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Results on *n*-tupled fixed points in complete asymptotically regular metric spaces

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Abstract The notion of n-tupled fixed point is introduced by Imdad, Soliman, Choudhury and Das, Jour. of Operators, Vol. 2013, Article ID 532867. In this manuscript, we prove some n-tupled fixed point theorems (for even n) for mappings having mixed monotone property in partially ordered complete asymptotically regular metric spaces. Our main theorem improves the corresponding results of Imdad, Sharma and Rao (M. Imdad, A. Sharma, K.P.R. Rao, Generalized n-tupled fixed point theorems for nonlinear contractions, preprint).

2010 MATHEMATICAL SUBJECT CLASSIFICATION: 47H10; 54H10; 54H25

1. Introduction and preliminaries

The Banach contraction principle is the most natural and significant result of fixed point theory. It has become one of the most fundamental and powerful tools of nonlinear analysis because of its wide range of applications to nonlinear equations arising of in physical and biological processes ensuring the existence and uniqueness of solutions. It is widely considered as the source of metric fixed point theory. Also, its significance lies in its vast applicability in a number of branches of mathematics. Generalization of the above principle has been a heavily branch of mathematics. Existence of a fixed point for contraction type mappings in partially ordered metric space and applications have been considered by many authors. There already exists an extensive literature on this topic, but keeping in view the relevance of this paper, we merely refer to [1–13,17–31].

In [6], Bhaskar and Lakshmikantham introduced the notion of a coupled fixed point and proved some coupled fixed point theorems in partially ordered complete metric spaces under certain conditions. Afterwards Lakshmikantham and Ćirić [17] extended these results by defining the g-monotone property, which indeed generalize the corresponding fixed point theorems contained in [6]. Since then Berinde and Borcut [8] introduced the concept of tripled fixed point and proved some related theorems.

Recently Imdad et al. [14] introduced the concept of n-tupled coincidence as well as n-tupled fixed point (for even n) and utilize these two definitions to obtain n-tupled coincidence

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as well as n-tupled common fixed point theorems for nonlinear ϕ-contraction mappings. For more details see [9,11,23,25]. Very recently, Soliman et al. [31] proved some n-tupled coincident point theorems for nonlinear ϕ-contraction mappings in partially ordered complete asymptotically regular metric spaces.

The purpose of this paper is to present some n-tupled coincidence and fixed point results for nonlinear contraction mappings in partially ordered complete asymptotically regular metric spaces. Our results generalize and improve the results of Imdad et al. [15]. As usual, this section is devoted to preliminaries which include some basic definitions and results related to coupled fixed point in metric spaces.

From now, $(X, \preceq, d)$ is a partially ordered complete metric space. Further, the product space $X \times X$ has the following partial order:

$$(u, v) \preceq (x, y) \iff x \succeq u, y \preceq v \quad \forall (x, y), (u, v) \in X \times X.$$

We summarize in the following the basic notions and results established in [6,8].

Definition 1 [6]. Let $(X, \preceq)$ be a partially ordered set and $F: X \times X \to X$ be a mapping. Then F is said to have mixed monotone property if for any $x, y \in X$, $F(x, y)$ is monotonically nondecreasing in first argument and monotonically nonincreasing in second argument, that is, for

$$x_1, x_2 \in X, x_1 \preceq x_2 \Rightarrow F(x_1, y) \preceq F(x_2, y)$$
$$y_1, y_2 \in X, y_1 \preceq y_2 \Rightarrow F(x, y_1) \succeq F(x, y_2).$$

Definition 2 [6]. An element $(x, y) \in X \times X$ is called a coupled fixed point of the mapping $F: X \times X \to X$ if

$$F(x, y) = x \text{ and } F(y, x) = y.$$

The main theoretical results of Bhaskar and Lakshmikantham [6] are the following two coupled fixed point theorems.

Theorem 1. *Let $(X, \preceq)$ be a partially ordered set and suppose there is a metric d on X such that (X, d) is a complete metric space. Let $F: X \times X \to X$ be a continuous mapping having the mixed monotone property on X. Assume that there exists $k \in [0, 1)$ with*

$$d(F(x, y), F(u, v)) \leqslant \frac{k}{2}[d(x, u) + d(y, v)] \text{ for each } x \succeq u \text{ and } y \preceq v.$$

If there exist x_0, $y_0 \in X$ such that $x_0 \preceq F(x_0, y_0)$ and $y_0 \succeq F(y_0, x_0)$, then there exist x, $y \in X$ such that $F(x, y) = x$ and $F(y, x) = y$.

