Journal of the Egyptian Mathematical Society (2014) 22, 466–470

Egyptian Mathematical Society
Journal of the Egyptian Mathematical Society
www.etms-eg.org
www.elsevier.com/locate/joems

ORIGINAL ARTICLE

On sequence ideal using Orlicz function and de la Vallée Poussin mean

Amit Maji *, P.D. Srivastava

Department of Mathematics, Indian Institute of Technology Kharagpur, Kharagpur 721 302, West Bengal, India

Received 26 October 2013; revised 29 November 2013; accepted 10 December 2013
Available online 17 January 2014

KEYWORDS
Sequence ideals;
Orlicz function;
de la Vallée Poussin mean

Abstract In this paper we have introduced a new sequence ideal using Orlicz function and the notion of de la Vallée Poussin mean. It is proved that the Cesáro-Orlicz sequence ideal is complete under a suitable norm. Moreover, it is shown that Cesáro-Orlicz sequence ideal is maximal, and if the Orlicz function satisfies Δ_2-condition at zero then it is also minimal.

2010 MATHEMATICS SUBJECT CLASSIFICATION: 47L20; 46A45; 46E30

1. Introduction

Sequence ideals are investigated by many authors. It has been turned out that the sequence ideals are the important tools to generate operator ideals. Operator ideals have the immense applications in spectral theory, geometry of Banach spaces, theory of eigenvalue distributions, etc. The theory of sequence ideals is an analogue of the theory of operator ideals. Some important examples of sequence ideals are l_p, $0 < p < \infty$, Lorentz sequence ideals, Sargent sequence ideals, Orlicz sequence ideals etc (see [1]). For some recent work on the relations between the sequence ideals and operator ideals, please refer [2,3].

* Corresponding author. Tel.: +91 9093929590.
E-mail addresses: amit.iitm07@gmail.com (A. Maji), pds@maths.iitkgp.ernet.in (P.D. Srivastava).

Peer review under responsibility of Egyptian Mathematical Society.

Production and hosting by Elsevier

In [1, p.189], Orlicz sequence ideals have been defined by using Orlicz function. It is defined as follows:

Let ϕ be an Orlicz function such that $\phi(1) = 1$. Then the Orlicz sequence ideal l_ϕ is defined by

$$l_\phi = \left\{ x = (x_n) \in l_\infty : \sum_{n=1}^{\infty} \phi\left(\frac{|x_n|}{\sigma}\right) \leqslant 1 \quad \text{for some } \sigma > 0 \right\}.$$

The sequence ideal l_ϕ is complete with respect to the norm α_ϕ, where

$$\alpha_\phi(x) = \inf\left\{ \sigma > 0 : \sum_{n=1}^{\infty} \phi\left(\frac{|x_n|}{\sigma}\right) \leqslant 1 \right\}.$$

Moreover, it has been shown that the normed sequence ideal $[l_\phi; \alpha_\phi]$ is maximal, and if the Orlicz function satisfies the Δ_2-condition at zero then $[l_\phi; \alpha_\phi]$ is also minimal.

In summability theory, de la Vallée Poussin mean is firstly used by Leindler [4] and this new idea generates new concepts. Recently, many mathematicians have introduced sequence spaces using de la Vallée Poussin mean and studied their geometrical properties. Let $\lambda = (\lambda_n)$ be a nondecreasing sequence

1110-256X © 2014 Production and hosting by Elsevier B.V. on behalf of Egyptian Mathematical Society. Open access under CC BY-NC-ND license.
http://dx.doi.org/10.1016/j.joems.2013.12.003

of positive real numbers such that $\lambda_1 = 1$, $\lambda_{n+1} \leqslant \lambda_n + 1$ for $n = 1, 2, \ldots$ and $\lambda_n \to \infty$ as $n \to \infty$. Then the generalized de la Vallée Poussin mean (refer [5] or [6]) is defined by

$$t_n(x) = \frac{1}{\lambda_n} \sum_{k \in I_n} x_k, \tag{1.1}$$

where $I_n = [n - \lambda_n + 1, n]$ and $x = (x_k)$ be a scalar sequence. If $\lambda_n = n$, then (1.1) reduces to the Cesáro mean.

