J_{rs}}(u_{mn}[f_{mn}(\|A_{mn}\mu_{mn}(x),(d(x_1),d(x_2),\ldots,d(x_{n-1}))\|^{V_2}_{p})]^{q_{mn}})^{1/H}\to\infty,$$

which is a contradiction. Therefore $A_{mn}\mu_{mn}(x)=0$. Let

$$\frac{1}{\varphi_{rs}}\sum_{m\in J_{rs}}\sum_{n\in J_{rs}}(u_{mn}[f_{mn}(\|A_{mn}\mu_{mn}(x),(d(x_1),d(x_2),\ldots,d(x_{n-1}))\|^{\varphi}_{p})]^{q_{mn}})^{1/H}\leqslant 1$$

and

$$\frac{1}{\varphi_{rs}}\sum_{m\in J_{rs}}\sum_{n\in J_{rs}}(u_{mn}[f_{mn}(\|A_{mn}\mu_{mn}(y),(d(x_1),d(x_2),\ldots,d(x_{n-1}))\|^{\varphi}_{p})]^{q_{mn}})^{1/H}\leqslant 1.$$

Then by using Minkowski's inequality, we have

$$\frac{1}{\varphi_{rs}}\sum_{m\in J_{rs}}\sum_{n\in J_{rs}}(u_{mn}[f_{mn}(\|A_{mn}\mu_{mn}(x+y),(d(x_1),d(x_2),\ldots,d(x_{n-1}))\|^{\varphi}_{p})]^{q_{mn}})^{1/H}$$
$$\leqslant\frac{1}{\varphi_{rs}}\sum_{m\in J_{rs}}\sum_{n\in J_{rs}}(u_{mn}[f_{mn}(\|A_{mn}\mu_{mn}(x),(d(x_1),d(x_2),\ldots,d(x_{n-1}))\|^{\varphi}_{p})]^{q_{mn}})^{1/H}$$
$$+\frac{1}{\varphi_{rs}}\sum_{m\in J_{rs}}\sum_{n\in J_{rs}}(u_{mn}[f_{mn}(\|A_{mn}\mu_{mn}(y),(d(x_1),d(x_2),\ldots,d(x_{n-1}))\|^{\varphi}_{p})]^{q_{mn}})^{1/H}.$$

So we have

$$g(x+y)=inf\left\{\frac{1}{\varphi_{rs}}\sum_{m\in J_{rs}}\sum_{n\in J_{rs}}(u_{mn}[f_{mn}(\|A_{mn}\mu_{mn}(x+y),(d(x_1),d(x_2),\ldots,d(x_{n-1}))\|^{\varphi}_{p})]^{q_{mn}})^{1/H}\leqslant 1\right\}$$
$$\leqslant inf\left\{\frac{1}{\varphi_{rs}}\sum_{m\in J_{rs}}\sum_{n\in J_{rs}}(u_{mn}[f_{mn}(\|A_{mn}\mu_{mn}(x),(d(x_1),d(x_2),\ldots,d(x_{n-1}))\|^{\varphi}_{p})]^{q_{mn}})^{1/H}\leqslant 1\right\}$$
$$+inf\left\{\frac{1}{\varphi_{rs}}\sum_{m\in J_{rs}}\sum_{n\in J_{rs}}(u_{mn}[f_{mn}(\|A_{mn}\mu_{mn}(y),(d(x_1),d(x_2),\ldots,d(x_{n-1}))\|^{\varphi}_{p})]^{q_{mn}})^{1/H}\leqslant 1\right\}.$$

Therefore,

$g(x+y)\leqslant g(x)+g(y)$.

