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SHORT COMMUNICATION

Refined Young inequalities with Specht's ratio

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Abstract In this paper, we show that the v-weighted arithmetic mean is greater than the product of the v-weighted geometric mean and Specht's ratio. As a corollary, we also show that the v-weighted geometric mean is greater than the product of the v-weighted harmonic mean and Specht's ratio. These results give the improvements for the classical Young inequalities, since Specht's ratio is generally greater than 1. In addition, we give an operator inequality for positive operators, applying our refined Young inequality.

1. Introduction

We start from the famous Young inequality:

$$(1-v)a+vb \geqslant a^{1-v}b^{v} \tag{1}$$

for positive numbers a, b and $v \in [0,1]$. The inequality (1) is also called v-weighted arithmetic-geometric mean inequality and its reverse inequality was given in [1] with Specht's ratio as follows:

$$S\left(\frac{a}{b}\right)a^{1-v}b^{v} \geqslant (1-v)a+vb \tag{2}$$

for positive numbers a, b and $v \in [0,1]$, where the Specht's ratio [2,3] was defined by

$$S(h) \equiv \frac{h^{\frac{1}{h-1}}}{e\log h^{\frac{1}{h-1}}}, \quad (h \neq 1)$$

for a positive real number h.

Recently, based on the refined Young inequality [4,5]:

$$(1-v)a+vb \geqslant a^{1-v}b^{v}+r\left(\sqrt{a}-\sqrt{b}\right)^{2}, \tag{3}$$

for positive numbers a, b and $v \in [0,1]$, where $r \equiv \min\{v, 1-v\}$, we proved the following operator inequalities:

Proposition 1 [6]. *For $v \in [0,1]$ and positive operators A and B, we have*

$$\begin{aligned}(1-v)A+vB &\geqslant A\sharp_{v}B+2r\left(\frac{A+B}{2}-A\sharp_{1/2}B\right)\\ &\geqslant A\sharp_{v}B\\ &\geqslant \left\{A^{-1}\sharp_{v}B^{-1}+2r\left(\frac{A^{-1}+B^{-1}}{2}-A^{-1}\sharp_{1/2}B^{-1}\right)\right\}^{-1}\\ &\geqslant \left\{(1-v)A^{-1}+vB^{-1}\right\}^{-1}\end{aligned}$$

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where $r \equiv min\{v, 1-v\}$ and $A \sharp_v B \equiv A^{1/2}(A^{-1/2}BA^{-1/2})^v A^{1/2}$ defined for $v \in [0,1]$.

The above inequalities can be regarded as an additive-type refinement for the Young inequalities [7,8]:

$$(1-v)A + vB \geqslant A \sharp_v B \geqslant \left\{(1-v)A^{-1} + vB^{-1}\right\}^{-1}. \tag{4}$$

In this short paper, we give a multiplicative-type refinement for the Young inequalities (4) with the Specht's ratio.

2. Main results

We here review the properties of the Specht's ratio. See [1–3] for example, as for the proof and the details.

Lemma 1. *The Specht's ratio*

$$S(h) \equiv \frac{h^{\frac{1}{h-1}}}{e \log h^{\frac{1}{h-1}}}, \quad (h \neq 1,\ h > 0)$$

has the following properties.

(i) *$S(1) = 1$ and $S(h) = S(1/h) > 1$ for $h > 0$.*
(ii) *$S(h)$ is a monotone increasing function on $(1, \infty)$.*
(iii) *$S(h)$ is a monotone decreasing function on $(0,1)$.*

We use the following lemmas to show our theorem.

Lemma 2. *For $x \geqslant 1$, we have*

$$\frac{2(x-1)}{x+1} \leqslant \log x \leqslant \frac{x-1}{\sqrt{x}}. \tag{5}$$

Proof 1. We firstly prove the second inequality of (5). We put $\sqrt{x} = t$ and

$$f(t) \equiv \frac{t^2-1}{t} - 2\log t, \quad (t \geqslant 1).$$

Then we have $f'(t) = \left(\frac{t-1}{t}\right)^2 \geqslant 0$ and $f(1) = 0$. Thus we have $f(t) \geqslant f(1) = 0$ and then we have $\log t^2 \leqslant \frac{t^2-1}{t}$, which implies the second inequality in (5).

We also put

$$g(x) \equiv (x+1)\log x - 2(x-1), \quad (x \geqslant 1).$$

Then we have $g(1) = 0$, $g'(x) = \log x + \frac{x+1}{x} - 2, g'(1) = 0$ and $g''(x) = \frac{x-1}{x^2} \geqslant 0$. Therefore we have $g(x) \geqslant g(1) = 0$, which implies the first inequality in (5). □

Note that Lemma 2 can be also proven by the following relation for three means:

$$\sqrt{xy} < \frac{x-y}{\log x - \log y} < \frac{x+y}{2}$$

for positive real numbers x and y, where $x \neq y$.