Theorem 2. *Let $(X, \preceq)$ be a partially ordered set and suppose there is a metric d on X such that (X, d) is a complete metric space. Assume that X has the following property:*

(i) *if nondecreasing sequence $\{x_n\} \to x$, then $x_n \preceq x$ for all n;*
(ii) *if nonincreasing sequence $\{y_n\} \to y$, then $y_n \succeq y$ for all n.* *Let $F: X \times X \to X$ be a mapping having the mixed monotone property on X. Assume that there exists $k \in [0, 1)$ with*

$$d(F(x, y), F(u, v)) \leqslant \frac{k}{2}[d(x, u) + d(y, v)] \text{ for each } x \succeq u \text{ and } y \preceq v.$$

If there exist x_0, $y_0 \in X$ such that $x_0 \preceq F(x_0, y_0)$ and $y_0 \succeq F(y_0, x_0)$, then there exist x, $y \in X$ such that $F(x, y) = x$ and $F(y, x) = y$.

Recently, Berinde and Borcut [8] introduced the following partial order on the product space $X \times X \times X$:

$$(u, v, w) \preceq (x, y, z) \iff x \succeq u, y \preceq v, z \succeq w$$
$$\forall (x, y, z), (u, v, w) \in X \times X \times X.$$

Definition 3 [8]. Let $(X, \preceq)$ be a partially ordered set and $F: X \times X \times X \to X$ be a mapping. Then F is said to have mixed monotone property if F is monotone nodecreasing in first and third argument and monotone noincreasing in second argument, that is, for any x, y, $z \in X$

$$x_1, x_2 \in X, x_1 \preceq x_2 \Rightarrow F(x_1, y, z) \preceq F(x_2, y, z)$$
$$y_1, y_2 \in X, y_1 \preceq y_2 \Rightarrow F(x, y_1, z) \succeq F(x, y_2, z)$$
$$z_1, z_2 \in X, z_1 \preceq z_2 \Rightarrow F(x, y, z_1) \preceq F(x, y, z_2).$$

Definition 4 [8]. An element $(x, y, z) \in X \times X \times X$ is called a tripled fixed point of the mapping $F: X \times X \times X \to X$ if $F(x, y, z) = x$, $F(y, x, y) = y$ and $F(z, y, x) = z$.

Inspired by the results on coupled fixed points, Karapinar [16] introduced the notion of quadrupled fixed point and proved some related fixed point theorems in partially ordered metric spaces.

Definition 5 [16]. An element $(x, y, z, w) \in X \times X \times X \times X$ is called a quadrupled fixed point of the mapping F: $X \times X \times X \times X \to X$ if $F(x, y, z, w) = x$, $F(y, z, w, x) = y$, $F(z, w, x, y) = z$ and $F(w, x, y, z) = w$.

The following concept of an n-fixed point was introduced by Gordji and Ramezani [12]. We suppose that the product space $X^n = X \times X \times \cdots \times X$($n$ times) is endowed with the following partial order, where n is the positive integer (odd or even): $(x^1, x^2, \ldots, x^n)$, $(y^1, y^2, \ldots, y^n) \in X^n$

$$(x^1, x^2, \ldots, x^n) \preceq (y^1, y^2, \ldots, y^n) \iff x^{2i-1} \preceq y^{2i-1} \ \forall i \in \left\{1, 2, \ldots, \left[\frac{n+1}{2}\right]\right\}$$
$$(x^1, x^2, \ldots, x^n) \preceq (y^1, y^2, \ldots, y^n) \iff x^{2i} \succeq y^{2i} \ \forall i \in \left\{1, 2, \ldots, \left[\frac{n}{2}\right]\right\}.$$

Definition 6 [12]. An element $(x^1, x^2, \ldots, x^n) \in X^n$ is called an n fixed point of the mapping $F: X^n \to X$ if

$$x^i = F(x^i, x^{i-1}, \ldots, x^2, x^1, x^2, \ldots, x^{n-i+1}) \ \forall i \in \{1, 2, \ldots, n\}.$$

Remark 1. The concept of n-tupled fixed point is given by Imdad et al. [14], which is quite different from the concept of Gordji and Ramezani [12]. A detailed version on n-tupled fixed point is given in next section.

Very recently, Soliman et al. [31] proved results on n-tupled coincidence point in complete asymptotically regular metric space and called this space as generalized complete metric space.

Some more definitions related to our script are as follows:

Definition 7. Let (X, d) be a metric space. Then a sequence $\{x_n\}$ in X is said to be Cauchy if $\lim_{n,m\to\infty} d(x_n, x_m) = 0$ for all $n, m \in \mathbb{N}$.