Finally, to prove that the scalar multiplication is continuous. Let λ be any complex number. By definition,

$$g(\lambda x)=inf\left\{\frac{1}{\varphi_{rs}}\sum_{m\in J_{rs}}\sum_{n\in J_{rs}}(u_{mn}[f_{mn}(\|A_{mn}\mu_{mn}(\lambda x),(d(x_1),d(x_2),\ldots,d(x_{n-1}))\|^{\varphi}_{p})]^{q_{mn}})^{1/H}\leqslant 1\right\}.$$

Then

$$g(\lambda\ x)=inf\left\{\frac{1}{\varphi_{rs}}\sum_{m\in J_{rs}}\sum_{n\in J_{rs}}((|\lambda|t)^{q_{mn}/H}:(u_{mn}[f_{mn}(\|A_{mn}\mu_{mn}(x),\left(d(x_1),d(x_2),\ldots,d(x_{n-1}))\|^{\varphi}_{p})\right]^{q_{mn}})^{1/H}\leqslant 1\right\},$$

where $t=\frac{1}{|\lambda|}$. Since $|\lambda|^{q_{mn}}\leqslant max(1,|\lambda|^{supp_{mn}})$, we have

$$g(\lambda\,x)\leqslant max(1,|\lambda|^{supp_{mn}})inf\left\{\frac{1}{\varphi_{rs}}\sum_{m\in J_{rs}}\sum_{n\in J_{rs}}t^{q_{mn}/H}:(u_{mn}[f_{mn}(\|A_{mn}\mu_{mn}(\lambda x),(d(x_1),d(x_2),\ldots,d(x_{n-1}))\|^{\varphi}_{p})]^{q_{mn}})^{1/H}\leqslant 1\right\}.$$

This completes the proof. □

Theorem 4.3. *The β-dual space of*

$$\left[\chi^{2quA}_{f\mu},\|A_{mn}\mu_{mn}(x),(d(x_1),d(x_2),\ldots,d(x_{n-1}))\|^{\varphi}_{p}\right]^{I\beta}_{N_\theta}=\left[\Lambda^{2quA}_{f\mu},\|A_{mn}\mu_{mn}(x),(d(x_1),d(x_2),\ldots,d(x_{n-1}))\|^{\varphi}_{p}\right]^{I}_{N_\theta}.$$

Proof. First, we observe that

$$\left[\chi^{2quA}_{f\mu},\|A_{mn}\mu_{mn}(x),(d(x_1),d(x_2),\ldots,d(x_{n-1}))\|^{\varphi}_{p}\right]^{I\beta}_{N_\theta}\subset\left[\Gamma^{2quA}_{f\mu},\|A_{mn}\mu_{mn}(x),(d(x_1),d(x_2),\ldots,d(x_{n-1}))\|^{\varphi}_{p}\right].$$

Therefore

$$\left[\Gamma^{2quA}_{f\mu},\|A_{mn}\mu_{mn}(x),(d(x_1),d(x_2),\ldots,d(x_{n-1}))\|^{\varphi}_{p}\right]^{I\beta}_{N_\theta}\subset\left[\chi^{2quA}_{f\mu},\|A_{mn}\mu_{mn}(x),(d(x_1),d(x_2),\ldots,d(x_{n-1}))\|^{\varphi}_{p}\right]^{I\beta}_{N_\theta}.$$

But

$$\left[\Gamma^{2quA}_{f\mu}\right]^{\beta}\underset{\neq}{\subset}\left[\Lambda^{2qA}_{f\mu},\|A_{mn}\mu_{mn}(x),(d(x_1),d(x_2),\ldots,d(x_{n-1}))\|^{\varphi}_{p}\right]^{I}_{N_\theta}.$$

Hence

$$\left[\Lambda^{2quA}_{f\mu},\|A_{mn}\mu_{mn}(x),(d(x_1),d(x_2),\ldots,d(x_{n-1}))\|^{\varphi}_{p}\right]^{I}_{N_\theta}\subset\left[\chi^{2quA}_{f\mu},\|A_{mn}\mu_{mn}(x),(d(x_1),d(x_2),\ldots,d(x_{n-1}))\|^{\varphi}_{p}\right]^{I\beta}_{N_\theta}. \quad (4.1)$$