Lemma 3. *For $t > 0$, we have*

$$e(t^2+1) \geqslant (t+1)t^{\frac{t}{t-1}}. \tag{6}$$

Proof 2. We firstly prove the inequality (6) for $t \geqslant 1$. We put

$$f(t) = e(t^2+1) - (t+1)t^{\frac{t}{t-1}}.$$

By using the first inequality of (5), we have

$$\begin{aligned} f'(t) &= \frac{2t(t-1)^2 e + 2t(1-t)t^{\frac{t}{t-1}} + t^{\frac{t}{t-1}}(t+1)\log t}{(t-1)^2} \\ &\geqslant \frac{2t(t-1)^2 e + 2t(1-t)t^{\frac{t}{t-1}} + 2(t-1)t^{\frac{t}{t-1}}}{(t-1)^2} \\ &= \frac{2t(t-1)^2 e - 2t(t-1)^2 t^{\frac{1}{t-1}}}{(t-1)^2} \\ &\geqslant \frac{2t(t-1)^2 t^{\frac{1}{t-1}} - 2t(t-1)^2 t^{\frac{1}{t-1}}}{(t-1)^2} = 0. \end{aligned}$$

In the last inequality, we have used the fact that $\lim_{t\to 1} t^{\frac{1}{t-1}} = e$ and the function $t^{\frac{1}{t-1}}$ is monotone decreasing on $t \in [1, \infty)$. We also have $f(1) = 0$ so that we have $f(t) \geqslant 0$ which proves the following inequality:

$$e(t^2+1) \geqslant (t+1)t^{\frac{t}{t-1}}, \quad t \geqslant 1.$$

Putting $t = \frac{1}{s}$ in the above inequality with simple calculations, we have

$$e(s^2+1) \geqslant (s+1)s^{\frac{s}{s-1}}, \quad 0 < s \leqslant 1. \qquad \square$$

Then we have the following inequality which improves the classical Young inequality between v-weighted geometric mean and v-weighted arithmetic mean.

Theorem 1. *For $a, b > 0$ and $v \in [0,1]$,*

$$(1-v)a + vb \geqslant S\left(\left(\frac{b}{a}\right)^r\right)a^{1-v}b^v, \tag{7}$$

where $r \equiv min\{v, 1-v\}$ ans $S(\cdot)$ is the Specht's ratio.

Proof 3. We prove the following inequality

$$\frac{(b-1)v+1}{b^v S(b^v)} = \frac{e\{(b-1)v+1\}\log b^v}{(b^v)^{\frac{b^v}{b^v-1}}(b^v-1)} \geqslant 1 \tag{8}$$

in the case of $0 \leqslant v \leqslant \frac{1}{2}$. From Lemma 2, we have

$$\frac{\log b^v}{b^v - 1} \geqslant \frac{2}{b^v+1}, \quad b > 0.$$

Therefore we have the following first inequality:

$$\frac{e\{(b-1)v+1\}\log b^v}{(b^v)^{\frac{b^v}{b^v-1}}(b^v-1)} \geqslant \frac{2e\{(b-1)v+1\}}{(b^v)^{\frac{b^v}{b^v-1}}(b^v+1)} \geqslant 1, \tag{9}$$

thus we have only to prove the above second inequality. For this purpose, we put the following function f_b on $v \in \left[0, \frac{1}{2}\right]$ for $b > 0$:

$$f_b(v) \equiv 2e\{(b-1)v+1\} - (b^v)^{\frac{b^v}{b^v-1}}(b^v+1).$$

Then we have

$$\begin{aligned} f_b''(v) = &-\frac{(\log b)^2}{(b^v-1)^4}(b^v)^{\frac{2b^v-1}{b^v-1}}\Big\{(b^v-1)^2(4b^{2v} - 5b^v - 1) \\ &- \Big(b^v-1)^2(3b^v+1)\log b^v + b^v(b^v+1)(\log b^v)^2\Big\}. \end{aligned}$$

For the case of $b \geqslant 1$, using the inequalities (5), we have

$$(b^v-1)^2(4b^{2v}-5b^v-1)-(b^v-1)^2(3b^v+1)\log b^v+b^v(b^v+1)(\log b^v)^2$$
$$\geqslant (b^v-1)^2(4b^{2v}-5b^v-1)-(b^v-1)^2(3b^v+1)\frac{b^v-1}{b^{v/2}}+b^v(b^v+1)\left(\frac{2(b^v-1)}{b^v+1}\right)^2$$
$$=\frac{(b^{v/2}-1)^4(b^{v/2}+1)^3(4b^{2v}+b^{3v/2}+4b^v+1)}{b^{v/2}(b^v+1)}\geqslant 0.$$