The Cesáro-Orlicz sequence space denoted by ces_ϕ, is defined as

$$ces_\phi = \left\{ x = (x_n) \in w : \sum_{n=1}^{\infty} \phi\left(\frac{1}{n} \sum_{k=1}^{n} \frac{|x_k|}{\sigma} \right) < \infty \quad \text{for some } \sigma > 0 \right\},$$

where w is the set of all real or complex sequences. The space ces_ϕ is complete under the Luxemburg norm

$$\|x\|_{ces_\phi} = \inf \left\{ \sigma > 0 : \sum_{n=1}^{\infty} \phi\left(\frac{1}{n} \sum_{k=1}^{n} \frac{|x_k|}{\sigma} \right) \leqslant 1 \right\}.$$

For the details of the sequence space ces_ϕ, one can see [7,8].

In particular, if $\phi(t) = t^p$, $1 < p < \infty$ then ces_ϕ coincides with the Cesáro sequence space ces_p [9].

The purpose of this paper is to introduce a new sequence ideal using the Orlicz function and the notion of de la Vallée Poussin mean. We show that the Cesáro-Orlicz sequence ideal is complete under a suitable norm. Moreover, we prove that the Cesáro-Orlicz sequence ideal is maximal, and if the Orlicz function satisfies Δ_2-condition at zero then the sequence ideal is also minimal.

2. Preliminaries

Definition 2.1 [10]. An Orlicz function $\phi : [0, \infty) \to [0, \infty)$ is a convex, nondecreasing, continuous function on $[0, \infty)$ such that $\phi(0) = 0$, $\phi(t) > 0$ for $t > 0$ and $\phi(t) \to \infty$ as $t \to \infty$.

For each Orlicz function ϕ, there always exists an integral representation, i.e., $\phi(x) = \int_0^x p(t)dt$, where p is called the kernel of ϕ. The kernel $p(t)$ is nondecreasing and right continuous for $t \geqslant 0$, and $p(0) = 0$, $p(t) > 0$ for $t > 0$, $p(t) \to \infty$ as $t \to \infty$.

Definition 2.2 [10]. An Orlicz function ϕ is said to satisfy Δ_2-condition at zero if there exist $K > 0$ and $t_0 > 0$ such that $\phi(2t) \leqslant K\phi(t)$ for all $t \in [0, t_0]$.

Throughout the paper we consider ϕ to be an Orlicz function and $\lambda = (\lambda_n)$ a nondecreasing sequence of positive real numbers satisfying $\lambda_1 = 1$, $\lambda_{n+1} \leqslant \lambda_n + 1$ for $n = 1, 2, \ldots$, and $\lambda_n \to \infty$ as $n \to \infty$. We denote by l_∞, c_0, $\mathbb{N}$ and $\mathbb{R}^+$ for the set of all bounded scalar sequences, the space of null sequences, the set of all natural numbers and the set of all non-negative real numbers respectively. e_k stand for k-th unit sequence.

Definition 2.3 [1, p. 23]. Let π be any one-to-one map from $\mathbb{N}$ into $\mathbb{N}$. Then the canonical injection $J_\pi : l_\infty \to l_\infty$ is defined by $J_\pi(x_n)_{n=1}^\infty = \left(x_{\pi^{-1}(m)}\right)_{m=1}^\infty$, where $\pi(n) = m$ and $x_\emptyset = 0$, ϕ is an empty set. The canonical surjection $Q_\pi : l_\infty \to l_\infty$ is defined by $Q_\pi(x_m)_{m=1}^\infty = (x_{\pi(n)})_{n=1}^\infty$. In particular, if $\pi(n) = 2n$ then $J_\pi(x_n)_{n=1}^\infty = (0, x_1, 0, x_2, 0, x_3, \ldots)$ and $Q_\pi(x_n)_{n=1}^\infty = (x_2, x_4, x_6, \ldots)$.

Definition 2.4 [1]. A sequence ideal $\mathfrak{a}$ on the scalar field (real or complex) is a subset of l_∞ satisfying the following conditions:

1. $e_1 \in \mathfrak{a}$, where e_1 is the 1-st unit sequence
2. if $x, y \in \mathfrak{a}$, then $x + y \in \mathfrak{a}$
3. if $z \in l_\infty$ and $x \in \mathfrak{a}$, then $zx \in \mathfrak{a}$
4. if π is any one to one map from $\mathbb{N}$ into $\mathbb{N}$, then $J_\pi x \in \mathfrak{a}$ and $Q_\pi x \in \mathfrak{a}$ for all $x \in \mathfrak{a}$.