Next we show that

$$\left[\chi^{2quA}_{f\mu},\|A_{mn}\mu_{mn}(x),(d(x_1),d(x_2),\ldots,d(x_{n-1}))\|^{\varphi}_{p}\right]^{I\beta}_{N_\theta}\subset\left[\Lambda^{2quA}_{f\mu},\|A_{mn}\mu_{mn}(x),(d(x_1),d(x_2),\ldots,d(x_{n-1}))\|^{\varphi}_{p}\right]^{I}_{N_\theta}.$$

Let

$$y = (y_{mn}) \in \left[\chi^{2quA}_{f\mu}, \|A_{mn}\mu_{mn}(x),(d(x_1),d(x_2),\ldots,d(x_{n-1}))\|^{\varphi}_{p}\right]^{I\beta}_{N_\theta}.$$

Consider $f(x) = \sum_{m=1}^{\infty}\sum_{n=1}^{\infty} x_{mn}y_{mn}$ with

$$x = (x_{mn}) \in \left[\chi^{2quA}_{f\mu}, \|A_{mn}\mu_{mn}(x),(d(x_1),d(x_2),\ldots,d(x_{n-1}))\|^{\varphi}_{p}\right]^{I}_{N_\theta},$$

$$x = [(\lambda_{mn} - \lambda_{mn+1}) - (\lambda_{m+1n} - \lambda_{m+1n+1})]$$

$$= \begin{pmatrix} 0 & 0 & \cdots 0 & 0 & \cdots & 0 \\ 0 & 0 & \cdots 0 & 0 & \cdots & 0 \\ . & & & & & \\ . & & & & & \\ . & & & & & \\ 0 & 0 & \cdots \frac{\varphi_{rs}}{a^{mn}_{k\ell}\Delta\lambda_{mn}(m+n)!} & \frac{-\varphi_{rs}}{a^{mn}_{k\ell}\Delta\lambda_{mn}(m+n)!} & \cdots & 0 \\ 0 & 0 & \cdots 0 & 0 & \cdots & 0 \end{pmatrix}$$

$$- \begin{pmatrix} 0 & 0 & \cdots 0 & 0 & \cdots & 0 \\ 0 & 0 & \cdots 0 & 0 & \cdots & 0 \\ . & & & & & \\ . & & & & & \\ . & & & & & \\ 0 & 0 & \cdots \frac{\varphi_{rs}}{a^{mn}_{k\ell}\Delta\lambda_{mn}(m+n)!} & \frac{-\varphi_{rs}}{a^{mn}_{k\ell}\Delta\lambda_{mn}(m+n)!} & \cdots & 0 \\ 0 & 0 & \cdots 0 & 0 & \cdots & 0 \end{pmatrix},$$

$$u_{mn}[f_{mn}(\|A_{mn}\mu_{mn}(x),(d(x_1),d(x_2),\ldots,d(x_{n-1}))\|^{\varphi}_{p})]$$

$$= \begin{pmatrix} 0 & 0 & \cdots 0 & 0 & \cdots & 0 \\ 0 & 0 & \cdots 0 & 0, & \cdots & 0 \\ . & & & & & \\ . & & & & & \\ . & & & & & \\ 0 & 0 & \cdots u_{mn}f_{mn}\left(\frac{\varphi_{rs}}{a^{mn}_{k\ell}\Delta\lambda_{mn}(m+n)!}\right) & u_{mn}f_{mn}\left(\frac{-\varphi_{rs}}{a^{mn}_{k\ell}\Delta\lambda_{mn}(m+n)!}\right) & \cdots & 0 \\ 0 & 0 & \cdots u_{mn}f_{mn}\left(\frac{-\varphi_{rs}}{a^{mn}_{k\ell}\Delta\lambda_{mn}(m+n)!}\right) & u_{mn}f_{mn}\left(\frac{\varphi_{rs}}{a^{mn}_{k\ell}\Delta\lambda_{mn}(m+n)!}\right) & \cdots & 0 \\ 0 & 0 & \cdots 0 & 0, & \cdots & 0 \end{pmatrix}.$$

Hence converges to zero.