For the case of $0<b\leqslant 1$, using the inequalities (5), we also have

$$(b^v-1)^2(4b^{2v}-5b^v-1)-(b^v-1)^2(3b^v+1)\log b^v+b^v(b^v+1)(\log b^v)^2$$
$$=(b^v-1)^2(4b^{2v}-5b^v-1)+(b^v-1)^2(3b^v+1)\log\frac{1}{b^v}+b^v(b^v+1)\left(\log\frac{1}{b^v}\right)^2$$
$$\geqslant (b^v-1)^2(4b^{2v}-5b^v-1)+(b^v-1)^2(3b^v+1)\left(\frac{2(\frac{1}{b^v}-1)}{\frac{1}{b^v}+1}\right)+b^v(b^v+1)\left(\frac{2(\frac{1}{b^v}-1)}{\frac{1}{b^v}+1}\right)^2$$
$$=\frac{(b^v-1)^4(4b^v+1)}{b^v+1}\geqslant 0.$$

Thus we have $f_b''(v)\leqslant 0$ for $b>0$. In addition, we have $f_b(0)=0$ and $f_b\left(\frac{1}{2}\right)=e(b+1)-\left(\sqrt{b}+1\right)\left(\sqrt{b}\right)^{\frac{\sqrt{b}}{\sqrt{b}-1}}\geqslant 0$, applying Lemma 3 with $t=\sqrt{b}>0$. Therefore we have $f_b(v)\geqslant 0$ for $v\in\left[0,\frac{1}{2}\right]$. Thus we have the following inequality

$$\frac{(b-1)v+1}{b^vS(b^v)}\geqslant 1,\quad 0\leqslant v\leqslant\frac{1}{2},\ b>0 \tag{10}$$

which implies

$$vb+(1-v)\geqslant S(b^v)b^v.$$

Replacing b by $\frac{b}{a}$ in the above inequality and then multiplying a to the both sides, we have

$$(1-v)a+vb\geqslant S\left(\left(\frac{b}{a}\right)^v\right)a^{1-v}b^v,\quad 0\leqslant v\leqslant\frac{1}{2},\ a,b>0.$$

Finally, from the inequality (10), we have

$$\frac{(a-1)\mu+1}{a^\mu S(a^\mu)}\geqslant 1,\quad 0\leqslant\mu\leqslant\frac{1}{2},\ a>0.$$

Putting $v=1-\mu$ in the above inequality we have

$$v+(1-v)a\geqslant a^{1-v}S(a^{1-v}),\quad \frac{1}{2}\leqslant v\leqslant 1,\ b>0.$$

Replacing a by $\frac{a}{b}$ in the above inequality and then multiplying b to the both sides, we have

$$(1-v)a+vb\geqslant S\left(\left(\frac{a}{b}\right)^{1-v}\right)a^{1-v}b^v,\quad \frac{1}{2}\leqslant v\leqslant 1,\ a,b>0,$$

since $S(1/h)=S(h)$ for $h>0$, ((i) of Lemma 1). Thus the proof of the present theorem was completed. □

Remark 1. Theorem 1 gives a tighter lower bound of the v-weighted arithmetic mean of two variables, since the Specht's ratio is greater than 1, ((i) of Lemma 1).

The following inequality also improves the relation between v-weighted geometric mean and v-weighted harmonic mean.

Corollary 1. *For positive numbers a, b and $v\in[0,1]$, we have*

$$S\left(\left(\frac{a}{b}\right)^r\right)\left((1-v)\frac{1}{a}+v\frac{1}{b}\right)^{-1}\leqslant a^{1-v}b^v, \tag{11}$$

where $r\equiv\min\{v,1-v\}$ and $S(\cdot)$ is the Specht's ratio.

Proof 4. Replace a and b in Theorem 1 by $\frac{1}{a}$ and $\frac{1}{b}$, respectively. □

Applying Theorem 1, we have the following operator inequality for positive operators.

Theorem 2. *For two positive operators A, B and positive real numbrs m,m',M,M' satisfying the following conditions (i) or (ii):*

(i) $0<m'I\leqslant A\leqslant mI<MI\leqslant B\leqslant M'I$
(ii) $0<m'I\leqslant B\leqslant mI<MI\leqslant A\leqslant M'I$

with $h\equiv\frac{M}{m}$ and $h'\equiv\frac{M'}{m'}$, we have

$$(1-v)A+vB\geqslant S(h^r)A\sharp_vB \tag{12}$$
$$\geqslant A\sharp_vB \tag{13}$$
$$\geqslant S(h^r)\left\{(1-v)A^{-1}+vB^{-1}\right\}^{-1} \tag{14}$$
$$\geqslant\left\{(1-v)A^{-1}+vB^{-1}\right\}^{-1}, \tag{15}$$

where $v\in[0,1]$, $r\equiv\min\{v,1-v\}$, $S(\cdot)$ is the Specht's ratio and $A\sharp_vB\equiv A^{1/2}(A^{-1/2}BA^{-1/2})^vA^{1/2}$ is the v-power mean for positive operators A and B [9].

