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SHORT COMMUNICATION

An instability theorem for a certain sixth order nonlinear delay differential equation

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KEYWORDS

Instability;
Lyapunov–Krasovskii functional;
Delay differential equation;
Sixth order

Abstract The aim of this paper is to establish an instability theorem for a certain sixth order nonlinear delay differential equation. The proof of the theorem is based on the use of Lyapunov–Krasovskii functional approach. By this work, we improve an instability result obtained in the literature for a certain sixth order nonlinear differential equation without delay to the instability of the zero solution of a certain sixth order nonlinear delay differential equation.

1. Introduction

In 1982, Ezeilo [2] proved an instability theorem for the sixth order nonlinear differential equation without delay,

$$\begin{aligned}&x^{(6)}(t)+a_1x^{(5)}(t)+a_2x^{(4)}(t)\\&\quad+e(x(t),x'(t),x''(t),x'''(t),x^{(4)}(t),x^{(5)}(t))x'''(t)\\&\quad+f(x'(t))x''(t)+g(x(t),x'(t),x''(t),x'''(t),x^{(4)}(t),x^{(5)}(t))x'(t)\\&\quad+h(x(t))=0.\end{aligned}\tag{1}$$

In this paper, instead of Eq. (1), we consider the sixth order nonlinear delay differential equation

$$\begin{aligned}&x^{(6)}(t)+a_1x^{(5)}(t)+a_2x^{(4)}(t)+e(x(t-r),x'(t-r),x''(t\\&\quad-r),x'''(t-r),x^{(4)}(t-r),x^{(5)}(t-r))x'''(t)\\&\quad+f(x'(t))x''(t)+g(x(t-r),x'(t-r),x''(t-r),x'''(t\\&\quad-r),x^{(4)}(t-r),x^{(5)}(t-r))x'(t)+h(x(t-r))=0.\end{aligned}\tag{2}$$

We write Eq. (2) in system form as

$$\begin{aligned}x_1'&=x_2,\\x_2'&=x_3,\\x_3'&=x_4,\\x_4'&=x_5,\\x_5'&=x_6,\\x_6'&=-a_1x_6-a_2x_5-e(x_1(t-r),x_2(t-r),x_3(t-r),\\&\quad x_4(t-r),x_5(t-r),x_6(t-r))x_4(t-r)-f(x_2)x_3\\&\quad-g(x_1(t-r),x_2(t-r),x_3(t-r),x_4(t-r),x_5(t-r),\\&\quad x_6(t-r))x_2-h(x_1)+\textstyle\int_{t-r}^{t}h'(x_1(s))x_2(s)ds,\end{aligned}\tag{3}$$

which is obtained as usual by setting $x=x_1$, $x'=x_2$, $x''=x_3$, $x'''=x_4$, $x^{(4)}=x_5$ and $x^{(5)}=x_6$ in (2), where r is a positive constant, a_1, a_2 are some constants, the primes in Eq. (2) denote differentiation with respect to t, $t\in\Re_+,\Re_+=[0,\infty)$; e,

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f, g and h are continuous functions on $\Re^6$, $\Re$, $\Re^6$ and $\Re$, respectively, with $h(0) = 0$, and satisfy a Lipschitz condition in their respective arguments. Hence, the existence and uniqueness of the solutions of Eq. (2) are guaranteed (see Èl'sgol'ts [1, pp. 14–15]). We assume in what follows that the function h is differentiable, and $x_1(t)$, $x_2(t)$, $x_3(t)$, $x_4(t)$, $x_5(t)$ and $x_6(t)$ are abbreviated as x_1, x_2, x_3, x_4, x_5 and x_6, respectively.

To the best of our observations, since 1982 by now, the instability of solutions of various sixth order nonlinear scalar and vector differential equations without delay has been discussed and is still being investigated in the literature (see, for example, Ezeilo [2], Tiryaki [5], Tunç [6–10] and Tunç and Tunç [11]). The motivation to write this paper comes from the mentioned papers done for sixth order nonlinear differential equations without delay. Our purpose is to obtain the result established in [2] to nonlinear delay differential equation given in (2) for the instability of its zero solution. On the other hand, to the best of our knowledge, we did not find any work on the instability of the solutions of sixth order linear and nonlinear delay differential equations in the literature. The basic reason for the lack of any paper on this topic may be the difficulty of the construction or definition of appropriate Lyapunov functions or functionals for the instability problems relative to the higher order delay differential equations. Here, by defining an appropriate Lyapunov functional we carry out our purpose. This paper is the first attempt and work on the topic in the literature.

