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Convergence theorems for three finite families of multivalued nonexpansive mappings

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Abstract In this paper, we obtain weak and strong convergence theorems of an iterative sequences associated with three finite families of multivalued nonexpansive mappings under some conditions in a uniformly convex real Banach space. Our results extend and improve several known results.

2000 MATHEMATICS SUBJECT CLASSIFICATION: 47H10; 54H25

1. Introduction and preliminaries

Let E be a Banach space with dim $E \geqslant 2$, the modulus of convexity of E is the function $\delta_E : (0,2] \to [0,1]$ defined by

$$\delta_E(\epsilon) = \inf\left\{1 - \frac{1}{2}\|x+y\| : \|x\| = 1, \|y\| = 1, \|x-y\| = \epsilon\right\}.$$

E is uniformly convex if and only if with $\delta_E(\epsilon) > 0$ for all $\epsilon \in (0,2]$.

A subset K is called proximinal if for each $x \in E$, there exists an element $k \in K$ such that

$$d(x,k) = \inf\{\|x-y\| : y \in K\} = d(x,K).$$

It is known that a weakly compact convex subsets of a Banach space and closed convex subsets of a uniformly convex Banach space are proximinal. We shall denote the family of nonempty bounded proximinal subsets of K by $P(K)$, $C(K)$ the family of nonempty compact subsets of K, and $CB(K)$ be the class of all nonempty bounded and closed subsets of K, Consistent with [1].

Let H be a Hausdorff metric induced by the metric d of K, given by

$$H(A,B) = \max\left\{\sup_{x\in A} d(x,B), \sup_{y\in B} d(y,A)\right\},$$

for every $A, B \in CB(K)$. It is obvious that $P(K) \in CB(K)$.

A multivalued mapping $T : K \to P(K)$ is said to be a contraction if there exists a constant $k \in [0,1)$ such that for any $x, y \in K$,

$$H(Tx,Ty) \leqslant k\|x-y\|,$$

and T is said to be nonexpansive if

$$H(Tx,Ty) \leqslant \|x-y\|,$$

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for all $x, y \in K$. A point $x \in K$ is called a fixed point of T if $x \in Tx$. Throughout the paper $\mathbb{N}$ denotes the set of all natural numbers and $F(T)$ the set of fixed points of T.

Let us recall the following definitions.

Definition 1.1 [2]. A Banach space E is said to satisfy Opial's condition if for any sequence $\{x_n\}$ in $E, x_n \rightharpoonup x$ ($\rightharpoonup$ denotes weak convergence) implies that $\limsup_{n\to\infty}\|x_n - x\| < \limsup_{n\to\infty}\|x_n - y\|$ for all $y \in E$ with $y \neq x$. Examples of Banach spaces satisfying this condition are Hilbert spaces and all $1 < p < \infty$.

Definition 1.2 [3]. The mapping $T : K \to K$ where K a subset of E, are said to satisfy condition (A) if there exists a nondecreasing function $f : [0,\infty) \to [0,\infty)$ with $f(0) = 0, f(r) > 0$ for all $r \in (0,\infty)$ such that either $d(x, Tx) \geqslant f(d(x, F(T))$ for all $x \in K$, where $d(x, F(T)) = \inf\{\|x - p\| : p \in F(T)\}$

The following is the multivalued version of condition (Á);

Definition 1.3. The three finite families of multivalued nonexpansive mappings $T_i, S_i, R_i : K \to CB(K), (i = 1, 2, 3, \dots, k)$, where K a subset of E, are said to satisfy condition $(\grave{A})$ if there exists a nondecreasing function $f : [0,\infty) \to [0,\infty)$ with $f(0) = 0, f(r) > 0$ for all $r \in (0,\infty)$ such that $d(x, T_i x) \geqslant f(d(x, \mathbf{F})$ or $d(x, S_i x) \geqslant f(d(x, \mathbf{F})$ or $d(x, R_i x) \geqslant f(d(x, \mathbf{F})$ for all $x \in K$, where $\mathbf{F} = \left(\bigcap_{i=1}^{k} F(T_i)\right) \cap \left(\bigcap_{i=1}^{k} F(S_i)\right) \cap \left(\bigcap_{i=1}^{k} F(R_i)\right)$, the set of all common fixed points of the mappings T_i, S_i and R_i.

Definition 1.4 [4]. A map $T : K \to CB(K)$, is called hemicompact if, for any sequence $\{x_n\}$ in E such that $d(x_n, Tx_n) \to 0$ as $n \to \infty$, there exists a subsequence $\{x_{n_r}\}$ of $\{x_n\}$ such that $x_{n_r} \to p \in K$. We note that if K is compact, then every multivalued mapping $T : K \to CB(K)$ is hemicompact.

Next we state the following useful lemma.

Lemma 1.1 [5]. *Let E be a uniformly convex Banach space, $r > 0$ a positive number and let $B_r(0) = \{x \in E : \|x\| \leqslant r\}$. Then, for any given sequence $\{x_i\} \subset B_r(0)$ and for any given sequence $\lambda_i \in [0,1]$ with $\sum_{i=0}^{k}\lambda_i = 1$, there exists a continuous strictly increasing and convex function $\varphi : [0, 2r) \to \mathbb{R}, \varphi(0) = 0$ such that for any positive integers m, j with $m < j$, the following inequality holds:*

$$\left\|\sum_{i=1}^{k}\lambda_i x_i\right\|^2 \leqslant \sum_{i=1}^{k}\lambda_i\|x_i\|^2 - \lambda_m\lambda_j\varphi(\|x_m - x_j\|).$$

The study of fixed points for multivalued contractions and nonexpansive mappings using the Hausdorff metric was initiated by Markin [6] and Nadler [1]. Later, an interesting and rich fixed point theory for such maps was developed which has applications in control theory, convex optimization, differential inclusion and economics.