Definition 2.5 [1]. Let $\mathfrak{a}$ be a sequence ideal on the scalar field. A map $\alpha : \mathfrak{a} \to \mathbb{R}^+$ is called a norm if the following conditions are satisfied:

1. $\alpha(e_1) = 1$
2. $\alpha(x + y) \leqslant \alpha(x) + \alpha(y)$ for all $x, y \in \mathfrak{a}$
3. if $z \in l_\infty$ and $x \in \mathfrak{a}$ then $\alpha(zx) \leqslant \|z\|_\infty \alpha(x)$, where $\|z\|_\infty = \sup_{k \geqslant 1} |z_k|$
4. if π is any one to one map from $\mathbb{N}$ into $\mathbb{N}$, then $\alpha(J_\pi x) \leqslant \alpha(x)$ and $\alpha(Q_\pi x) \leqslant \alpha(x)$ for all $x \in \mathfrak{a}$.

Definition 2.6 [1]. A normed sequence ideal $[\mathfrak{a}; \alpha]$ is a sequence ideal $\mathfrak{a}$ such that $\mathfrak{a}$ is complete under the norm α.

Definition 2.7 [1]. Let $[\mathfrak{a}; \alpha]$ be a normed sequence ideal. A sequence x belongs to minimal kernel $\mathfrak{a}^{min}$ if $x = zx_0$ for some $z \in c_0$ and $x_0 \in \mathfrak{a}$. The sequence ideal $\mathfrak{a}^{min}$ is complete under the norm α^{min}, where α^{min} is defined by

$$\alpha^{min}(x) = \inf\{\|z\|_\infty \alpha(x_0)\},$$

here the infimum is taken over all possible factorizations.

A normed sequence ideal $[\mathfrak{a}; \alpha]$ is called minimal if $[\mathfrak{a}; \alpha] = [\mathfrak{a}; \alpha]^{min}$, where $[\mathfrak{a}; \alpha]^{min} = [\mathfrak{a}^{min}; \alpha^{min}]$.

Definition 2.8 [1]. Let $[\mathfrak{a}; \alpha]$ be a normed sequence ideal. A sequence x belongs to maximal kernel $\mathfrak{a}^{max}$ if $zx \in \mathfrak{a}$ for all $z \in c_0$. The sequence ideal $\mathfrak{a}^{max}$ is complete with respect to the norm α^{max}, where α^{max} is defined by

$$\alpha^{max}(x) = \sup\{\alpha(zx) : z \in c_0 \quad \text{and} \quad \|z\|_\infty \leqslant 1\}.$$

A normed sequence ideal $[\mathfrak{a}; \alpha]$ is called maximal if $[\mathfrak{a}; \alpha] = [\mathfrak{a}; \alpha]^{max}$, where $[\mathfrak{a}; \alpha]^{max} = [\mathfrak{a}^{max}; \alpha^{max}]$.

Lemma 2.1 [1]. *Let $[\mathfrak{a}; \alpha]$ be a normed sequence ideal. Then $\mathfrak{a}^{max}$ consists of those sequences $x = (x_n) \in l_\infty$ such that $P_m x$ is α-bounded, and*

$$\alpha^{max}(x) = \sup_{m \geqslant 1} \alpha(P_m x),$$

where $P_m(x_1, x_2, \ldots, x_m, x_{m+1}, \ldots) = (x_1, x_2, \ldots, x_m, 0, 0, \ldots)$.

Lemma 2.2 [1]. *Let $[\mathfrak{a}; \alpha]$ be a normed sequence ideal. Then $\mathfrak{a}^{min}$ consists of those sequences $x \in \mathfrak{a}$ such that $x = \alpha - \lim_m P_m x$.*

3. Main results

Let $\mathfrak{a}_{\phi,\lambda}$ be a set consisting of all the sequences $x = (x_n) \in l_\infty$ such that

$$\sum_{n=1}^{\infty}\phi\left(\frac{1}{\lambda_n}\sum_{k\in I_n}\frac{|x_k|}{\sigma}\right) \leqslant 1$$

for some $\sigma > 0$.

Theorem 3.1. *If $\sum_{n=1}^{\infty}\phi\left(\frac{1}{\lambda_n}\right) \leqslant 1$, then the set $\mathfrak{a}_{\phi,\lambda}$ is a sequence ideal.*

Proof. Since $\sum_{n=1}^{\infty}\phi\left(\frac{1}{\lambda_n}\right) \leqslant 1$, we have $e_1 = (1,0,0,\dots,) \in \mathfrak{a}_{\phi,\lambda}$.