Let $r \geqslant 0$ be given, and let $C = C([-r, 0], \Re^n)$ with

$$\|\phi\| = \max_{-r \leqslant s \leqslant 0} | \phi(s) |, \quad \phi \in C.$$

For $H > 0$, define $C_H \subset C$ by

$$C_H = \{\phi \in C : \|\phi\| < H\}.$$

If $x : [-r, a] \to \Re^n$ is continuous, $0 < A \leqslant \infty$, then, for each t in $[0, A)$, x_t in C is defined by

$$x_t(s) = x(t + s), \quad -r \leqslant s \leqslant 0, \quad t \geqslant 0.$$

Let G be an open subset of C and consider the general autonomous differential system with finite delay

$$\dot{x} = F(x_t), x_t = x(t + \theta), \quad -r \leqslant \theta \leqslant 0, \quad t \geqslant 0,$$

where $F : G \to \Re^n$ is a continuous function that maps closed and bounded sets into bounded sets. It follows from the conditions on F that each initial value problem

$$\dot{x} = F(x_t), \quad x_0 = \phi \in G$$

has a unique solution defined on some interval $[0, A)$, $0 < A \leqslant \infty$. This solution will be denoted by $x(\phi)(.)$ so that $x_0(\phi) = \phi$.

Definition 1. The zero solution, $x = 0$, of $\dot{x} = F(x_t)$ is stable if for each $\varepsilon > 0$ there exists $\delta = \delta(\varepsilon) > 0$ such that $\|\phi\| < \delta$ implies that $|x(\phi)(t)| < \varepsilon$ for all $t \geqslant 0$. The zero solution is said to be unstable if it is not stable.

Theorem A. *Suppose there exists a Lyapunov function $V : G \to \Re_+$ such that $V(0) = 0$ and $V(x) > 0$ if $x \neq 0$. If either*

(i) $\dot{V}(\phi) > 0$ for all ϕ in G for which

$$V[\phi(0)] = \max_{-s \leqslant t \leqslant 0} V[\phi(s)] > 0$$

or

(ii) $\dot{V}(\phi) > 0$ for all ϕ in G for which

$$V[\phi(0)] = \min_{-s \leqslant t \leqslant 0} V[\phi(s)] > 0,$$

then the solution $x = 0$ of $\dot{x} = F(x_t)$ is unstable (see Haddock and Ko [3]).

2. The main results

Our main result is the following theorem.

Theorem. *Together with all the assumptions imposed to the functions e, f, g and h in Eq. (2), we assume that there exist constants a_1 ($\neq 0$), a_6 and δ such that the following conditions hold:*

$$h(0) = 0, h(x_1) \neq 0 \quad \text{for } x_1 \neq 0, 0 < h'(x_1) \leqslant a_6,$$
$$g(.)sgna_1 - \frac{1}{4|a_1|} e^2(.) \geqslant \delta > 0.$$

Then, the zero solution, $x = 0$, of Eq. (2) is unstable provided that

$$r < \frac{\delta}{a_6}.$$

Proof. Consider the function $V = V(x_1, x_2, x_3, x_4, x_5, x_6)$ given by

$$V = -x_2x_6 - a_1x_2x_5 - a_2x_2x_4 + \frac{1}{2}a_2x_3^2 + x_3x_5 + a_1x_3x_4 - \frac{1}{2}x_4^2 - \int_0^{x_2} f(s)sds - \int_0^{x_1} h(\xi)d\xi. \tag{4}$$

We define the Lyapunov functional

$$V_1 = Vsgna_1 - \lambda_1 \int_{-r}^{0} \int_{t+s}^{t} x_2^2(\theta)d\theta ds, \tag{5}$$

where s is a real variable such that the integral $\int_{-r}^{0} \int_{t+s}^{t} x_2^2(\theta)d\theta ds$ is non-negative, and λ_1 is a positive constant which will be determined later in the proof.