The theory of multivalued nonexpansive mappings is harder than the corresponding theory of single valued nonexpansive mappings. Different iterative processes have been used to approximate the fixed points of multivalued nonexpansive mappings. In particular in 2005, Sastry and Babu [7] proved the convergence of Mann and Ishikawa iteration process for multivalued mapping T with a fixed point p converge to a fixed point q of T under certain conditions. They also claimed that the fixed point q may be different from p. Under some conditions Panyanak [8] extended result of Sastry and Babu to uniformly convex Banach spaces. Song and Wang [9] noted that there was a gap in the proof of the main result in [8]. They further revised the gap and also gave the affirmative answer to Panyanak's open question.

Abbas et al. [10] established weak and strong convergence theorems of two multivalued nonexpansive mappings in a uniformly convex real Banach space by one-step iterative process to approximate common fixed points under some basic boundary conditions. Rashwan and Altwqi [11] introduced a new one-step iterative process to approximate the common fixed points of three multivalued nonexpansive mappings.

Recently Eslamian and Abkar [12] introduced a new one-step iterative process for approximate the common fixed points of finitely many multivalued mappings satisfying some conditions. They proved some weak and strong convergence theorems for such iterative process in uniformly convex Banach spaces as follows. Let E be a Banach space, K be a nonempty convex subset of E and $T_i : K \to CB(K)$ $(i = 1, 2, \dots, m)$ be finitely many given mappings. Then, for $x_0 \in K$ and they defined:

$$x_{n+1} = a_{n,0}x_n + \sum_{i=1}^{m}a_{n,i}z_{n,i}, \quad n \in \mathbb{N}, \tag{1.1}$$

where $z_{n,i} \in T_i(x_n)$ and $\{a_{n,i}\}$ are sequences of numbers in $[0,1]$ such that for every natural number $n \in \mathbb{N}$ and $\sum_{i=0}^{m}a_{n,i} = 1$.

We now introduce the following iteration scheme which attend (1.1). Let E be Banach space, K be a nonempty closed convex subset of E and let $T_i, S_i, R_i : K \to CB(K)$, $(i = 1, 2, \dots, k)$ be three finite families of multivalued mappings. Then for $x_0 \in K$, define the sequences $\{x_n\}_{n=1}^{\infty}, \{y_n\}_{n=1}^{\infty}$ and $\{z_n\}_{n=1}^{\infty}$ by:

$$\begin{aligned} x_{n+1} &= \alpha_{n,0}x_n + \sum_{i=1}^{k}\alpha_{n,i}u_{n,i}, \\ y_n &= \beta_{n,0}x_n + \sum_{i=1}^{k}\beta_{n,i}v_{n,i}, \quad n \in \mathbb{N} \\ z_n &= \gamma_{n,0}x_n + \sum_{i=1}^{k}\gamma_{n,i}w_{n,i}, \end{aligned} \tag{1.2}$$

where $u_{n,i} \in T_i y_n, v_{n,i} \in S_i z_n, w_{n,i} \in R_i x_n$ and $\{\alpha_{n,i}\}, \{\beta_{n,i}\}$ and $\{\gamma_{n,i}\}$ are sequence of numbers in $[0,1]$ satisfying $\sum_{i=0}^{k}\alpha_{n,i} = \sum_{i=0}^{k}\beta_{n,i} = \sum_{i=0}^{k}\gamma_{n,i} = 1$.

Remark 1.1

1. If $\beta_{n,0} = 1$ $\gamma_{n,0} = 1$ and $\sum_{i=1}^{k}\beta_{n,i} = \sum_{i=1}^{k}\gamma_{n,i} \equiv 0$. The iterative scheme (1.2) reduce to iterative scheme defined by (1.1).
2. If $\sum_{i=2}^{k}\alpha_{n,i} = \sum_{i=2}^{k}\beta_{n,i} = \sum_{i=2}^{k}\gamma_{n,i} \equiv 0$. The iterative scheme (1.2) reduce to Noor iterative scheme defined by

$$\begin{aligned} x_{n+1} &= \alpha_{n,0}x_n + \alpha_{n,1}u_{n,1}, \\ y_n &= \beta_{n,0}x_n + \beta_{n,1}v_{n,1}, \quad n \in \mathbb{N} \\ z_n &= \gamma_{n,0}x_n + \gamma_{n,1}w_{n,1}, \end{aligned} \tag{1.3}$$

where $u_{n,1} \in T_1 y_n$, $v_{n,1} \in S_1 z_n$ and $w_{n,1} \in R_1 x_n$.

The following is an example of three finite families of multivalued nonexpansive mappings with a common fixed point.

Example 1.1. Let $X = [0, 1]$. Define $T_n, S_n, R_n : X \to CB(X)$ for each $n \in \mathbb{N}$ as follows:

$$T_n x = \left[0, \frac{2x + n - 1}{2n}\right],$$

$$S_n x = \left[0, \frac{n^2 - 2x + 1}{2n^2}\right],$$

and

$$R_n x = \left[0, \frac{2x - n}{2(1 - n)}\right].$$

Then clearly T_n, S_n and R_n are three finite families of multivalued nonexpansive mappings and have a common fixed point at $\{\frac{1}{2}\}$.