A metric space (X, d) is said to be complete if every Cauchy sequence converges.

Definition 8. A sequence $\{x_n\}$ in a metric space (X, d) is said to be asymptotically regular if $\lim_{n\to\infty} d(x_{n+1}, x_n) = 0$.

Every Cauchy sequence is asymptotically regular but the converse may not be true in metric spaces.

Example 1. Let $X = \mathbb{R}$ and $d(x, y) = |x - y|$ for all $x, y \in X$. Then $\{x_n\}$ in X defined by $x_n = \sum_{k=1}^{n} \frac{1}{k}$ is asymptotically regular but not Cauchy.

Definition 9. A metric space is called complete asymptotically regular if every asymptotically regular sequence $\{x_n\}$ in X converges to some point in X.

Let Φ denote the set of all functions $\phi:[0,\infty) \to [0,\infty)$ which satisfy that $\lim_{t\to r}\phi(t) > 0$ for all $r > 0$ and $\lim_{t\to 0^+}\phi(t) = 0$. Let Ψ denote the set of all functions $\psi:[0,\infty) \to [0,\infty)$ which satisfy

(i) $\psi(t) = 0$ if and only if $t = 0$,
(ii) ψ is continuous and nondecreasing,
(iii) $\psi(s + t) \leqslant \psi(s) + \psi(t)$ for all $s, t \in [0,\infty)$.
Examples of typical functions ϕ and ψ are given in [19].

2. Main results

In this paper we used the new definitions of n-tupled fixed point and mixed monotone property given by Imdad et al. [14]. Throughout the paper n stands for a general even natural number. In the sequel we have the following definitions:

Definition 10 [14]. Let $(X, \preceq)$ be a partially ordered set and $F: X^n \to X$ be a mapping. The mapping F is said to have the mixed monotone property if F is nondecreasing in its odd position arguments and nonincreasing in its even position arguments, that is,

$$\begin{cases} \text{for all } x_1^1, x_2^1 \in X, x_1^1 \preceq x_2^1 \Rightarrow F(x_1^1, x^2, x^3, \ldots, x^n) \preceq F(x_2^1, x^2, x^3, \ldots, x^n) \\ \text{for all } x_1^2, x_2^2 \in X, x_1^2 \preceq x_2^2 \Rightarrow F(x^1, x_1^2, x^3, \ldots, x^n) \succeq F(x^1, x_2^2, x^3, \ldots, x^n) \\ \text{for all } x_1^3, x_2^3 \in X, x_1^3 \preceq x_2^3 \Rightarrow F(x^1, x^2, x_1^3, \ldots, x^n) \preceq F(x^1, x^2, x_2^3, \ldots, x^n) \\ \vdots \\ \text{for all } x_1^n, x_2^n \in X, x_1^n \preceq x_2^n \Rightarrow F(x^1, x^2, x^3, \ldots, x_1^n) \succeq F(x^1, x^2, x^3, \ldots, x_2^n). \end{cases} \tag{2.1}$$

Definition 11 [14]. An element $(x^1, x^2, \ldots, x^n) \in X^n$ is called an n-tupled fixed point of the mapping $F:X^n \to X$ if

$$\begin{cases} F(x^1, x^2, x^3, \ldots, x^n) = x^1 \\ F(x^2, x^3, \ldots, x^n, x^1) = x^2 \\ F(x^3, \ldots, x^n, x^1, x^2) = x^3 \\ \vdots \\ F(x^n, x^1, x^2, \ldots, x^{n-1}) = x^n. \end{cases} \tag{2.2}$$

Now we are equipped to prove our main results as fo

Theorem 3. *Let $(X, \preceq)$ be a partially ordered set and (X, d) be a complete asymptotically regular metric space. Let $F:X^n \to X$ be a mapping such that F has the mixed monotone property on X. Assume that for all $x^1, x^2, \ldots, x^n, y^1, y^2, \ldots, y^n \in X$ for which $x^1 \succeq y^1$, $x^2 \preceq y^2$, $x^3 \succeq y^3, \ldots, x^n \preceq y^n$,*

$$\psi(d(F(x^1, x^2, \ldots, x^n), F(y^1, y^2, \ldots, y^n))) \leqslant \frac{1}{n}\psi(d(x^1, y^1) + d(x^2, y^2) + \ldots + d(x^n, y^n)) - \phi(d(x^1, y^1) + d(x^2, y^2) + \cdots + d(x^n, y^n)), \tag{2.3}$$

where $\phi \in \Phi$ and $\psi \in \Psi$. Suppose that there exist $x_0^1, x_0^2, x_0^3, \ldots, x_0^n \in X$ such that

$$\begin{cases} x_0^1 \preceq F(x_0^1, x_0^2, x_0^3, \ldots, x_0^n) \\ x_0^2 \succeq F(x_0^2, x_0^3, \ldots, x_0^n, x_0^1) \\ x_0^3 \preceq F(x_0^3, \ldots, x_0^n, x_0^1, x_0^2) \\ \vdots \\ x_0^n \succeq F(x_0^n, x_0^1, x_0^2, \ldots, x_0^{n-1}). \end{cases}$$