Let $k \equiv 1 + |a_2|$. Then, it is obvious that

$$V_1(0, 0, \varepsilon sgna_1, 0, k\varepsilon, 0) = \varepsilon^2 \left(k + \frac{1}{2}a_2sgna_1\right) > 0$$

for arbitrary ε so that every neighborhood of the origin in the $(x_1, x_2, x_3, x_4, x_5, x_6)-$ space contains points $(\xi_1, \xi_2, \xi_3, \xi_4, \xi_5, \xi_6)$ such that $V_1(\xi_1, \xi_2, \xi_3, \xi_4, \xi_5, \xi_6) > 0$.

Let

$$(x_1, x_2, x_3, x_4, x_5, x_6) = (x_1(t), x_2(t), x_3(t), x_4(t), x_5(t), x_6(t))$$

be an arbitrary solution of (3). By an elementary differentiation, time derivative of the functional V_1 in (5) along the solutions of (3) gives that

$$\begin{aligned}\dot{V}_1 &\equiv \frac{d}{dt}V_1(x_{1t}, x_{2t}, x_{3t}, x_{4t}, x_{5t}, x_{6t})\\ &= \{a_1x_4^2 + e(x_1(t-r),\ldots,x_6(t-r))x_2x_4\\ &\quad + g(x_1(t-r),\ldots,x_6(t-r))x_2^2\}sgna_1\\ &\quad - x_2\int_{t-r}^{t} h'(x_1(s))x_2(s)ds - \lambda_1 r x_2^2 + \lambda_1\int_{t-r}^{t} x_2^2(s)ds\\ &= |a_1|\left(x_4 + \frac{1}{2|a_1|}e(.)x_2\right)^2 + \{g(.)sgna_1 - \frac{1}{4|a_1|}e^2(.)\}x_2^2\\ &\quad - x_2\int_{t-r}^{t} h'(x_1(s))x_2(s)ds - \lambda_1 r x_2^2\\ &\quad + \lambda_1\int_{t-r}^{t} x_2^2(s)ds.\end{aligned}$$

The assumption $a_6 \geqslant h'(x_1) > 0$ and the estimate $2|mn| \leqslant m^2 + n^2$ imply that

$$\begin{aligned}-x_2\int_{t-r}^{t} h'(x_1(s))x_2(s)ds &\geqslant -|x_2|\int_{t-r}^{t} h'(x_1(s))|x_2(s)|ds\\ &\geqslant -\frac{1}{2}a_6 r x_2^2 - \frac{1}{2}a_6\int_{t-r}^{t} x_2^2(s)ds\end{aligned}$$

so that

$$\dot{V}_1 \geqslant \{\delta - (2^{-1}a_6 + \lambda_1)r\}x_2^2 + \left(\lambda_1 - \frac{1}{2}a_6\right)\int_{t-r}^{t} x_2^2(s)ds.$$

Let $\lambda_1 = \frac{1}{2}a_6$. Hence

$$\dot{V}_1 \geqslant (\delta - a_6 r)x_2^2 > 0$$

provided that $r < \frac{\delta}{a_6}$.

On the other hand, $\frac{d}{dt}V_1(x_{1t}, x_{2t}, x_{3t}, x_{4t}, x_{5t}, x_{6t}) = 0$ if and only if $x_2 = 0$, which implies that

$$x_1 = \xi_1(\text{constant}), \quad x_2 = x_3 = x_4 = x_5 = x_6 = 0.$$

Furthermore, in view of $\frac{d}{dt}V_1(x_{1t}, x_{2t}, x_{3t}, x_{4t}, x_{5t}, x_{6t})$ and the system (3), we can easily obtain $x_1 = x_2 = x_3 = x_4 = x_5 = x_6 = 0$ since $h(\xi_1) = 0$ if and only if $\xi_1 = 0$. The functional V_1 thus has all the requisite Krasovskii properties subject to the conditions in the theorem, which now follows (see Krasovskii [4]). By the above discussion, we conclude that the zero solution of Eq. (2) is unstable. The proof of the theorem is completed.

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