The main purpose of this paper is to prove weak and strong convergence of the iterative scheme (1.2) to a common fixed point of T_i, S_i and R_i.

2. Main results

In this section, we prove that the iterative process defined by (1.2) converges weakly and strongly to a common fixed point.

At first, we shall prove the following lemmas.

Lemma 2.1. *Let E be a uniformly convex Banach space and K a nonempty closed convex subset. Let $T_i, S_i, R_i : K \to CB(K)$ for each $(i = 1, 2, \ldots, k)$ be three finite families of multivalued nonexpansive mappings. Let $\{x_n\}$ be the sequence as defined in (1.2). If $\mathbf{F} \neq \emptyset$ and $T_i p = S_i p = R_i p = \{p\}$ for any $p \in \mathbf{F}$ then $\lim_{n\to\infty} \|x_n - p\|$ exists.*

Proof. Let $p \in \mathbf{F}$. Then from (1.2) we have,

$$\begin{aligned}
\|x_{n+1} - p\| &= \|\alpha_{n,0} x_n + \sum_{i=1}^{k} \alpha_{n,i} u_{n,i} - p\| \\
&= \|\alpha_{n,0}(x_n - p) + \sum_{i=1}^{k} \alpha_{n,i}(u_{n,i} - p)\| \\
&\leqslant \alpha_{n,0}\|x_n - p\| + \sum_{i=1}^{k} \alpha_{n,i}\|u_{n,i} - p\| \\
&\leqslant \alpha_{n,0}\|x_n - p\| + \sum_{i=1}^{k} \alpha_{n,i} d(u_{n,i}, T_i p) \\
&\leqslant \alpha_{n,0}\|x_n - p\| + \sum_{i=1}^{k} \alpha_{n,i} H(T_i y_n, T_i p) \\
&\leqslant \alpha_{n,0}\|x_n - p\| + \sum_{i=1}^{k} \alpha_{n,i}\|y_n - p\|, \qquad (2.1)
\end{aligned}$$

and

$$\begin{aligned}
\|y_n - p\| &= \|\beta_{n,0} x_n + \sum_{i=1}^{k} \beta_{n,i} v_{n,i} - p\| \\
&= \|\beta_{n,0}(x_n - p) + \sum_{i=1}^{k} \beta_{n,i}(v_{n,i} - p)\| \\
&\leqslant \beta_{n,0}\|x_n - p\| + \sum_{i=1}^{k} \beta_{n,i}\|v_{n,i} - p\| \\
&\leqslant \beta_{n,0}\|x_n - p\| + \sum_{i=1}^{k} \beta_{n,i} d(v_{n,i}, S_i p) \\
&\leqslant \beta_{n,0}\|x_n - p\| + \sum_{i=1}^{k} \beta_{n,i} H(S_i z_n, S_i p) \\
&\leqslant \beta_{n,0}\|x_n - p\| + \sum_{i=1}^{k} \beta_{n,i}\|z_n - p\|, \qquad (2.2)
\end{aligned}$$

and

$$\begin{aligned}
\|z_n - p\| &= \|\gamma_{n,0} x_n + \sum_{i=1}^{k} \gamma_{n,i} w_{n,i} - p\| \\
&= \|\gamma_{n,0}(x_n - p) + \sum_{i=1}^{k} \gamma_{n,i}(w_{n,i} - p)\| \\
&\leqslant \gamma_{n,0}\|x_n - p\| + \sum_{i=1}^{k} \gamma_{n,i}\|w_{n,i} - p\| \\
&\leqslant \gamma_{n,0}\|x_n - p\| + \sum_{i=1}^{k} \gamma_{n,i} d(w_{n,i}, R_i p) \\
&\leqslant \gamma_{n,0}\|x_n - p\| + \sum_{i=1}^{k} \gamma_{n,i} H(R_i x_n, R_i p) \\
&\leqslant \gamma_{n,0}\|x_n - p\| + \sum_{i=1}^{k} \gamma_{n,i}\|x_n - p\| \\
&= \sum_{i=0}^{k} \gamma_{n,i}\|x_n - p\| = \|x_n - p\|. \qquad (2.3)
\end{aligned}$$

Substituting (2.3) into (2.2) we obtain,

$$\begin{aligned}
\|y_n - p\| &\leqslant \beta_{n,0}\|x_n - p\| + \sum_{i=1}^{k} \beta_{n,i}\|x_n - p\| \\
&= \sum_{i=0}^{k} \beta_{n,i}\|x_n - p\| = \|x_n - p\|. \qquad (2.4)
\end{aligned}$$

Substituting (2.4) into (2.1) we have

$$\begin{aligned}
\|x_{n+1} - p\| &\leqslant \alpha_{n,0}\|x_n - p\| + \sum_{i=1}^{k} \alpha_{n,i}\|y_n - p\| \\
&= \alpha_{n,0}\|x_n - p\| + \sum_{i=1}^{k} \alpha_{n,i}\|x_n - p\| \\
&= \sum_{i=0}^{k} \alpha_{n,i}\|x_n - p\| = \|x_n - p\|. \qquad (2.5)
\end{aligned}$$

Thus $\lim_{n\to\infty} \|x_n - p\|$ exists for each $p \in \mathbf{F}$, hence $\{x_n\}$ is bounded. □

Lemma 2.2. *Let E be a uniformly convex Banach space and K be nonempty closed convex subset. Let $T_i, S_i, R_i : K \to CB(K)$ for each $(i = 1, 2, \ldots, k)$ be three finite families of multivalued nonexpansive mappings and $\{x_n\}$ be the sequence as defined in (1.2). If $\mathbf{F} \neq \emptyset$ and $T_i p = S_i p = R_i p = \{p\}$ for any $p \in \mathbf{F}$ then* $\lim_{n\to\infty} d(x_n, T_i y_n) = \lim_{n\to\infty} d(x_n, S_i z_n) = 0 = \lim_{n\to\infty} d(x_n, R_i x_n)$.