Let $x, y \in \mathfrak{a}_{\phi,\lambda}$. Then there exist $\sigma_1 > 0$ and $\sigma_2 > 0$ such that

$$\sum_{n=1}^{\infty}\phi\left(\frac{1}{\lambda_n}\sum_{k\in I_n}\frac{|x_k|}{\sigma_1}\right) \leqslant 1 \quad \text{and} \quad \sum_{n=1}^{\infty}\phi\left(\frac{1}{\lambda_n}\sum_{k\in I_n}\frac{|y_k|}{\sigma_2}\right) \leqslant 1.$$

Now

$$\begin{aligned}\sum_{n=1}^{\infty}\phi\left(\frac{1}{\lambda_n}\sum_{k\in I_n}\frac{|x_k+y_k|}{\sigma_1+\sigma_2}\right) &\leqslant \sum_{n=1}^{\infty}\phi\left(\frac{1}{\lambda_n}\sum_{k\in I_n}\frac{|x_k|}{\sigma_1+\sigma_2}+\frac{1}{\lambda_n}\sum_{k\in I_n}\frac{|y_k|}{\sigma_1+\sigma_2}\right)\\ &= \sum_{n=1}^{\infty}\phi\left(\frac{\sigma_1}{\sigma_1+\sigma_2}\left(\frac{1}{\lambda_n}\sum_{k\in I_n}\frac{|x_k|}{\sigma_1}\right)+\frac{\sigma_2}{\sigma_1+\sigma_2}\left(\frac{1}{\lambda_n}\sum_{k\in I_n}\frac{|y_k|}{\sigma_2}\right)\right)\\ &\leqslant \frac{\sigma_1}{\sigma_1+\sigma_2}\sum_{n=1}^{\infty}\phi\left(\frac{1}{\lambda_n}\sum_{k\in I_n}\frac{|x_k|}{\sigma_1}\right)+\frac{\sigma_2}{\sigma_1+\sigma_2}\sum_{n=1}^{\infty}\phi\left(\frac{1}{\lambda_n}\sum_{k\in I_n}\frac{|y_k|}{\sigma_2}\right)\\ &\leqslant \frac{\sigma_1}{\sigma_1+\sigma_2}+\frac{\sigma_2}{\sigma_1+\sigma_2} = 1.\end{aligned}$$

Thus $x + y \in \mathfrak{a}_{\phi,\lambda}$.

Let $x \in \mathfrak{a}_{\phi,\lambda}$. Then there exists some $\sigma > 0$ such that $\sum_{n=1}^{\infty}\phi\left(\frac{1}{\lambda_n}\sum_{k\in I_n}\frac{|x_k|}{\sigma}\right) \leqslant 1$.

Let $z = (z_k) \in l_\infty$. Then there exists $M > 0$ such that $|z_k| \leqslant M$ for all k. As ϕ is a nondecreasing function and $\frac{|z_k x_k|}{M\sigma} \leqslant \frac{|x_k|}{\sigma}$, we have

$$\sum_{n=1}^{\infty}\phi\left(\frac{1}{\lambda_n}\sum_{k\in I_n}\frac{|z_k x_k|}{M\sigma}\right) \leqslant \sum_{n=1}^{\infty}\phi\left(\frac{1}{\lambda_n}\sum_{k\in I_n}\frac{|x_k|}{\sigma}\right) \leqslant 1.$$

Hence $zx \in \mathfrak{a}_{\phi,\lambda}$.

Let $\pi : \mathbb{N} \to \mathbb{N}$ be any one to one map. Choose suitable $M > 0$ such that

$$\sum_{n=1}^{\infty}\phi\left(\frac{1}{\lambda_n}\sum_{k\in I_n}\frac{|x_{\pi^{-1}(k)}|}{M\sigma}\right) \leqslant \sum_{n=1}^{\infty}\phi\left(\frac{1}{\lambda_n}\sum_{k\in I_n}\frac{|x_k|}{\sigma}\right) \leqslant 1.$$

Hence $J_\pi x \in \mathfrak{a}_{\phi,\lambda}$.

Similarly, we have $Q_\pi x \in \mathfrak{a}_{\phi,\lambda}$. Thus $\mathfrak{a}_{\phi,\lambda}$ is a sequence ideal. □

Remark 3.1. In particular, if $\lambda_n = n$ for $n = 1, 2, \dots$, and if $\sum_{n=1}^{\infty}\phi\left(\frac{1}{n}\right) \leqslant 1$ then the sequence ideal $\mathfrak{a}_{\phi,\lambda}$ reduces to a sequence ideal

$$\left\{x = (x_n) \in l_\infty : \sum_{n=1}^{\infty}\phi\left(\frac{1}{n}\sum_{k=1}^{n}\frac{|x_k|}{\sigma}\right) \leqslant 1\right\}.$$

We shall call it Cesáro-Orlicz sequence ideal and denote it by $\mathfrak{a}_{ces_\phi}$.