Suppose that either

(a) *F is continuous or*
(b) *X has the following property:*
 (i) *if nondecreasing sequence $\{x_m\} \to x$, then $x_m \preceq x$ for all $m \geqslant 0$;*
 (ii) *if nonincreasing sequence $\{x_m\} \to x$, then $x_m \succeq x$ for all $m \geqslant 0$,*
 then there exist $x^1, x^2, \ldots, x^n \in X$ such that

$$F(x^1, x^2, \ldots, x^n) = x^1, F(x^2, \ldots, x^n, x^1) = x^2, \ldots, F(x^n, x^1, \ldots, x^{n-1}) = x^n.$$

Proof. Let $x_0^1, x_0^2, x_0^3, \ldots, x_0^n$ in X such that

$$\begin{cases} x_0^1 \preceq F(x_0^1, x_0^2, x_0^3, \ldots, x_0^n) \\ x_0^2 \succeq F(x_0^2, x_0^3, \ldots, x_0^n, x_0^1) \\ x_0^3 \preceq F(x_0^3, \ldots, x_0^n, x_0^1, x_0^2) \\ \vdots \\ x_0^n \succeq F(x_0^n, x_0^1, x_0^2, \ldots, x_0^{n-1}). \end{cases}$$

We construct sequences $\{x_m^1\}, \{x_m^2\}, \{x_m^3\}, \ldots, \{x_m^n\}$ in X as follows:

$$\begin{cases} x_m^1 = F(x_{m-1}^1, x_{m-1}^2, x_{m-1}^3, \ldots, x_{m-1}^n) \\ x_m^2 = F(x_{m-1}^2, x_{m-1}^3, \ldots, x_{m-1}^n, x_{m-1}^1) \\ x_m^3 = F(x_{m-1}^3, \ldots, x_{m-1}^n, x_{m-1}^1, x_{m-1}^2) \\ \vdots \\ x_m^n = F(x_{m-1}^n, x_{m-1}^1, x_{m-1}^2, \ldots, x_{m-1}^{n-1}) \text{ for } m \geqslant 1. \end{cases} \tag{2.4}$$

By the mixed monotone property of F, it is easy to show that

$$\begin{cases} x_0^1 \preceq x_1^1 \preceq x_2^1 \preceq \cdots \preceq x_{m-1}^1 \preceq x_m^1 \preceq \cdots \\ x_0^2 \succeq x_1^2 \succeq x_2^2 \succeq \cdots \succeq x_{m-1}^2 \succeq x_m^2 \succeq \cdots \\ x_0^3 \preceq x_1^3 \preceq x_2^3 \preceq \cdots \preceq x_{m-1}^3 \preceq x_m^3 \preceq \cdots \\ \vdots \\ x_0^n \succeq x_1^n \succeq x_2^n \succeq \cdots \succeq x_{m-1}^n \succeq x_m^n \succeq \cdots. \end{cases} \tag{2.5}$$

Due to (2.3)–(2.5), we have

$$\begin{aligned}\psi(d(x^1_{m+1},x^1_{m+2})) &= \psi(d(F(x^1_m,x^2_m,\ldots,x^n_m),F(x^1_{m+1},x^2_{m+1},\ldots,x^n_{m+1})))\\ &\leqslant \frac{1}{n}\psi(d(x^1_m,x^1_{m+1})+d(x^2_m,x^2_{m+1})+\cdots+d(x^n_m,x^n_{m+1}))\\ &\quad -\phi(d(x^1_m,x^1_{m+1})+d(x^2_m,x^2_{m+1})+\cdots+d(x^n_m,x^n_{m+1})),\end{aligned}$$