Proof. Let $p \in \mathbf{F}$. By Lemma (2.1), $\lim_{n\to\infty}\|x_n - p\|$ exists, $\{x_n\}$ is bounded and so $\{y_n\}$ and $\{z_n\}$ are bounded. Therefore, there exists $r > 0$ such that $x_n - p,\ y_n - p,\ z_n - p \in B_r(0)$ for all $n \geqslant 0$. Applying lemma (1.1) and using (1.2) we have

$$\begin{aligned}
\|x_{n+1} - p\|^2 &= \left\|\alpha_{n,0}x_n + \sum_{i=1}^{k}\alpha_{n,i}u_{n,i} - p\right\|^2 \\
&= \left\|\alpha_{n,0}(x_n - p) + \sum_{i=1}^{k}\alpha_{n,i}(u_{n,i} - p)\right\|^2 \\
&\leqslant \alpha_{n,0}\|x_n - p\|^2 + \sum_{i=1}^{k}\alpha_{n,i}\|u_{n,i} - p\|^2 \\
&\quad - \alpha_{n,0}\alpha_{n,i}\varphi(\|u_{n,i} - x_n\|) \\
&\leqslant \alpha_{n,0}\|x_n - p\|^2 + \sum_{i=1}^{k}\alpha_{n,i}d(u_{n,i}, T_i p)^2 \\
&\quad - \alpha_{n,0}\alpha_{n,i}\varphi(\|u_{n,i} - x_n\|) \\
&\leqslant \alpha_{n,0}\|x_n - p\|^2 + \sum_{i=1}^{k}\alpha_{n,i}H(T_i y_n, T_i p)^2 \\
&\quad - \alpha_{n,0}\alpha_{n,i}\varphi(\|u_{n,i} - x_n\|) \\
&\leqslant \alpha_{n,0}\|x_n - p\|^2 + \sum_{i=1}^{k}\alpha_{n,i}\|y_n - p\|^2 \\
&\quad - \alpha_{n,0}\alpha_{n,i}\varphi(\|u_{n,i} - x_n\|).
\end{aligned} \tag{2.6}$$

$$\begin{aligned}
\|y_n - p\|^2 &= \left\|\beta_{n,0}x_n + \sum_{i=1}^{k}\beta_{n,i}v_{n,i} - p\right\|^2 \\
&= \left\|\beta_{n,0}(x_n - p) + \sum_{i=1}^{k}\beta_{n,i}(v_{n,i} - p)\right\|^2 \\
&\leqslant \beta_{n,0}\|x_n - p\|^2 + \sum_{i=1}^{k}\beta_{n,i}\|v_{n,i} - p\|^2 \\
&\quad - \beta_{n,0}\beta_{n,i}\varphi(\|v_{n,i} - x_n\|) \\
&\leqslant \beta_{n,0}\|x_n - p\|^2 + \sum_{i=1}^{k}\beta_{n,i}d(v_{n,i}, S_i p)^2 \\
&\quad - \beta_{n,0}\beta_{n,i}\varphi(\|v_{n,i} - x_n\|) \\
&\leqslant \beta_{n,0}\|x_n - p\|^2 + \sum_{i=1}^{k}\beta_{n,i}H(S_i z_n, S_i p)^2 \\
&\quad - \beta_{n,0}\beta_{n,i}\varphi(\|v_{n,i} - x_n\|) \\
&\leqslant \beta_{n,0}\|x_n - p\|^2 + \sum_{i=1}^{k}\beta_{n,i}\|z_n - p\|^2 \\
&\quad - \beta_{n,0}\beta_{n,i}\varphi(\|v_{n,i} - x_n\|),
\end{aligned} \tag{2.7}$$