Let $\mathfrak{a}_{ces_\phi}$ be a sequence ideal. Define a function $\alpha_{ces_\phi} : \mathfrak{a}_{ces_\phi} \to \mathbb{R}^+$ by

$$\alpha_{ces_\phi}(x) = \inf\left\{\sigma > 0 : \sum_{n=1}^{\infty}\phi\left(\frac{1}{n}\sum_{k=1}^{n}\frac{|x_k|}{\sigma}\right) \leqslant 1\right\}.$$

Theorem 3.2. *If $\sum_{n=1}^{\infty}\phi\left(\frac{1}{n}\right) = 1$, then the function α_{ces_ϕ} is a norm on the sequence ideal $\mathfrak{a}_{ces_\phi}$.*

Proof. Since $\sum_{n=1}^{\infty}\phi\left(\frac{1}{n}\right) = 1$, we have $\alpha_{ces_\phi}(e_1) = \inf\left\{\sigma > 0 : \sum_{n=1}^{\infty}\phi\left(\frac{1}{\sigma n}\right) \leqslant 1\right\} = 1$.

Let $x, y \in \mathfrak{a}_{ces_\phi}$ and ϵ be any positive real number. Choose $\sigma_1 > 0$ and $\sigma_2 > 0$ such that

$$\sigma_1 \leqslant \alpha_{ces_\phi}(x) + \frac{\epsilon}{2} \qquad \text{and} \qquad \sigma_2 \leqslant \alpha_{ces_\phi}(y) + \frac{\epsilon}{2}.$$

Now

$$\begin{aligned}\sum_{n=1}^{\infty}\phi\left(\frac{1}{n}\sum_{k=1}^{n}\frac{|x_k+y_k|}{\sigma_1+\sigma_2}\right) &\leqslant \frac{\sigma_1}{\sigma_1+\sigma_2}\sum_{n=1}^{\infty}\phi\left(\frac{1}{n}\sum_{k=1}^{n}\frac{|x_k|}{\sigma_1}\right)\\ &\quad+\frac{\sigma_2}{\sigma_1+\sigma_2}\sum_{n=1}^{\infty}\phi\left(\frac{1}{n}\sum_{k=1}^{n}\frac{|y_k|}{\sigma_2}\right)\\ &\leqslant \frac{\sigma_1}{\sigma_1+\sigma_2}+\frac{\sigma_2}{\sigma_1+\sigma_2} = 1.\end{aligned}$$

Therefore

$$\alpha_{ces_\phi}(x+y) \leqslant \sigma_1+\sigma_2 \leqslant \alpha_{ces_\phi}(x)+\alpha_{ces_\phi}(y)+\epsilon.$$

As $\epsilon > 0$ is any arbitrary number, we have

$$\alpha_{ces_\phi}(x+y) \leqslant \alpha_{ces_\phi}(x)+\alpha_{ces_\phi}(y)$$

for all $x, y \in \mathfrak{a}_{ces_\phi}$.

Let $z = (z_k) \in l_\infty$ and $x \in \mathfrak{a}_{ces_\phi}$. Suppose that ϵ is any positive real number. Then there exists some $\sigma > 0$ such that

$$\sum_{n=1}^{\infty}\phi\left(\frac{1}{n}\sum_{k=1}^{n}\frac{|x_k|}{\sigma}\right) \leqslant 1 \quad \text{and} \quad \sigma \leqslant \alpha_{ces_\phi}(x)+\epsilon.$$

Since $|z_k| \leqslant \|z\|_\infty$ for all k, we get $\frac{1}{n}\sum_{k=1}^{n}\frac{|z_k x_k|}{\sigma\|z\|_\infty} \leqslant \frac{1}{n}\sum_{k=1}^{n}\frac{|x_k|}{\sigma}$.