$$\begin{aligned}\psi(d(x^2_{m+1},x^2_{m+2})) &= \psi(d(F(x^2_m,\ldots,x^n_m,x^1_m),F(x^2_{m+1},\ldots,x^n_{m+1},x^1_{m+1})))\\ &\leqslant \frac{1}{n}\psi(d(x^2_m,x^2_{m+1})+\cdots+d(x^n_m,x^n_{m+1})+d(x^1_m,x^1_{m+1}))\\ &\quad -\phi(d(x^2_m,x^2_{m+1})+\cdots+d(x^n_m,x^n_{m+1})+d(x^1_m,x^1_{m+1})),\end{aligned}$$

$$\vdots$$

$$\begin{aligned}\psi(d(x^n_{m+1},x^n_{m+2})) &= \psi(d(F(x^n_m,x^1_m,\ldots,x^{n-1}_m),F(x^n_{m+1},x^1_{m+1},\ldots,x^{n-1}_{m+1})))\\ &\leqslant \frac{1}{n}\psi(d(x^n_m,x^n_{m+1})+d(x^1_m,x^1_{m+1})+\cdots+d(x^{n-1}_m,x^{n-1}_{m+1}))\\ &\quad -\phi(d(x^n_m,x^n_{m+1})+d(x^1_m+x^1_{m+1})+\cdots+d(x^{n-1}_m,x^{n-1}_{m+1})).\end{aligned}$$

Due to above inequalities, we conclude that

$$\begin{aligned}&\psi(d(x^1_{m+1},x^1_{m+2}))+\psi(d(x^2_{m+1},x^2_{m+2}))+\cdots+\psi(d(x^n_{m+1},x^n_{m+2}))\\ &\leqslant \psi(d(xm^1,x^1_{m+1})+d(xm^2,x^2_{m+1})+\cdots+d(xm^n,x^n_{m+1}))\\ &\quad -n\phi(d(xm^1,x^1_{m+1})+d(xm^2,x^2_{m+1})+\cdots+d(xm^n,x^n_{m+1})).\end{aligned} \tag{2.6}$$

From property (iii) of ψ, we have

$$\begin{aligned}&\psi(d(x^1_{m+1},x^1_{m+2})+d(x^2_{m+1},x^2_{m+2})+\cdots+d(x^n_{m+1},x^n_{m+2}))\\ &\leqslant \psi(d(x^1_{m+1},x^1_{m+2}))+\psi(d(x^2_{m+1},x^2_{m+2}))+\cdots\\ &\quad +\psi(d(x^n_{m+1},x^n_{m+2})).\end{aligned} \tag{2.7}$$

Combining (2.6) and (2.7), we get that

$$\begin{aligned}&\psi(d(x^1_{m+1},x^1_{m+2}))+d(x^2_{m+1},x^2_{m+2})+\ldots+d(x^n_{m+1},x^n_{m+2})\\ &\leqslant \psi(d(x^1_m,x^1_{m+1})+d(x^2_m,x^2_{m+1})+\cdots+d(x^n_m,x^n_{m+1}))\\ &\quad -n\phi(d(x^1_m,x^1_{m+1})+d(x^2_m,x^2_{m+1})+\cdots+d(x^n_m,x^n_{m+1})).\end{aligned} \tag{2.8}$$

Set $\delta_m = d(x^1_m,x^1_{m+1})+d(x^2_m,x^2_{m+1})+\cdots+d(x^n_m,x^n_{m+1})$, then we have

$$\psi(\delta_{m+1}) \leqslant \psi(\delta_m) - n\phi(\delta_m), \quad \text{for all } m, \tag{2.9}$$

which yields that

$$\psi(\delta_{m+1}) \leqslant \psi(\delta_m), \quad \text{for all } m. \tag{2.10}$$

Since ψ is nondecreasing, we get that $\delta_{m+1} \leqslant \delta_m$ for all m. Hence $\{\delta_m\}$ is a nonincreasing sequence. Since it is bounded below, there is some $\delta \geqslant 0$ such that

$$\lim_{m\to\infty}\delta_m = \delta. \tag{2.11}$$

We shall show that $\delta = 0$. Suppose on contrary that $\delta > 0$. Letting $m \to \infty$ in (2.9) and having in mind that we suppose $\lim_{t\to r}\phi(t) > 0$ for all $r > 0$ and $\lim_{t\to 0^+}\phi(t) = 0$, we have

$$\delta \leqslant \delta - n\phi(\delta) < \delta, \tag{2.12}$$

which is a contradiction. Thus $\delta = 0$, that is,

$$\lim_{m\to\infty}\delta_m = \lim_{m\to\infty}[d(xm^1,x^1_{m+1})+d(xm^2,x^2_{m+1})+\cdots+d(xm^n,x^n_{m+1})] = 0, \tag{2.13}$$

which implies that $\{x^1_m\}, \{x^2_m\}, \ldots, \{x^n_m\}$ are asymptotically regular sequences in (X,d). Since (X,d) is a complete asymptotically regular metric space, there exist $x^1, x^2, \ldots, x^n \in X$ such that

$$\lim_{m\to\infty}x^1_m = x^1, \lim_{m\to\infty}x^2_m = x^2, \ldots, \lim_{m\to\infty}x^n_m = x^n. \tag{2.14}$$