and

$$\begin{aligned}
\|z_n - p\|^2 &= \left\|\gamma_{n,0}x_n + \sum_{i=1}^{k}\gamma_{n,i}w_{n,i} - p\right\|^2 \\
&= \left\|\gamma_{n,0}(x_n - p) + \sum_{i=1}^{k}\gamma_{n,i}(w_{n,i} - p)\right\|^2 \\
&\leqslant \gamma_{n,0}\|x_n - p\|^2 + \sum_{i=1}^{k}\gamma_{n,i}\|w_{n,i} - p\|^2 \\
&\quad - \gamma_{n,0}\gamma_{n,i}\varphi(\|w_{n,i} - x_n\|) \\
&\leqslant \gamma_{n,0}\|x_n - p\|^2 + \sum_{i=1}^{k}\gamma_{n,i}d(w_{n,i}, R_i p)^2 \\
&\quad - \gamma_{n,0}\gamma_{n,i}\varphi(\|w_{n,i} - x_n\|) \\
&\leqslant \gamma_{n,0}\|x_n - p\|^2 + \sum_{i=1}^{k}\gamma_{n,i}H(R_i x_n, R_i p)^2 \\
&\quad - \gamma_{n,0}\gamma_{n,i}\varphi(\|w_{n,i} - x_n\|) \\
&\leqslant \gamma_{n,0}\|x_n - p\|^2 + \sum_{i=1}^{k}\gamma_{n,i}\|x_n - p\|^2 \\
&\quad - \gamma_{n,0}\gamma_{n,i}\varphi(\|w_{n,i} - x_n\|) \\
&\leqslant \sum_{i=0}^{k}\gamma_{n,i}\|x_n - p\|^2 - \gamma_{n,0}\gamma_{n,i}\varphi(\|w_{n,i} - x_n\|) \\
&= \|x_n - p\|^2 - \gamma_{n,0}\gamma_{n,i}\varphi(\|w_{n,i} - x_n\|).
\end{aligned} \tag{2.8}$$

From (2.7) and (2.8) we have

$$\begin{aligned}
\|y_n - p\|^2 &\leqslant \beta_{n,0}\|x_n - p\|^2 + \sum_{i=1}^{k}\beta_{n,i}\left[\|x_n - p\|^2 - \gamma_{n,0}\gamma_{n,i}\varphi(\|w_{n,i} - x_n\|)\right] \\
&\quad - \beta_{n,0}\beta_{n,i}\varphi(\|v_{n,i} - x_n\|) \\
&\leqslant \sum_{i=0}^{k}\beta_{n,i}\|x_n - p\|^2 - \gamma_{n,0}\sum_{i=1}^{k}\gamma_{n,i}\beta_{n,i}\varphi(\|w_{n,i} - x_n\|) \\
&\quad - \beta_{n,0}\beta_{n,i}\varphi(\|v_{n,i} - x_n\|) \\
&\leqslant \|x_n - p\|^2 - \gamma_{n,0}\sum_{i=1}^{k}\gamma_{n,i}\beta_{n,i}\varphi(\|w_{n,i} - x_n\|) \\
&\quad - \beta_{n,0}\beta_{n,i}\varphi(\|v_{n,i} - x_n\|).
\end{aligned} \tag{2.9}$$

From (2.6) and (2.9) we obtain

$$\begin{aligned}
\|x_{n+1} - p\|^2 &\leqslant \alpha_{n,0}\|x_n - p\|^2 \\
&\quad + \sum_{i=1}^{k}\alpha_{n,i}\left[\|x_n - p\|^2 - \gamma_{n,0}\sum_{i=1}^{k}\gamma_{n,i}\beta_{n,i}\varphi(\|w_{n,i} - x_n\|)\right. \\
&\quad \left.-\beta_{n,0}\beta_{n,i}\varphi(\|v_{n,i} - x_n\|)\right] - \alpha_{n,0}\alpha_{n,i}\varphi(\|u_{n,i} - x_n\|) \\
&\leqslant \|x_n - p\|^2 - \gamma_{n,0}\sum_{i=1}^{k}\gamma_{n,i}\beta_{n,i}\alpha_{n,i}\varphi(\|w_{n,i} - x_n\|) \\
&\quad - \beta_{n,0}\sum_{i=1}^{k}\alpha_{n,i}\beta_{n,i}\varphi(\|v_{n,i} - x_n\|) \\
&\quad - \alpha_{n,0}\alpha_{n,i}\varphi(\|u_{n,i} - x_n\|).
\end{aligned} \tag{2.10}$$

From (2.10) we obtain,

$$\begin{aligned}
\alpha_{n,0}\alpha_{n,i}\varphi(\|u_{n,i} - x_n\|) &\leqslant \|x_n - p\|^2 - \|x_{n+1} - p\|^2 \\
&\quad - \gamma_{n,0}\sum_{i=1}^{k}\gamma_{n,i}\beta_{n,i}\alpha_{n,i}\varphi(\|w_{n,i} - x_n\|) \\
&\quad - \beta_{n,0}\sum_{i=1}^{k}\alpha_{n,i}\beta_{n,i}\varphi(\|v_{n,i} - x_n\|).
\end{aligned} \tag{2.11}$$

Thus,

$$\alpha_{n,0}\alpha_{n,i}\varphi(\|u_{n,i}-x_n\|) \leqslant \|x_n-p\|^2 - \|x_{n+1}-p\|^2,$$

this implies,

$$\alpha_{n,0}\alpha_{n,i}\varphi(\|u_{n,i}-x_n\|) \leqslant \|x_1-p\|^2 \leqslant \infty.$$

Let $(M=\alpha_{n,0}\alpha_{n,i})$, then

$$\sum_{n=1}^{\infty} M\varphi(\|u_{n,i}-x_n\|) \leqslant \|x_1-p\|^2 \leqslant \infty.$$