Thus

$$\sum_{n=1}^{\infty}\phi\left(\frac{1}{n}\sum_{k=1}^{n}\frac{|z_k x_k|}{\sigma\|z\|_\infty}\right) \leqslant \sum_{n=1}^{\infty}\phi\left(\frac{1}{n}\sum_{k=1}^{n}\frac{|x_k|}{\sigma}\right) \leqslant 1.$$

Therefore

$$\alpha_{ces_\phi}(zx) \leqslant \|z\|_\infty\sigma \leqslant \|z\|_\infty(\alpha_{ces_\phi}(x)+\epsilon).$$

As $\epsilon > 0$ is any arbitrary number, we have $\alpha_{ces_\phi}(zx) \leqslant \|z\|_\infty\alpha_{ces_\phi}(x)$.

Finally, let $\pi : \mathbb{N} \to \mathbb{N}$ be any one-to-one map, and $x \in \mathfrak{a}_{ces_\phi}$. Then $\alpha_{ces_\phi}(J_\pi x) \leqslant \alpha_{ces_\phi}(x)$ and $\alpha_{ces_\phi}(Q_\pi x) \leqslant \alpha_{ces_\phi}(x)$. This completes the proof. □

Theorem 3.3. *The sequence ideal $\mathfrak{a}_{ces_\phi}$ is complete with the norm α_{ces_ϕ}, i.e., $[\mathfrak{a}_{ces_\phi}; \alpha_{ces_\phi}]$ is a normed sequence ideal.*

Proof. Let $(x^m)_{m=1}^{\infty}$ be a Cauchy sequence in $\mathfrak{a}_{ces_\phi}$, where $x^m = (x_k^m)_{k=1}^{\infty} = (x_1^m, x_2^m, \ldots) \in \mathfrak{a}_{ces_\phi}$. Then for $\epsilon > 0$, there exists $n_0 \in \mathbb{N}$ such that

$$\alpha_{ces_\phi}(x^l - x^m) < \epsilon \qquad \text{for all } m, l \geqslant n_0. \tag{3.1}$$

Since ϕ is nondecreasing in nature, we have

$$\sum_{n=1}^{\infty}\phi\left(\frac{1}{n\epsilon}\sum_{k=1}^{n}|x_k^l - x_k^m|\right) \leqslant \sum_{n=1}^{\infty}\phi\left(\frac{1}{n}\frac{\sum_{k=1}^{n}|x_k^l - x_k^m|}{\alpha_{ces_\phi}(x^l - x^m)}\right) \leqslant 1.$$

Thus

$$\sum_{n=1}^{\infty}\phi\left(\frac{1}{n\epsilon}\sum_{k=1}^{n}|x_k^l - x_k^m|\right) \leqslant 1 \tag{3.2}$$

for all $l, m \geqslant n_0$.

Now, we consider the integral representation of the Orlicz function ϕ, i.e., $\phi(x) = \int_0^x p(t)dt$. Choose $s_0 > 0$ such that $\frac{s_0}{2}p(\frac{s_0}{2}) \geqslant 1$. We have from (3.2)

$$\sum_{n=1}^{\infty}\phi\left(\frac{1}{n\epsilon}\sum_{k=1}^{n}|x_k^l - x_k^m|\right) \leqslant \frac{s_0}{2}p\left(\frac{s_0}{2}\right). \tag{3.3}$$

Now we establish that

$$\frac{1}{n\epsilon}\sum_{k=1}^{n}|x_k^l - x_k^m| \leqslant s_0. \tag{3.4}$$

Suppose that (3.4) is not true, i.e.,

$$\frac{1}{n\epsilon}\sum_{k=1}^{n}|x_k^l - x_k^m| > s_0 > \frac{s_0}{2}.$$

Then

$$\phi\left(\frac{1}{n\epsilon}\sum_{k=1}^{n}|x_k^l - x_k^m|\right) = \int_0^{\frac{1}{n\epsilon}\sum_{k=1}^{n}|x_k^l - x_k^m|} p(t)dt$$
$$\geqslant \int_{\frac{s_0}{2}}^{\frac{1}{n\epsilon}\sum_{k=1}^{n}|x_k^l - x_k^m|} p(t)dt > \frac{s_0}{2}p\left(\frac{s_0}{2}\right),$$

which is a contradiction to (3.3). Hence

$$\frac{1}{n\epsilon}\sum_{k=1}^{n}|x_k^l - x_k^m| \leqslant s_0 \Rightarrow \sum_{k=1}^{n}|x_k^l - x_k^m| \leqslant ns_0\epsilon \quad \Rightarrow |x_k^l - x_k^m| \leqslant ns_0\epsilon \to 0 \text{ as } \epsilon \to 0.$$

Thus $(x_k^m)_{m=1}^{\infty}$ is a Cauchy sequence of scalars for each k and hence convergent. Let $x = (x_k)_{k=1}^{\infty}$, where $x_k = \lim_{m\to\infty} x_k^m$. Using the continuity of ϕ and taking $m \to \infty$ in (3.1), we have

$$\alpha_{ces_\phi}(x^l - x) \to 0 \text{ as } \epsilon \to 0.$$

This shows that the sequence (x^l) converges to x under the norm α_{ces_ϕ}.