Suppose that assumption (a) holds. Then by (2.4) and (2.14), we have

$$\begin{aligned}x^1 &= \lim_{m\to\infty}x^1_m = \lim_{m\to\infty}(x^1_{m-1},x^2_{m-1},\ldots,x^n_{m-1})\\ &= F\left(\lim_{m\to\infty}x^1_{m-1},\lim_{m\to\infty}x^2_{m-1},\ldots,\lim_{m\to\infty}x^n_{m-1}\right) = F(x^1,x^2,\ldots,x^n).\end{aligned}$$

Analogously, we also observe that

$$\begin{aligned}x^2 &= \lim_{m\to\infty}x^2_m = \lim_{m\to\infty}(x^2_{m-1},\ldots,x^n_{m-1},x^1_{m-1})\\ &= F\left(\lim_{m\to\infty}x^2_{m-1},\ldots,\lim_{m\to\infty}x^n_{m-1},\lim_{m\to\infty}x^1_{m-1}\right) = F(x^2,\ldots,x^n,x^1).\end{aligned}$$

$$\vdots$$

$$\begin{aligned}x^n &= \lim_{m\to\infty}x^n_m = \lim_{m\to\infty}(x^n_{m-1},x^1_{m-1},\ldots,x^{n-1}_{m-1})\\ &= F\left(\lim_{m\to\infty}x^n_{m-1},\lim_{m\to\infty}x^1_{m-1},\ldots,\lim_{m\to\infty}x^{n-1}_{m-1}\right) = F(x^n,x^1,\ldots,x^{n-1}).\end{aligned}$$

Thus we have

$$\begin{aligned}F(x^1,x^2,\ldots,x^n) &= x^1, F(x^2,\ldots,x^n,x^1)\\ &= x^2,\ldots,F(x^n,x^1,\ldots,x^{n-1}) = x^{n-1}.\end{aligned}$$

Let us assume that assumption (b) holds. Since $\{x^1_m\}, \{x^3_m\}, \ldots, \{x^{n-1}_m\}$ are nondecreasing and $x^1_m \to x^1, x^3_m \to x^3, \ldots, x^{n-1}_m \to x^{n-1}$ and also $\{x^2_m\}, \{x^4_m\}, \ldots, \{x^n_m\}$ are nonincreasing and $x^2_m \to x^2, x^4_m \to x^4, \ldots, x^n_m \to x^n$, we have

$$x^1_m \succeq x^1, x^2_m \preceq x^2, x^3_m \succeq x^3, \ldots, x^n_m \preceq x^n \quad \forall m, \text{ by assumption (b)}.$$

Consider now

$$\begin{aligned}d(x^1,F(x^1,x^2,\ldots,x^n)) &\leqslant d(x^1,x^1_{m+1})+d(x^1_{m+1},F(x^1,x^2,\ldots,x^n))\\ &= d(x^1,x^1_{m+1})+d(F(x^1_m,x^2_m,\ldots,x^n_m),\\ &\quad F(x^1,x^2,\ldots,x^n))\\ &< d(x^1,x^1_{m+1})+\frac{1}{n}\psi(d(x^1_m,x^1)+d(x^2_m,x^2)\\ &\quad +\cdots+d(x^n_m,x^n))-\phi(d(x^1_m,x^1)\\ &\quad +d(x^2_m,x^2)+\cdots+d(x^n_m,x^n)).\end{aligned} \tag{2.15}$$

Taking $m \to \infty$ in (2.15) and using (2.14), we get that

$$d(x^1,F(x^1_m,x^2_m,\ldots,x^n_m)) = 0 \Rightarrow x^1 = F(x^1_m,x^2_m,\ldots,x^n_m).$$

Analogously, we get that

$$x^2 = F(x_m^2, \ldots, x_m^n, x_m^1), \ldots, x^n = F(x_m^n, x_m^1, \ldots, x_m^{n-1}).$$