Therefore

$$[(\lambda_{mn} - \lambda_{mn+1}) - (\lambda_{m+1n} - \lambda_{m+1n+1})] \in \left[\chi^{2quA}_{f\mu}, \|A_{mn}\mu_{mn}(x),(d(x_1),d(x_2),\ldots,d(x_{n-1}))\|^{\varphi}_{p}\right]^{I}_{N_\theta}.$$

Hence $d(a^{mn}_{k\ell}(\lambda_{mn} - \lambda_{mn+1}) - (\lambda_{m+1n} - \lambda_{m+1n+1}), 0) = 1$. But $|y_{mn}| \leqslant \|f\| d(a^{mn}_{k\ell}(\lambda_{mn} - \lambda_{mn+1}) - (\lambda_{m+1n} - \lambda_{m+1n+1}), 0) \leqslant \|f\| \cdot 1 < \infty$ for each m, n. Thus (y_{mn}) is a double analytic sequence and hence an p-metric Musielak modulus function of double analytic sequence.

In other words $y \in \left[\Lambda^{2quA}_{f\mu}, \|\mu_{mn}(x),(d(x_1),d(x_2),\ldots, d(x_{n-1}))\|^{\varphi}_{p}\right]^{I}_{N_\theta}$. But $y = (y_{mn})$ is arbitrary in $\left[\chi^{2qu}_{f\mu}, \|A_{mn}\mu_{mn}(x), (d(x_1),d(x_2),\ldots,d(x_{n-1}))\|^{\varphi}_{p}\right]^{I\beta}_{N_\theta}$. Therefore

$$\left[\chi^{2quA}_{f\mu}, \|A_{mn}\mu_{mn}(x),(d(x_1),d(x_2),\ldots,d(x_{n-1}))\|^{\varphi}_{p}\right]^{I\beta}_{N_\theta} \subset \left[\Lambda^{2quA}_{f\mu}, \|A_{mn}\mu_{mn}(x),(d(x_1),d(x_2),\ldots,d(x_{n-1}))\|^{\varphi}_{p}\right]^{I}_{N_\theta}. \quad (4.2)$$

From (4.1) and (4.2) we get

$$\left[\chi^{2quA}_{f\mu}, \|A_{mn}\mu_{mn}(x),(d(x_1),d(x_2),\ldots,d(x_{n-1}))\|^{\varphi}_{p}\right]^{I\beta}_{N_\theta} = \left[\Lambda^{2quA}_{f\mu}, \|A_{mn}\mu_{mn}(x),(d(x_1),d(x_2),\ldots,d(x_{n-1}))\|^{\varphi}_{p}\right]^{I}_{N_\theta}.$$

□

Theorem 4.4. *The dual space of* $\left[\chi^{2quA}_{f\mu}, \|A_{mn}\mu_{mn}(x),(d(x_1), d(x_2),\ldots,d(x_{n-1}))\|^{\varphi}_{p}\right]^{I}_{N_\theta}$ *is* $\left[\Lambda^{2quA}_{f\mu}, \|A_{mn}\mu_{mn}(x),(d(x_1),d(x_2), \ldots,d(x_{n-1}))\|^{\varphi}_{p}\right]^{I}_{N_\theta}$. *In other words*

$$\left[\chi^{2quA}_{f\mu}, \|A_{mn}\mu_{mn}(x),(d(x_1),d(x_2),\ldots,d(x_{n-1}))\|^{\varphi}_{p}\right]^{I*} = \left[\Lambda^{2quA}_{f\mu}, \|A_{mn}\mu_{mn}(x),(d(x_1),d(x_2),\ldots,d(x_{n-1}))\|^{\varphi}_{p}\right]^{I}_{N_\theta}.$$