Now we show that $x \in \mathfrak{a}_{ces_\phi}$.

Clearly for $l \geqslant n_0$, $x^l - x \in \mathfrak{a}_{ces_\phi}$. So, $x = x^{n_0} - (x^{n_0} - x) \in \mathfrak{a}_{ces_\phi}$. This establishes the proof. □

Let $[\mathfrak{a}_{ces_\phi}; \alpha_{ces_\phi}]$ be a normed sequence ideal. The maximal and minimal kernel of $\mathfrak{a}_{ces_\phi}$ are defined as

$$\mathfrak{a}_{ces_\phi}^{max} = \{x = (x_n) \in l_\infty : zx \in \mathfrak{a}_{ces_\phi} \quad \text{for all} \quad z \in c_o\},$$

and

$$\mathfrak{a}_{ces_\phi}^{min} = \{x = (x_n) \in l_\infty : x = zx_0 \quad \text{for some} \quad z \in c_o \text{ and } x_0 \in \mathfrak{a}_{ces_\phi}\}.$$

Then $\mathfrak{a}_{ces_\phi}^{max}$ and $\mathfrak{a}_{ces_\phi}^{min}$ are normed sequence ideals equipped with the norms $\alpha_{ces_\phi}^{max}$ and $\alpha_{ces_\phi}^{min}$ respectively, where

$$\alpha_{ces_\phi}^{max}(x) = \sup\{\alpha_{ces_\phi}(zx) : z \in c_0 \text{ and } \|z\|_\infty \leqslant 1\}$$

and

$$\alpha_{ces_\phi}^{min} = \inf\{\|z\|_\infty \alpha_{ces_\phi}(x_0)\}.$$

Note 3.1. Let $\mathfrak{a}$ be any normed sequence ideal. We claim that $\mathfrak{a} \subseteq \mathfrak{a}^{max}$.

Let $x \in \mathfrak{a}$ and $z \in c_0 \subset l_\infty$. Then $zx \in \mathfrak{a}$ for all $z \in c_0$ as $\mathfrak{a}$ is a sequence ideal. Hence $x \in \mathfrak{a}^{max}$. Thus $\mathfrak{a} \subseteq \mathfrak{a}^{max}$. Similarly, one can see that $\mathfrak{a}^{min} \subseteq \mathfrak{a}$.

For a Cesáro -Orlicz sequence ideal $\mathfrak{a}_{ces_\phi}$, we have $\mathfrak{a}_{ces_\phi} \subseteq \mathfrak{a}_{ces_\phi}^{max}$ and $\mathfrak{a}_{ces_\phi}^{min} \subseteq \mathfrak{a}_{ces_\phi}$.

The next results concern about maximal and minimal normed sequence ideal.

Theorem 3.4. *The Cesáro-Orlicz sequence ideal* $[\mathfrak{a}_{ces_\phi}; \alpha_{ces_\phi}]$ *is a maximal normed sequence ideal.*

Proof. To prove that the normed sequence ideal $[\mathfrak{a}_{ces_\phi}; \alpha_{ces_\phi}]$ is maximal, we show

$$\mathfrak{a}_{ces_\phi}^{max} = \mathfrak{a}_{ces_\phi}.$$

Clearly $\mathfrak{a}_{ces_\phi} \subseteq \mathfrak{a}_{ces_\phi}^{max}$.

Now applying Lemma 2.1, we establish the reverse inclusion, i.e., $\mathfrak{a}_{ces_\phi}^{max} \subseteq \mathfrak{a}_{ces_\phi}$.

Let $x \in \mathfrak{a}_{ces_\phi}^{max}$. Then $\sup_{m\geqslant 1}\alpha_{ces_\phi}(P_m x) < \infty$. We show that for some $\sigma > 0$

$$\sum_{n=1}^{\infty}\phi\left(\frac{1}{n}\sum_{k=1}^{n}\frac{|x_k|}{\sigma}\right) \leqslant 1.$$

Now, choose $\sigma > \sup_{m\geqslant 1}\alpha_{ces_\phi}(P_m x)$. Then we get

$$\sum_{n=1}^{m}\phi\left(\frac{1}{n}\sum_{k=1}^{n}\frac{|x_k|}{\sigma}\right) + \sum_{n=m+1}^{\infty}\phi\left(\frac{1}{n}\sum_{k=1}^{m}\frac{|x_k|}{\sigma}\right) \leqslant 1$$

for all $m = 1, 2, \ldots$.