Thus we proved that F has an n-tupled fixed point. □

Corollary 1. *Let $(X, \preceq)$ be a partially ordered set and (X, d) be a complete asymptotically regular metric space. Let $F: X^n \to X$ be a mapping such that F has the mixed monotone property on X. Assume that there exists $k \in [0,1)$ such that for all $x^1, x^2, \ldots, x^n, y^1, y^2, \ldots, y^n \in X$ for which $x^1 \succeq y^1$, $x^2 \preceq y^2$, $x^3 \succeq y^3, \ldots, x^n \preceq y^n$,*

$$d(F(x^1, x^2, \ldots, x^n), F(y^1, y^2, \ldots, y^n)) \leqslant \frac{k}{n}[d(x^1, y^1) + d(x^2, y^2) + \cdots + d(x^n, y^n)]. \quad (2.16)$$

Suppose that there exist $x_0^1, x_0^2, x_0^3, \ldots, x_0^n \in X$ such that

$$\begin{cases} x_0^1 \preceq F(x_0^1, x_0^2, x_0^3, \ldots, x_0^n) \\ x_0^2 \succeq F(x_0^2, x_0^3, \ldots, x_0^n, x_0^1) \\ x_0^3 \preceq F(x_0^3, \ldots, x_0^n, x_0^1, x_0^2) \\ \vdots \\ x_0^n \succeq F(x_0^n, x_0^1, x_0^2, \ldots, x_0^{n-1}). \end{cases}$$

Suppose that either

(a) *F is continuous or*
(b) *X has the following property:*
 (i) *if nondecreasing sequence $\{x_m\} \to x$, then $x_m \preceq x$ for all $m \geqslant 0$;*
 (ii) *if nonincreasing sequence $\{x_m\} \to x$, then $x_m \succeq x$ for all $m \geqslant 0$,*

then there exist $x^1, x^2, \ldots, x^n \in X$ such that

$$F(x^1, x^2, \ldots, x^n) = x^1, \; F(x^2, \ldots, x^n, x^1) = x^2, \ldots, F(x^n, x^1, \ldots, x^{n-1}) = x^n.$$

Proof. It is sufficient to take $\psi(t) = t$ and $\phi(t) = \frac{1-k}{n} t$ in the previous theorem. □

Now we shall prove the uniqueness of n-tupled fixed point. For a product X^n of a partial ordered set $(X, \preceq)$, we define a partial ordering in the following way:

$$\forall (x^1, x^2, \ldots, x^n), (y^1, y^2, \ldots, y^n) \in X^n \quad (x^1, x^2, \ldots, x^n) \preceq (y^1, y^2, \ldots, y^n) \iff x^1 \preceq y^1, x^2 \succeq y^2, x^3 \preceq y^3, \ldots, x^n \succeq y^n. \quad (2.17)$$

We say that $(x^1, x^2, \ldots, x^n)$ is equal to $(y^1, y^2, \ldots, y^n)$ if and only if $x^1 = y^1$, $x^2 = y^2, \ldots, x^n = y^n$.

Theorem 4. *In addition to the hypotheses of Theorem 3, suppose that for real $(x^1, x^2, \ldots, x^n), (y^1, y^2, \ldots, y^n) \in X^n$ there exists, $(z^1, z^2, \ldots, z^n) \in X^n$ that is comparable to $(x^1, x^2, \ldots, x^n)$ and $(y^1, y^2, \ldots, y^n)$, then F has a unique n-tupled fixed point.*

Proof. The set of n-tupled fixed point is non empty due to Theorem 3. Assume that $(x^1, x^2, x^3, \ldots, x^n)$ and $(y^1, y^2, y^3, \ldots, y^n)$ are two n-tupled fixed points of F, that is,

$$F(x^1, x^2, x^3, \ldots, x^n) = x^1, \; F(y^1, y^2, y^3, \ldots, y^n) = y^1$$
$$F(x^2, x^3, \ldots, x^n, x^1) = x^2, \; F(y^2, y^3, \ldots, y^n, y^1) = y^2$$
$$\vdots$$
$$F(x^n, x^1, x^2, \ldots, x^{n-1}) = x^n, \; F(y^n, y^1, y^2, \ldots, y^{n-1}) = y^n.$$

We shall show that $(x^1, x^2, x^3, \ldots, x^n)$ and $(y^1, y^2, y^3, \ldots, y^n)$ are equal. By the assumption of the theorem, there exists $(z^1, z^2, z^3, \ldots, z^n) \in X^n$ that is comparable to $(x^1, x^2, x^3, \ldots, x^n)$ and $(y^1, y^2, y^3, \ldots, y^n)$. Define sequences $\{z_m^1\}, \{z_m^2\}, \ldots, \{z_m^n\}$ such that $z_0^1 = z^1, z_0^2 = z^2, \ldots, z_0^n = z^n$ and

$$\begin{cases} z_m^1 = F(z_{m-1}^1, z_{m-1}^2, z_{m-1}^3, \ldots, z_{m-1}^n) \\ z_m^2 = F(z_{m-1}^2, z_{m-1}^3, \ldots, z_{m-1}^n, z_{m-1}^1) \\ \vdots \\ z_m^n = F(z_{m-1}^n, z_{m-1}^1, z_{m-1}^2, \ldots, z_{m-1}^{n-1}) \forall m. \end{cases} \quad (2.18)$$