Proof. We recall that

$$\lambda_{mn} = \begin{pmatrix} 0 & 0 & \cdots 0 & 0 & \cdots \\ 0 & 0 & \cdots 0 & 0 & \cdots \\ . & & & & \\ . & & & & \\ . & & & & \\ 0 & 0 & \cdots \frac{\varphi_{rs}}{a^{mn}_{k\ell}\Delta\lambda_{mn}(m+n)!} & 0 & \cdots \\ 0 & 0 & \cdots 0 & 0 & \cdots \end{pmatrix}$$

with $\frac{\varphi_{rs}}{a^{mn}_{k\ell}\Delta\lambda_{mn}(m+n)!}$ in the $(m,n)th$ position and zero's elsewhere,

$$\left[\chi^{2quA}_{f\mu}, \|A_{mn}\mu_{mn}(x),(d(x_1),d(x_2),\ldots,d(x_{n-1}))\|^{\varphi}_{p}\right]^{I}_{N_\theta}$$

$$= \begin{pmatrix} 0. & . & . & 0 \\ . & & & \\ . & & & \\ . & & & \\ 0 & uf\left(\frac{\varphi_{rs}}{a^{mn}_{k\ell}\Delta\lambda_{mn}(m+n)!}\right)^{1/m+n} & . & 0 \\ & (m,n)^{th} & & \\ 0 & . & . & 0 \end{pmatrix},$$

which is a p-metric of double gai sequence. Hence,

$$x \in \left[\chi^{2quA}_{f\mu}, \|A_{mn}\mu_{mn}(x),(d(x_1),d(x_2),\ldots,d(x_{n-1}))\|^{\varphi}_{p}\right]^{I}_{N_\theta}.$$

$$f(x) = \sum_{m,n=1}^{\infty} x_{mn}y_{mn}$$

with

$$x \in \left[\chi^{2quA}_{f\mu}, \|A_{mn}\mu_{mn}(x),(d(x_1),d(x_2),\ldots,d(x_{n-1}))\|^{\varphi}_{p}\right]^{I}_{N_\theta}$$

and

$$f \in \left[\chi^{2quA}_{f\mu}, \|A_{mn}\mu_{mn}(x),(d(x_1),d(x_2),\ldots,d(x_{n-1}))\|^{\varphi}_{p}\right]^{I*}_{N_\theta},$$

where

$$\left[\chi^{2quA}_{f\mu}, \|A_{mn}\mu_{mn}(x),(d(x_1),d(x_2),\ldots,d(x_{n-1}))\|^{\varphi}_{p}\right]^{I*}_{N_\theta}$$

is the dual space of

$$\left[\chi^{2quA}_{f\mu}, \|A_{mn}\mu_{mn}(x),(d(x_1),d(x_2),\ldots,d(x_{n-1}))\|^{\varphi}_{p}\right]^{I}_{N_\theta}.$$

Take $x = (x_{mn}) \in [\chi_{f\mu}^{2quA}, \|A_{mn}\mu_{mn}(x), (d(x_1), d(x_2), \ldots, d(x_{n-1}))\|_p^{\varphi}]_{N_\theta}^I$. Then,

$$|y_{mn}| \leqslant \|f\| d(\varphi_{rs}, 0) < \infty \ \forall m, n. \tag{4.3}$$

Thus, (y_{mn}) is a double analytic sequence and hence an p-metric Musielak modulus function of double analytic sequence. In other words,

$$y \in \left[\Lambda_{f\mu}^{2quA}, \|A_{mn}\mu_{mn}(x), (d(x_1), d(x_2), \ldots, d(x_{n-1}))\|_p^{\varphi}\right]_{N_\theta}^I.$$

Therefore

$$\left[\chi_{f\mu}^{2quA}, \|A_{mn}\mu_{mn}(x), (d(x_1), d(x_2), \ldots, d(x_{n-1}))\|_p^{\varphi}\right]_{N_\theta}^{I*}$$
$$= \left[\Lambda_{f\mu}^{2qu}, \|A_{mn}\mu_{mn}(x), (d(x_1), d(x_2), \ldots, d(x_{n-1}))\|_p^{\varphi}\right]_{N_\theta}^I.$$

This completes the proof. □

5. Competing interests

Authors have declared that no competing interests exist.