Thus we have

$$\sum_{n=1}^{\infty}\phi\left(\frac{1}{n}\sum_{k=1}^{n}\frac{|x_k|}{\sigma}\right) \leqslant 1.$$

Therefore $x \in \mathfrak{a}_{ces_\phi}$. Hence $\mathfrak{a}_{ces_\phi}^{max} \subseteq \mathfrak{a}_{ces_\phi}$. This completes the proof. □

Theorem 3.5. *If the Orlicz function* ϕ *satisfies the* Δ_2*-condition at zero, then the normed sequence ideal* $[\mathfrak{a}_{ces_\phi}; \alpha_{ces_\phi}]$ *is minimal.*

Proof. Clearly $\mathfrak{a}_{ces_\phi}^{min} \subseteq \mathfrak{a}_{ces_\phi}$.

Now applying Lemma 2.2, we prove that $\mathfrak{a}_{ces_\phi} \subseteq \mathfrak{a}_{ces_\phi}^{min}$.

Let $x \in \mathfrak{a}_{ces_\phi}$. Since ϕ satisfies Δ_2-condition at zero, we have

$$\sum_{n=1}^{\infty} \phi\left(\frac{1}{n}\sum_{k=1}^{n} \frac{|x_k|}{\sigma}\right) < \infty$$

for all $\sigma > 0$.

Now choose $m \in \mathbb{N}$ such that

$$\sum_{n=m+1}^{\infty} \phi\left(\frac{1}{n}\sum_{k=m+1}^{n} \frac{|x_k|}{\sigma}\right) \leqslant 1.$$

Thus $\alpha_{ces_\phi}(x - P_m x) \leqslant \sigma$. Therefore $x = \alpha_{ces_\phi} - \lim_m P_m x$ and hence $x \in \mathfrak{a}_{ces_\phi}^{min}$.

Thus $\mathfrak{a}_{ces_\phi} = \mathfrak{a}_{ces_\phi}^{min}$. This proves the theorem. □

Acknowledgments

The authors are thankful to the referees for their valuable comments and suggestions which improved the presentation of the paper. The first author is grateful to CSIR, New Delhi, Govt. of India for the financial support award no. 09/081(1120)/2011-EMR-I.

References

[1] A. Pietsch, Operator Ideals, VEB Deutscher Verlag der Wissenschaften, Berlin, 1978.
[2] G. Levitina, A. Pietsch, FA. Sukochev, D. Zanin, Completeness of quasi-normed operator ideals generated by s-numbers, Indag. Math. (2013), http://dx.doi.org/10.1016/j.indag.2013.07.005.
[3] A. Pietsch, Traces on operator ideals and related linear forms on sequence ideals (part I), Indag. Math. (2012), http://dx.doi.org/10.1016/j.indag.2012.08.008.
[4] L. Leindler, Über die verallgemeinerte de la Vallée-Poussinsche summierbarkeit allgemeiner Orthogonalreihen, Acta Math. Acad. Sci. Hungar. 16 (1965) 375–387.
[5] E. Savaş, R. Savaş, Some λ-sequence spaces defined by Orlicz functions, Indian J. Pure Appl. Math. 34 (12) (2003) 1673–1680.
[6] R. Çolak, Ç.A. Bektaş, λ-Statistical convergence of order α, Acta Math. Sci. Ser. B: Engl. Ed. 31 (3) (2011) 953–959.
[7] A. Kaminska, D. Kubiak, On isometric copies of l_∞ and James constants in Cesáro-Orlicz sequence spaces, J. Math. Anal. Appl. 372 (2) (2010) 574–584.
[8] D. Kubiak, A note on Cesáro-Orlicz sequence space, J. Math. Anal. Appl. 349 (1) (2009) 291–296.
[9] J.S. Shiue, On the Cesáro sequence spaces, Tamkang J. Math. 1 (1) (1970) 19–25.
[10] J. Lindenstrauss, L. Tzafriri, Classical Banach Spaces I. Sequence Spaces, Springer-Verlag, Berlin, 1977.