Proof 5. From Theorem 1, we have

$$vx+(1-v)\geqslant S(x^r)x^v$$

for any $x>0$.

Therefore we have

$$vX+(1-v)I\geqslant\min_{m'\leqslant x\leqslant M'}S(x^r)X^v$$

for the positive operator X such that $0<m'I\leqslant X\leqslant M'I$. We here put $X=A^{-1/2}BA^{-1/2}$.

In the case of (i), we have $h=\frac{M}{m}\leqslant A^{-1/2}BA^{-1/2}\leqslant\frac{M'}{m'}=h'$. Then we have

$$vA^{-1/2}BA^{-1/2}+(1-v)I\geqslant\min_{h\leqslant x\leqslant h'}S(x^r)\left(A^{-1/2}BA^{-1/2}\right)^v.$$

Since $S(x)$ is an increasing function for $x>1$, ((ii) of Lemma 1) we have

$$vA^{-1/2}BA^{-1/2}+(1-v)I\geqslant S(h^r)\left(A^{-1/2}BA^{-1/2}\right)^v. \tag{16}$$

In the case of (ii), we also have $\frac{1}{h'}=\frac{m'}{M'}\leqslant A^{-1/2}BA^{-1/2}\leqslant\frac{m}{M}=\frac{1}{h}$. Then we also have

$$vA^{-1/2}BA^{-1/2}+(1-v)I\geqslant\min_{\frac{1}{h'}\leqslant x\leqslant\frac{1}{h}}S(x^r)\left(A^{-1/2}BA^{-1/2}\right)^v.$$

Since $S(x)$ is a decreasing function for $0 < x < 1$ ((iii) of Lemma 1), we have

$$vA^{-1/2}BA^{-1/2} + (1-v)I \geqslant S\left(\frac{1}{h^r}\right)\left(A^{-1/2}BA^{-1/2}\right)^v.$$

By the property $S(x) = S(1/x)$ for $x > 0$ ((i) of Lemma 1), the above inequality is the same to (16). Multiplying $A^{1/2}$ from the both sides to the inequality (16), we have the inequality (12).

The inequality (14) can be proven by replacing A and B by A^{-1} and B^{-1}, respectively in the first inequality and taking its inverse.

The inequality (13) and the inequality (15) are trivial, due to the property of the Specht's ratio $S(x) \geqslant 1$ for $x > 0$. □

3. Conclusion

We have shown the refined Young inequalities for a real number with Specht ratio. Applying these inequalities we have obtained their operator version inequalities which refine the classical Young operator inequalities as our previous results have done in Proposition 1 (See [6]). Therefore we have two different refinements for the classical Young inequalities (4). Two kinds of the operator inequalities are based on the scalar inequalities (3) and (7).

In our previous paper [6], we have proved the additive-type refined Young inequality for n real numbers.

Proposition 2 [6]. *Let $a_1, \cdots, a_n \geqslant 0$ and $p_1, \cdots, p_n > 0$ with $\sum_{j=1}^n p_j = 1$ and $\lambda \equiv min\{p_1, \cdots, p_n\}$. If we assume that the multiplicity attaining λ is 1, then we have*

$$\sum_{i=1}^n p_i a_i - \prod_{i=1}^n a_i^{p_i} \geqslant n\lambda\left(\frac{1}{n}\sum_{i=1}^n a_i - \prod_{i=1}^n a_i^{1/n}\right), \tag{17}$$

with equality if and only if $a_1 = \cdots = a_n$.

See [10,11] for recent developments based on the above inequality (or Jensen-type inequality [12]). It is also notable that we do not need the assumption that the multiplicity attaining λ is 1, to prove only inequality (17). This assumption connects with the equality condition.

Closing this section, we give comments on the multiplicative-type refined Young inequality for n real numbers. We have not yet found its proof. We also have not found any counter-examples for the following 3-variables case:

$$w_1a_1 + w_2a_2 + w_3a_3 \geqslant S(h^r)a_1^{w_1}a_2^{w_2}a_3^{w_3}, \tag{18}$$

for $a_i \in [m, M]$ where $0 < m < M$ with $h \equiv \frac{\max\{a_1,a_2,a_3\}}{\min\{a_1,a_2,a_3\}}$ and $r \equiv \min\{w_1,w_2,w_3\}$, where $w_i > 0$ and $w_1 + w_2 + w_3 = 1$.

The problem on the multiplicative-type refined Young inequality for n real numbers will be our future work.

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