Since φ is continuous at 0 and is strictly increasing, we have

$$\lim_{n\to\infty}\|u_{n,i}-x_n\|=0.$$

Similarly from (2.11) we obtain that

$$\lim_{n\to\infty}\|v_{n,i}-x_n\| = \lim_{n\to\infty}\|w_{n,i}-x_n\|=0.$$

Hence we obtain

$$\begin{aligned} d(x_n, T_i y_n) &\leqslant d(x_n, u_{n,i}) + d(u_{n,i}, T_i y_n) \\ &\to 0 \ as\ n\to\infty, \end{aligned}$$

also

$$\begin{aligned} d(x_n, S_i z_n) &\leqslant d(x_n, v_{n,i}) + d(v_{n,i}, S_i z_n) \\ &\to 0 \ as\ n\to\infty, \end{aligned}$$

and

$$\begin{aligned} d(x_n, R_i x_n) &\leqslant d(x_n, w_{n,i}) + d(w_{n,i}, R_i x_n) \\ &\to 0 \ as\ n\to\infty. \end{aligned}$$

This completes the proof of the lemma. □

2.1. A weak convergence theorem

Now we approximate common fixed points of the sequences mappings T_i, S_i and $R_i, (i=1,2,3,\ldots,k)$ through weak convergence of the sequence $\{x_n\}$ defined in (1.2) as follows:

Theorem 2.1. *Let E be a uniformly convex Banach space satisfying the Opial's condition. Let K be a nonempty compact convex subset of E and $T_i, S_i, R_i : K\to C(K), (i=1,2,3,\ldots,k)$ be three finite families of multivalued nonexpansive mappings. If $\mathbf{F}\neq\emptyset$ and $T_ip=S_ip=R_ip=\{p\}$ for any $p\in\mathbf{F}$. Let $\{x_n\}$ be a sequence defined in (1.2), then $\{x_n\}$ converges weakly to a common fixed point of T_i, S_i and R_i.*

Proof. Then as proved in lemma (2.1), $\lim_{n\to\infty}\|x_n-p\|$ exists and $\{x_n\}$ bounded and by lemma (2.2) $\lim_{n\to\infty}d(x_n,T_iy_n) = \lim_{n\to\infty}d(x_n,S_iz_n)=0=\lim_{n\to\infty}d(x_n,R_ix_n)$, for $i=1,2,3,\ldots,k$. Since E is uniformly convex, there exists a subsequence $\{x_{n_r}\}$ of $\{x_n\}$ such that $x_{n_r}\to q$ weakly as $r\to\infty$ for some $q\in K$. First we show that $q\in\mathbf{F}$. Since $T_iq, i=1,2,3,\ldots,k$ is compact, for each x_{n_r} in E there exists $y_r\in T_iq$ such that

$$d(x_{n_r}, y_r) = d(x_{n_r}, T_iq).$$

Since T_iq is compact the sequence $\{y_{n_r}\}$ be the subsequence of $\{y_n\}$ such that $\lim_{r\to\infty}y_{n_r}=z\in T_iq$. Now we show that $z=q$. If not, then we have:

$$\begin{aligned} \limsup_{r\to\infty}\|x_{n_r}-z\| &\leqslant \limsup_{r\to\infty}\|x_{n_r}-y_{n_r}\| + \limsup_{r\to\infty}\|y_{n_r}-z\| \\ &= \limsup_{r\to\infty}\|x_{n_r}-y_{n_r}\| \\ &= \limsup_{r\to\infty} d(x_{n_r}, T_iq) \\ &\leqslant \limsup_{r\to\infty} d(x_{n_r}, T_ix_{n_r}) + \limsup_{r\to\infty} d(T_ix_{n_r}, T_iq) \\ &\leqslant \limsup_{r\to\infty}\|x_{n_r}-q\| < \limsup_{r\to\infty}\|x_{n_r}-z\|. \end{aligned}$$

Which gives a contradiction and hence $q=z\in T_iq$. Similarly, it can be shown that $q\in S_iq$ and $q\in R_iq$. Now we prove that $\{x_n\}$ has a unique weak subsequential limit in $\mathbf{F}$. To prove this, let z_1, z_1 and z_3 be weak limits of the subsequences $\{x_{n_i}\},\{x_{n_j}\}$ and $\{x_{n_l}\}$ of $\{x_n\}$, respectively and $z_1\neq z_2\neq z_3$ as above $z_1,z_2,z_3\in\mathbf{F}$ and by lemma (2.1) the limits, $\lim_{n\to\infty}\|x_n-z_1\|$, $\lim_{n\to\infty}\|x_n-z_2\|, \lim_{n\to\infty}\|x_n-z_3\|$ exist. Then by the Opial's condition,

$$\begin{aligned} \lim_{n\to\infty}\|x_n-z_1\| &= \lim_{n_i\to\infty}\|x_{n_i}-z_1\| \\ &< \lim_{n_i\to\infty}\|x_{n_i}-z_2\| \\ &= \lim_{n\to\infty}\|x_n-z_2\| \\ &= \lim_{n_j\to\infty}\|x_{n_j}-z_2\| \\ &< \lim_{n_j\to\infty}\|x_{n_j}-z_3\| \\ &= \lim_{n\to\infty}\|x_n-z_3\| \\ &= \lim_{n_l\to\infty}\|x_{n_l}-z_3\| \\ &< \lim_{n_l\to\infty}\|x_{n_l}-z_1\| \\ &= \lim_{n\to\infty}\|x_n-z_1\|. \end{aligned}$$

This is a contradiction. Hence $z_1=z_2=z_3$. This implies that $\{x_n\}$ converges weakly to a common fixed point in $\mathbf{F}$. □

Corollary 2.1. *Let E be a uniformly convex Banach space satisfying the Opial's condition. Let K be a nonempty compact convex subset of E and $T,S,R : K\to C(K)$, be three of multivalued nonexpansive mappings. If $\mathbf{F}\neq\emptyset$ and $\{x_n\}$ be the sequence as defined in (1.3), then $\{x_n\}$ converges weakly to a common fixed point of T,S and R.*

2.2. Strong convergence theorems

The following result gives a necessary and sufficient condition for strong convergence of the sequence (1.2) to a common fixed point of three finite families mappings on a real Banach space.