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References

[1] T.J.I'A. Bromwich, An introduction to the theory of infinite series, Macmillan and Co.Ltd., New York, 1965.

[2] G.H. Hardy, On the convergence of certain multiple series, Proc. Camb. Phil. Soc. 19 (1917) 86–95.

[3] F. Moricz, Extentions of the spaces c and c_0 from single to double sequences, Acta Math. Hung. 57 (1–2) (1991) 129–136.

[4] F. Moricz, B.E. Rhoades, Almost convergence of double sequences and strong regularity of summability matrices, Math. Proc. Camb. Phil. Soc. 104 (1988) 283–294.

[5] M. Basarir, O. Solancan, On some double sequence spaces, J. Indian Acad. Math. 21 (2) (1999) 193–200.

[6] B.C. Tripathy, On statistically convergent double sequences, Tamkang J. Math. 34 (3) (2003) 231–237.

[7] A. Turkmenoglu, Matrix transformation between some classes of double sequences, J. Inst. Math. Comp. Sci. Math. Ser. 12 (1) (1999) 23–31.

[8] A. Gökhan, R. Çolak, The double sequence spaces $c_2^P(p)$ and $c_2^P B(p)$, Appl. Math. Comput. 157 (2) (2004) 491–501.

[9] A. Gökhan, R. Çolak, Double sequence spaces ℓ_2^∞, Appl. Math. Comput. 160 (1) (2005) 147–153.

[10] M. Zeltser, Investigation of Double Sequence Spaces by Soft and Hard Analitical Methods, Dissertationes Mathematicae Universitatis Tartuensis 25, Tartu University Press, Univ. of Tartu, Faculty of Mathematics and Computer Science, Tartu, 2001.

[11] M. Mursaleen, O.H.H. Edely, Statistical convergence of double sequences, J. Math. Anal. Appl. 288 (1) (2003) 223–231.

[12] B. Altay, F. Baaar, Some new spaces of double sequences, J. Math. Anal. Appl. 309 (1) (2005) 70–90.

[13] F. BaÇsar, Y. Sever, The space $\mathcal{L}_p$ of double sequences, Math. J. Okayama Univ. 51 (2009) 149–157.

[14] N. Subramanian, U.K. Misra, The semi normed space defined by a double gai sequence of modulus function, Fasciculi Math. 46 (2010).

[15] I.J. Maddox, Sequence spaces defined by a modulus, Math. Proc. Cambridge Philos. Soc. 100 (1) (1986) 161–166.

[16] J. Cannor, On strong matrix summability with respect to a modulus and statistical convergence, Canad. Math. Bull. 32 (2) (1989) 194–198.

[17] A. Pringsheim, Zurtheorie derzweifach unendlichen zahlenfolgen, Math. Ann. 53 (1900) 289–321.

[18] H.J. Hamilton, Transformations of multiple sequences, Duke Math. J. 2 (1936) 29–60.

[19] H.J. Hamilton, A generalization of multiple sequences transformation, Duke Math. J. 4 (1938) 343–358.

[20] H.J. Hamilton, Preservation of partial limits in multiple sequence transformations, Duke Math. J. 4 (1939) 293–297.

[21] P.K. Kamthan, M. Gupta, Sequence Spaces and Series, Lecture Notes, Pure and Applied Mathematics, vol. 65, Marcel Dekker, Inc., New York, 1981.

[22] J. Lindenstrauss, L. Tzafriri, On Orlicz sequence spaces, Israel J. Math. 10 (1971) 379–390.