Since $(x^1, x^2, \ldots, x^n)$ is comparable with $(z_0^1, z_0^2, \ldots, z_0^n)$, we may assume that

$$(x^1, x^2, \ldots, x^n) \succeq (z^1, z^2, \ldots, z^n) = (z_0^1, z_0^2, \ldots, z_0^n).$$

Recursively, we get that

$$(x^1, x^2, \ldots, x^n) \succeq (z_m^1, z_m^2, \ldots, z_m^n) \quad \forall m. \quad (2.19)$$

By (2.3) and (2.19), we have

$$\begin{aligned} \psi(d(x^1, z_{m+1}^1)) &= \psi(d(F(x^1, x^2, \ldots, x^n), F(z_m^1, z_m^2, \ldots, z_m^n))) \\ &\leqslant \frac{1}{n}\psi(d(x^1, z_m^1) + d(x^2, z_m^2) + \ldots + d(x^n, z_m^n)) \\ &\quad - \phi(d(x^1, z_m^1) + d(x^2, z_m^2) + \ldots + d(x^n, z_m^n)), \end{aligned}$$

$$\begin{aligned} \psi(d(x^2, z_{m+1}^2)) &= \psi(d(F(x^2, \ldots, x^n, x^1), F(z_m^2, \ldots, z_m^n, z_m^1))) \\ &\leqslant \frac{1}{n}\psi(d(x^2, z_m^2) + \cdots + d(x^n, z_m^n) + d(x^1, z_m^1)) \\ &\quad - \phi(d(x^2, z_m^2) + \cdots + d(x^n, z_m^n) + d(x^1, z_m^1)), \end{aligned}$$

$$\vdots$$

$$\begin{aligned} \psi(d(x^n, z_{m+1}^n)) &= \psi(d(F(x^n, x^1, \ldots, x^{n-1}), F(z_m^n, z_m^1, \ldots, z_m^{n-1}))) \\ &\leqslant \frac{1}{n}\psi(d(x^n, z_m^n) + d(x^1, z_m^1) + \cdots + d(x^{n-1}, z_m^{n-1})) \\ &\quad - \phi(d(x^n, z_m^n) + d(x^1, z_m^1) + \cdots + d(x^{n-1}, z_m^{n-1})). \end{aligned}$$

Set $\delta_m = d(x^1, z_m^1) + d(x^2, z_m^2) + \ldots + d(x^n, z_m^n)$. Then due to above inequalities, we have

$$\psi(\delta_{m+1}) \leqslant \psi(\delta_m) - n\phi(\delta_m) \quad \text{for all } m, \quad (2.20)$$

which implies $\delta_{m+1} \leqslant \delta_m$. Hence the sequence $\{\delta_m\}$ is decreasing and bounded below. Thus there exists $\delta \geqslant 0$ such that $\lim_{m\to\infty}\delta_m = \delta$. Now, we shall show that $\delta = 0$. Suppose to the contrary that $\delta > 0$. Letting $m \to \infty$ in (2.20), we obtain

$$\psi(\delta) \leqslant \psi(\delta) - n \lim_{m\to\infty} \phi(\delta_m) < \psi(\delta),$$

which is a contradiction. Therefore $\delta = 0$. That is, $\lim_{m\to\infty}\delta_m = 0$. Consequently, we have

$$\lim_{m\to\infty} d(x^1, z_m^1) = 0,\ \lim_{m\to\infty} d(x^2, z_m^2) = 0, \ldots, \lim_{m\to\infty} d(x^n, z_m^n) = 0. \tag{2.21}$$

Similarly, we show that

$$\lim_{m\to\infty} d(y^1, z_m^1) = 0,\ \lim_{m\to\infty} d(y^2, z_m^2) = 0, \ldots, \lim_{m\to\infty} d(y^n, z_m^n) = 0. \tag{2.22}$$

Combining (2.21) and (2.22) yields that $(x^1, x^2, x^3, \ldots, x^n)$ and $(y^1, y^2, y^3, \ldots, y^n)$ are equal. □

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