Theorem 2.2. *Let E be a uniformly convex Banach space and $K,\{x_n\}$ be as in lemma (2.2). Let $T_i,S_i,R_i : K\to CB(K)$, $(i=1,2,3,\ldots,k)$ be three finite families of multivalued nonexpansive mappings satisfying condition $(\grave{A})$. If $\mathbf{F}\neq\emptyset$ and $T_ip=S_ip=R_ip=\{p\}$ for any $p\in\mathbf{F}$, then $\{x_n\}$ converges strongly to a common fixed point of T_i, S_i and R_i.*

Proof. Since T_i, S_i and $R_i, i=1,2,3,\ldots,k$, satisfies condition $(\grave{A})$, we have $\lim_{n\to\infty}f(d(x_n,\mathbf{F}))=0$. Thus there is a subsequence $\{x_{n_r}\}$ of $\{x_n\}$ and a sequence $\{p_r\}\subset\mathbf{F}$ such that

$$\|x_{n_r} - p_r\| < \frac{1}{2^r},$$

for all $r > 0$. By lemma (2.1) we obtain that

$$\|x_{n_{r+1}} - p_r\| \leqslant \|x_{n_r} - p_r\| < \frac{1}{2^r}.$$

We now show that $\{p_r\}$ is a Cauchy sequence in K. Observes that

$$\|p_{r+1} - p_r\| \leqslant \|p_{r+1} - x_{n_{r+1}}\| + \|x_{n_{r+1}} - p_r\| < \frac{1}{2^{r+1}} + \frac{1}{2^r} < \frac{1}{2^{r-1}}.$$

This shows that $\{p_r\}$ is a Cauchy sequence in K and thus converges to $p \in K$. Since

$$d(p_r, T_ip) \leqslant H(T_ip, T_ip_r) \leqslant \|p - p_r\|,$$

and $p_r \to p$ as $r \to \infty$, it follows that $d(p, T_ip) = 0, i = 1, 2, 3, \ldots, k$ which implies that $p \in T_ip$.

Similarly,

$$d(p_r, S_ip) \leqslant H(S_ip, S_ip_r) \leqslant \|p - p_r\|,$$

and $p_r \to p$ as $r \to \infty$, it follows that $d(p, S_ip) = 0, i = 1, 2, 3, \ldots, k$ which implies that $p \in S_ip$.

Similarly,

$$d(p_r, R_ip) \leqslant H(R_ip, R_ip_r) \leqslant \|p - p_r\|,$$

and $p_r \to p$ as $r \to \infty$, it follows that $d(p, R_ip) = 0, i = 1, 2, 3, \ldots, k$ which implies that $p \in R_ip$. Consequently, $p \in \mathbf{F} \neq \emptyset$. $\lim_{n\to\infty}\|x_n - p\|$ exists, we conclude that $\{x_n\}$ converges strongly to a common fixed point p. □

Theorem 2.3. *Let E be a real Banach space and $K, \{x_n\}, T_i, S_i, R_i$, $(i = 1, 2, 3, \ldots, k)$ be as in Lemma (2.2). If $\mathbf{F} \neq \emptyset$ and $T_ip = S_ip = R_ip = \{p\}$ for any $p \in \mathbf{F}$, then $\{x_n\}$ converges strongly to a common fixed point of T_i, S_i and R_i iff $\liminf_{n\to\infty} d(x_n, \mathbf{F}) = 0$.*

Proof. The necessity is obvious. Conversely, suppose that $\liminf_{n\to\infty} d(x_n, \mathbf{F}) = 0$. As proved in lemma (2.1),

$$\|x_{n+1} - p\| \leqslant \|x_n - p\|.$$

This gives

$$d(x_{n+1}, \mathbf{F}) \leqslant d(x_n, \mathbf{F}),$$

so that $\lim_{n\to\infty} d(x_n, \mathbf{F})$ exists. But, by hypothesis, $\liminf_{n\to\infty} d(x_n, \mathbf{F}) = 0$. Therefore we must have $\lim_{n\to\infty} d(x_n, \mathbf{F}) = 0$. Next we show that $\{x_n\}$ is a Cauchy sequence in K. Let $\varepsilon > 0$ be arbitrarily chosen. Since $\lim_{n\to\infty} d(x_n, \mathbf{F}) = 0$, there exists a constant n_0 such that for all $n \geqslant n_0$, we have

$$\lim_{n\to\infty} d(x_n, \mathbf{F}) < \frac{\epsilon}{4}.$$

In particular, $\inf\{\|x_{n_0} - p\| : p \in \mathbf{F}\} < \frac{\epsilon}{4}$. There must exist a $p^* \in \mathbf{F}$ such that

$$\|x_{n_0} - p^*\| < \frac{\epsilon}{2}.$$

Now for $m, n \geqslant n_0$, we have

$$\|x_{n+m} - x_n\| \leqslant \|x_{n+m} - p^*\| + \|x_n - p^*\| \leqslant 2\|x_{n_0} - p^*\| < 2\left(\frac{\epsilon}{2}\right) = \epsilon.$$