[23] A. Wilansky, Summability through Functional Analysis, North-Holland Mathematical Studies, vol. 85, North-Holland Publishing, Amsterdam, 1984.

[24] P. Kostyrko, T. Salat, W. Wilczynski, I-convergence, Real Anal. Exchange 26 (2) (2000,2001) 669–686, MR 2002e:54002.

[25] V. Kumar, K. Kumar, On the ideal convergence of sequences of fuzzy numbers, Inform. Sci. 178 (24) (2008) 4670–4678.

[26] V. Kumar, On I and I^*-convergence of double sequences, Math. Commun. 12 (2007) 171–181.

[27] B. Hazarika, On Fuzzy Real Valued I-Convergent Double Sequence Spaces, J. Nonlinear Sci. Its Appl., in press.

[28] B. Hazarika, On fuzzy real valued I-convergent double sequence spaces defined by Musielak-Orlicz function, J. Intell. Fuzzy Syst. 25 (1) (2013) 9–15, http://dx.doi.org/10.3233/IFS-2012-0609.

[29] B. Hazarika, Lacunary difference ideal convergent sequence spaces of fuzzy numbers, J. Intell. Fuzzy Syst. 25 (1) (2013) 157–166, http://dx.doi.org/10.3233/IFS-2012-0622.

[30] B. Hazarika, On σ-uniform density and ideal convergent sequences of fuzzy real numbers, J. Intell. Fuzzy Systems, http://dx.doi.org/10.3233/IFS-130769.

[31] B. Hazarika, Fuzzy real valued lacunary I-convergent sequences, Appl. Math. Lett. 25 (3) (2012) 466–470.

[32] B. Hazarika, Lacunary I-convergent sequence of fuzzy real numbers, Pacific J. Sci. Technol. 10 (2) (2009) 203–206.

[33] B. Hazarika, On generalized difference ideal convergence in random 2-normed spaces, Filomat 26 (6) (2012) 1265–1274.

[34] B. Hazarika, Some classes of ideal convergent difference sequence spaces of fuzzy numbers defined by Orlicz function, Fasciculi Math. 52 (2014) (accepted for publication).

[35] B. Hazarika, I-convergence and summability in topological group, J. Inform. Math. Sci. 4 (3) (2012) 269–283.

[36] B. Hazarika, Classes of generalized difference ideal convergent sequence of fuzzy numbers, Ann. Fuzzy Math. Inform. 7 (1) (2014) 155–172.

[37] B. Hazarika, On ideal convergent sequences in fuzzy normed linear spaces, Afrika Matematika, http://dx.doi.org/10.1007/s13370-013-0168-0.

[38] B. Hazarika, E. Savas, Some I-convergent lambda-summable difference sequence spaces of fuzzy real numbers defined by a sequence of Orlicz functions, Math. Comput. Modell. 54 (11–12) (2011) 2986–2998.

[39] B. Hazarika, K. Tamang, B.K. Singh, Zweier Ideal Convergent Sequence Spaces Defined by Orlicz Function, The J. Math. Comput. Sci. 8 (3) (2014) 309–318.

[40] B. Hazarika and V. Kumar, Fuzzy real valued I-convergent double sequences in fuzzy normed spaces, J. Intell. Fuzzy Syst., in press, http://dx.doi.org/10.3233/IFS-13905.

[41] B.C. Tripathy, B. Hazarika, *I*-convergent sequence spaces associated with multiplier sequences, Math. Ineq. Appl. 11 (3) (2008) 543–548.
[42] B.C. Tripathy, B. Hazarika, Paranorm *I*-convergent sequence spaces, Math. Slovaca 59 (4) (2009) 485–494.
[43] B.C. Tripathy, B. Hazarika, Some *I*-convergent sequence spaces defined by Orlicz functions, Acta Math. Appl. Sinica 27 (1) (2011) 149–154.