Hence $\{x_n\}$ is a Cauchy sequence in a closed subset K of a Banach space E, and therefore it must converge in K. Let $\lim_{n\to\infty} x_n = \lim_{n\to\infty} y_n = \lim_{n\to\infty} z_n = p$ and

$$d(x_n, y_n) \leqslant d(x_n, p) + d(p, y_n) \to 0 \text{ as } n \to \infty,$$

and

$$d(x_n, z_n) \leqslant d(x_n, p) + d(p, z_n) \to 0 \text{ as } n \to \infty.$$

Now for each $i = 1, 2, 3, \ldots, k$ we obtain,

$$d(p, T_ip) \leqslant d(p, y_n) + d(y_n, x_n) + d(x_n, T_iy_n) + H(T_iy_n, T_ip) \leqslant d(p, y_n) + d(y_n, x_n) + d(x_n, u_{n,i}) + d(y_n, p) \to 0 \text{ as } n \to \infty,$$

gives that $d(p, T_ip) = 0, i = 1, 2, 3, \ldots, k$ which implies that $p \in T_ip$.

Similarly, let $\lim_{n\to\infty} z_n = p$. Now for each $i = 1, 2, 3, \ldots, k$ we obtain

$$d(p, S_ip) \leqslant d(p, z_n) + d(z_n, x_n) + d(x_n, S_iz_n) + H(S_iz_n, S_ip) \leqslant d(p, z_n) + d(z_n, x_n) + d(x_n, v_{n,i}) + d(z_n, p) \to 0 \text{ as } n \to \infty,$$

gives that $d(p, S_ip) = 0, i = 1, 2, 3, \ldots, k$ which implies that $p \in S_ip$. Similarly, let $\lim_{n\to\infty} x_n = p$. Now for each $i = 1, 2, 3, \ldots, k$ we obtain

$$d(p, R_ip) \leqslant d(p, x_n) + d(x_n, R_ix_n) + H(R_ix_n, R_ip) \leqslant d(p, x_n) + d(x_n, w_{n,i}) + d(x_n, p) \to 0 \text{ as } n \to \infty,$$

gives that $d(p, R_ip) = 0, i = 1, 2, 3, \ldots, k$ implies that $p \in R_ip$. Consequently, $p \in \mathbf{F} \neq \emptyset$. □

Corollary 2.2. *Let E be a real Banach space and K a nonempty closed convex subset of E. Let T, S, R, be three multivalued nonexpansive mappings and $\{x_n\}$ be the sequence as defined in (1.3). If $\mathbf{F} \neq \emptyset$ and $Tp = Sp = Rp = \{p\}$ for any $p \in \mathbf{F}$, then $\{x_n\}$ converges strongly to a common fixed point of T, S and R iff $\liminf_{n\to\infty} d(x_n, \mathbf{F}) = 0$.*

Theorem 2.4. *Let E be a uniformly convex Banach space and $K, \{x_n\}$ be as in Lemma (2.2). Let $T_i, S_i, R_i : K \to CB(K)$, $(i = 1, 2, 3, \ldots, k)$ be three finite families multivalued nonexpansive mappings and T_i, S_i and R_i are hemicompact and continuous. If $\mathbf{F} \neq \emptyset$ and $T_ip = S_ip = R_ip = \{p\}$ for any $p \in \mathbf{F}$ then $\{x_n\}$ converges strongly to a common fixed point of T_i, S_i and R_i.*

Proof. Since $\lim_{n\to\infty} d(x_n, T_iy_n) = \lim_{n\to\infty} d(x_n, S_iz_n) = 0 = \lim_{n\to\infty} d(x_n, R_ix_n)$, and T_i, S_i and R_i are hemicompact for each $(i = 1, 2, 3, \ldots, k)$, there is a subsequence $\{x_{n_r}\}$ of $\{x_n\}$ such that $x_{n_r} \to p$ as $r \to \infty$ for some $p \in K$. Since T_i, S_i and R_i are continuous for each $(i = 1, 2, 3, \ldots, k)$, we have

$$d(x_{n_r}, T_i x_{n_r}) \to d(p, T_i p),$$

$$d(x_{n_r}, S_i x_{n_r}) \to d(p, S_i p),$$

and

$$d(x_{n_r}, R_i x_{n_r}) \to d(p, R_i p).$$

As a result, we have that $d(p, T_i p) = d(p, S_i p) = d(p, R_i p) = 0$, for each $(i = 1, 2, 3, \ldots, k)$ and $p \in \mathbf{F}$. Since $\lim_{n\to\infty}\|x_n - p\|$ exists, it follows that $\{x_n\}$ converges strongly to p. This completes the proof.

Corollary 2.3. *Let E be a uniformly convex Banach space and K a nonempty closed convex subset of E. Let T, S, R, be three multivalued nonexpansive mappings and $\{x_n\}$ be the sequence as defined in (1.3) and T, S and R are hemicompact and continuous. If $\mathbf{F} \neq \emptyset$ and $Tp = Sp = Rp = \{p\}$ for any $p \in \mathbf{F}$ then $\{x_n\}$ converges strongly to a common fixed point of T, S and R.*

3. Conclusion

In this work, we obtained a three-step iterative process to approximate the common fixed points of three finite families of multivalued nonexpansive mappings in a uniformly convex real Banach space and establish strong and weak convergence theorems for the proposed iteration process. Our results extend and generalized the recent results in [10,12,13].

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